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Projective Parallelians and Related Porisms

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ABSTRACT

We give a projective generalization of the construction of parallelians and the thus defined conics. To any properly chosen point P and line g in the plane of a triangle $\Delta = ABC$, we construct six points that always lie on a conic $\mathcal{P}$, the parallelian conic $\mathcal{P}$ of the pivot P with respect to Δ. Further, we find the parallelian tangent conic $\mathcal{T}$, the parallelian inconic I, and two further conics $\mathcal{D}$ and $\mathcal{J}$ that are related in a natural way with Δ and P. Any pair out of these conics gives rise to a certain porism and even a chain of porisms by means of polarization. We study the regularity and singularity as well as the relative position of these conics with respect to the line g depending on the choice of P and g. We also give a detailed study of the sets of possible pivot points changing the triangle or hexagon porisms of any pair of conics into such with one-parameter families of quadrangles and pentagons.

Key words: parallelian, parallelian conic, porism, triangle cubic, triangle center, algebraic transformation

MSC2020: 51M15, 51M04, 14E05

Projektivne paralelijane i s njima povezane porizme

SAŽETAK

Dajemo projektivno poopćenje konstrukcije paralelijana i tako definiranih konika. Za bilo koju dobro odabranu točku P i pravac g u ravnini trokuta $\Delta = ABC$ konstruiramo šest točaka koje uvijek leže na jednoj konici $\mathcal{P}$, paralelijanskoj konici $\mathcal{P}$ točke P s obzirom na trokut Δ. Nadalje, nalazimo paralelijansku tangentnu koniku $\mathcal{T}$, paralelijansku upisanu koniku I i dvije daljnje konike $\mathcal{D}$ i $\mathcal{J}$ koje su prirodno povezane s Δ i P. Bilo koji par ovih konika rezultira određenom porizmom, pa čak i lancem porizmi pomoću polarizacije. Proučavamo regularnost i singularnost kao i posebne položaje ovih konika prema pravcu g ovisno o izboru polazne točke P i pravca g. Također, dajemo detaljno istraživanje skupova mogućih polaznih točaka koje mijenjaju trokutaste ili šesterokutne porizme bilo kojeg para konika u porizme s jednoparametarskim familijama četverokuta i peterokuta.

Ključne riječi: paralelijana, paralelijanska konika, porizma, kubika trokuta, središte trokuta, algebarska transformacija

1 Introduction

In the present paper, we shall construct several chains of porisms that are attached to a triangle $\Delta = ABC$, a point P, and a line g in a natural way. The initial steps of the construction can be done in a purely synthetic way and the same holds true for the proofs of the existence of the conics involved. The construction (synthetic or algebraic) are exclusively done in the framework of projective geometry. At a later stage, we have to deploy the analytical approach. This allows us to deduce some algebraic properties of the porisms and some conditions on the choice of the pivot point P.

1.1 Prior and related work

In recent years, porisms were studied mainly within the framework of Euclidean geometry, focusing on invariants [14, 15, 25], traces [6, 9, 11, 16, 21, 23], closure conditions [7, 8], and relations to billards and Poncelet grids [10, 26, 27, 28, 29]. Nevertheless, results concerning the projective nature of porisms are also given in [24, 29]. An excellent overview on the history and various approaches towards the classical forms of porisms can be found in [3, 5].

Occasionally, the article [1] disclosed the relations between Euclidean parallelian conics and the related proisms. As we shall see, all the results from [1] allow an explanation from the superordinate standpoint of projective geometry as is the case with some results from the Euclidean geometry of the triangle (cf. [20]).

1.2 Contributions of the present paper

In Section 2, we show the existence of the projectivized parallelian conic $\mathcal{P}$ and the parallelian inconic $\mathcal{I}$. This can be done in a purely synthetic way. Further, the first porisms are described and extended to the chains of porisms that are obtained by means of polarization or by tracing the (discrete) exponential pencil of conics spanned by $\mathcal{I}$ and $\mathcal{P}$.

Section 3 is to show that the parallelian conic $\mathcal{P}$ is enclosed by two triangles Δ_U and Δ_V whose six vertices lie on a conic $\mathcal{D}$. This gives rise to a triangle porism between $\mathcal{P}$ and $\mathcal{D}$, and consequently, this gives rise to a chain of triangle porisms. Moreover, the six vertices of Δ_U and Δ_V form a hexagon with an inconic $\mathcal{J}$ (provided a certain ordering of points). Thus, we also find a further hexagon porism independent of the hexagon porisms discovered so far.

Section 4 mentions the relations to already existing and Euclidean cases.

Finally, in Section 5, we discuss all possible pairings of projectivized parallelian conics and the thus defined porisms (and chains of porisms). In particular, we derive conditions on the pivot point P such that certain types of porisms can be found in between the chosen pairs of conics. We shall not discuss whether these porisms do really exist between regular conics or not.

2 Projective parallelians

2.1 The first porism

In the projective plane, we choose a triangle $\Delta = ABC$ and called it henceforth the *base triangle*. The union of the three side lines $[A,B]$, $[B,C]$, $[C,A]$ shall be denoted by $\Delta^\star$. Further a point P which is not incident with any line of $\Delta^\star$ is chosen and called the *pivot point*. Then, we assume that $g \notin \Delta^\star$ is a line neither incident with a vertex of Δ nor passing through P.

We shall label the three intersection points of $\Delta^\star$'s lines with g with $C^\star := g \cap [A,B]$ (cyclic). Now, we call the lines $[P,A^\star]$, $[P,B^\star]$, and $[P,C^\star]$ the *g-parallels* of $[B,C]$, $[C,A]$, and $[A,B]$ through P. The projections of P from the points $A^\star$, $B^\star$, $C^\star$ onto the non-incident sides of Δ are defined as

$$\begin{aligned} P_1 &:= [P,C^\star] \cap [B,C], P_2 := [P,C^\star] \cap [C,A], \\ P_3 &:= [P,A^\star] \cap [C,A], P_4 := [P,A^\star] \cap [A,B], \\ P_5 &:= [P,B^\star] \cap [A,B], P_6 := [P,B^\star] \cap [B,C]. \end{aligned} \tag{1}$$

The points $P_1, \ldots, P_6$ are the projectivized versions of the elementary geometric parallelians (cf. [18]), and therefore, we call them the *g-parallelians* of P with respect to Δ.

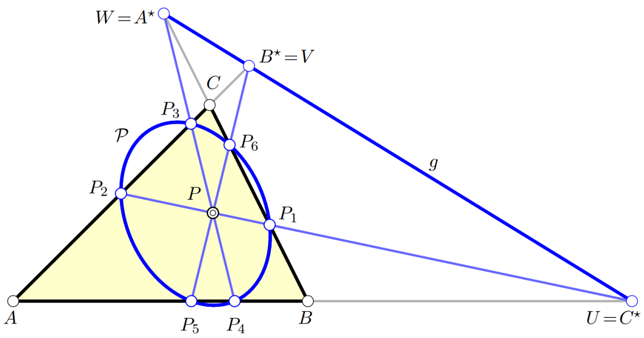

Figure 1: *The conic $\mathcal{P}$ on the six g-parallelians.*

With these preparations, it is rather elementary to show the following:

Theorem 1 *The g-parallelians $P_1, \ldots, P_6$ are located on a single conic $\mathcal{P}$, the* parallelian conic *of P.*

Proof. According to PASCAL's theorem (cf. [12, p. 220]), six points $P_1, \ldots, P_6$ lie on a single conic if, and only if, the three point $U := [P_1,P_2] \cap [P_4,P_5]$, $V := [P_2,P_3] \cap [P_5,P_6]$, $W := [P_3,P_4] \cap [P_6,P_1]$ are collinear. By construction, $U = C^\star$, $V = B^\star$, and $W = A^\star$, which are collinear (located on g). □

Fig. 1 illustrates the contents of Thm. 1.

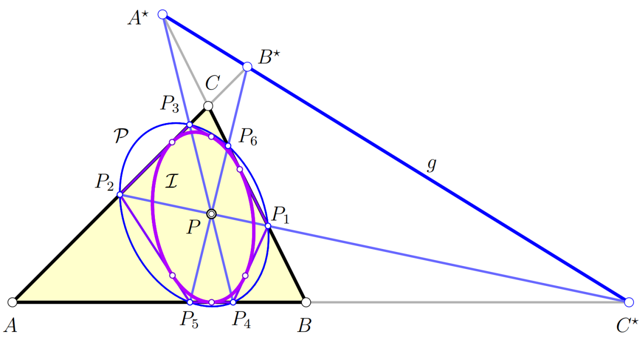

Figure 2: *The inconic $\mathcal{I}$ that comes along with $\mathcal{P}$.*

Further, we can confirm the existence of an inscribed conic:

Theorem 2 *The hexagon $H_1 := P_2P_3P_6P_1P_4P_5$ is tangent to a single conic $\mathcal{I}$, the* parallelian inconic *of P.*

Proof. We define $l_1 := [P_2,P_3]$, $l_2 := [P_3,P_6]$, $l_3 := [P_6,P_1]$, $l_4 := [P_1,P_4]$, $l_5 := [P_4,P_5]$, and $l_6 := [P_5,P_2]$, apply BRIANCHON's theorem (see [12, p. 222]), and find $[l_1,l_2] \cap [l_4,l_5] = [A^\star,P]$, $[l_2,l_3] \cap [l_5,l_6] = [C^\star,P]$, $[l_3,l_4] \cap [l_6,l_1] = [B^\star,P]$. The latter three lines are incident with P, *i.e.*, P equals the Brianchon point of the six lines $l_1, \ldots, l_6$. Therefore, $l_1, \ldots, l_6$ are tangents of a single conic. □

The results of Thms. 1 and 2 give rise to a porism:

Theorem 3 *The pair $(\mathcal{P}, \mathcal{I})$ allows for a poristic family of hexagons with vertices on $\mathcal{P}$ and edges tangent to $\mathcal{I}$.*

Proof. The existence of a single hexagon interscribed between $\mathcal{P}$ and $\mathcal{I}$ is necessary and sufficient in order to guarantee the existence of a one-parameter family of interscribed hexagons (cf. [3, 5]). □

2.2 The first chain of porisms

For what follows, we shall describe points and lines by homogeneous coordinates. It appears useful to assume that the vertices of Δ are the base points of the projective frame. Hence, $A = 1:0:0$, $B = 0:1:0$, $C = 0:0:1$. The pivot point shall be given by $P = \xi:\eta:\zeta \neq 0:0:0$ (and, since P is not contained in any line of $\Delta^\star$, we have $\xi\eta\zeta \neq 0$). The line g can be represented by its homogeneous equation as $lx+my+nz=0$, or equivalently, by its homogeneous coordinates $l:m:n \neq 0:0:0$ which also satisfy $lmn \neq 0$, since g shall not be incident with any vertex of Δ. Further, we have $l\xi+m\eta+n\zeta \neq 0$, for $P \notin g$.

It is a matter of elementary linear algebra to determine the equations (up to non-zero multiples) of $\mathcal{P}$ and $\mathcal{I}$ (cf. [12, p. 254]). So, we find

$$\mathcal{P}: \sum_{\text{cyclic}} l\eta\zeta(m\eta+n\zeta)x^2 - \xi(2\eta\zeta mn+\zeta\xi nl+\xi\eta lm+l^2\xi^2)yz = 0,$$

$$\mathcal{I}: \sum_{\text{cyclic}} l^2(m\eta+n\zeta)^2x^2 - 2mn(l\xi+m\eta)(l\xi+n\zeta)yz = 0. \qquad (2)$$

Here, in the following, $\sum_{\text{cyclic}} f(l,m,n,\xi,\eta,\zeta,x,y,z)$ means the *cyclic sum* of $f(\ldots)$, *i.e.*,

$$\sum_{\text{cyclic}} f(l,m,n,\xi,\eta,\zeta,x,y,z) := f(l,m,n,\xi,\eta,\zeta,x,y,z) + f(m,n,l,\eta,\zeta,\xi,y,z,x) + f(n,l,m,\zeta,\xi,\eta,z,x,y).$$

The variables in the argument (function) are shifted twice cyclically and the three functions are summed up.

The conics $\mathcal{I}$ and $\mathcal{P}$ span a pencil of the third kind. The common pole equals the pivot point P, the common polar line p has the homogeneous coordinates

$$\eta\zeta(m\eta+n\zeta) : \zeta\xi(n\zeta+l\xi) : \xi\eta(l\xi+m\eta). \qquad (3)$$

It is easily verified that the three harmonic conjugates of P with respect to the pairs (P_1,P_2), (P_3,P_4), and (P_5,P_6) are collinear and line up on p. This fact yields a linear construction of p as shown in Fig. 3.

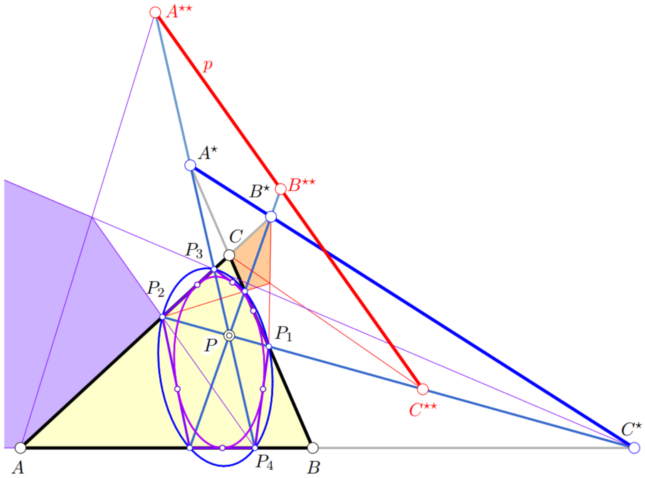

Figure 3: *Linear construction of the common polar p of $\mathcal{P}$ and $\mathcal{I}$: $C^{\star\star}$ is the harmonic conjugate of P with respect to the pair (P_1,P_2) and, in like manner, $A^{\star\star}$ is the harmonic conjugate of P with respect to (P_3,P_4).*

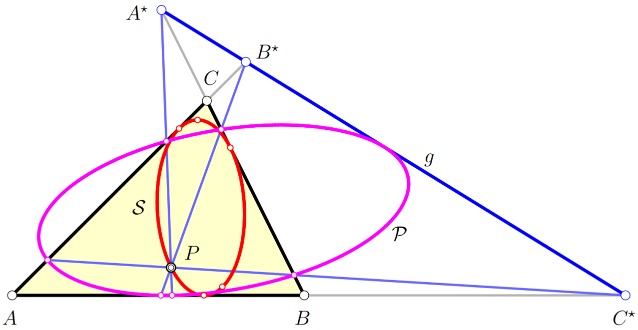

Figure 4: *The conic $\mathcal{P}$ touches g if P is chosen on the inconic $\mathcal{S}$.*

Fig. 4 demonstrates that a properly chosen pivot point P leads to a parallelian conic $\mathcal{P}$ that touches g. The equations (2) of the parallelian conic and the parallelian inconic allow us to state:

Theorem 4
1. The g-parallelian conic $\mathcal{P}$ touches the line g if, and only if, the pivot point P is chosen on the g-Steiner inconic $\mathcal{S}$ of Δ with the equation

$$\mathcal{S}: \sum_{\text{cyclic}} l^2x^2 - 2mnyz = 0. \qquad (4)$$

$\mathcal{P}$ is singular if P is chosen on the g-Steiner circumconic

$$\mathcal{S}': \sum_{\text{cyclic}} lmxy = 0 \qquad (5)$$

and the 'centers' of the singular conics fulfill the g-Steiner Deltoid

$$\mathcal{Q}: \sum_{\text{cyclic}} l^2m^2x^2y^2 - 2l^2mnx^2yz = 0. \quad (6)$$

2. The g-parallelian inconic $\mathcal{I}$ cannot touch g for any suitable choice of P, i.e.*, P may not lie on the sides of the g-anticomplementary triangle* $\Delta_a^\star = \{[A,A^\star],[B,B^\star],[C,C^\star]\}$.

Proof. 1. We compute the resultant of $\mathcal{P}$'s equation (2) and g's equation with respect to any of the variables x, y, or z. The determinant of the coefficient matrix of the remaining quadratic form is the product of the fourth power of g's equation and the quadratic form

$$\sum_{\text{cyclic}} l^2\xi^2 - 2mn\eta\zeta$$

which (set equal to zero) yields the equation of $\mathcal{S}$ after replacing ξ, η, ζ by x, y, z.

The regularity of $\mathcal{P}$ is equivalent to the regularity of its coefficient matrix $\mathbf{P}$. Hence, $\mathcal{P}$ is regular if the homogeneous coordinates $\xi:\eta:\zeta$ of P do not satisfy

$$\underbrace{\xi\eta\zeta}_{=\Delta^\star}\cdot\underbrace{(lm\xi\eta+mn\eta\zeta+nl\zeta\xi)}_{=\mathcal{S}'}\cdot\underbrace{(l\xi+m\eta+n\zeta)^4}_{=g}=0.$$

Since admissible positions of P are off $\Delta^\star$ and off g, P may only be chosen on $\mathcal{S}'$. With the parametrization of $\mathcal{S}' = n(\alpha l+\beta m)\alpha : n(\alpha l+\beta m)\beta : -lm\alpha\beta$ $(\alpha:\beta\neq 0:0)$ inserted of $\xi:\eta:\zeta$ into the first equation of (2), we are able to factor $\mathcal{P}$'s equation and find

$$\begin{aligned}&\mathcal{P}_{\text{singular}}:\\ &\big(\beta^2m^2(\alpha l+\beta m)x-\alpha^2\beta lm^2y-\alpha n(\alpha l+\beta m)^2z\big)\cdot\\ &\big(\alpha\beta^2l^2mx-\alpha^2l^2(\alpha l+\beta m)y+\beta n(\alpha l+\beta m)^2z\big)=0.\end{aligned}$$

The latter equation describes a pair of lines (as long as $l:m:n\neq 0:0:0$) that always intersect in

$$\alpha^2ln(\alpha l+\beta m)^2 : \beta^2mn(\alpha l+\beta m)^2 : \alpha^2\beta^2l^2m^2$$

which parametrizes the quartic $\mathcal{S}'$ given by (6).

2. In the same way, we proceed with $\mathcal{I}$ and find

$$\underbrace{(n\zeta+l\xi)(n\zeta+m\eta)(m\eta+l\xi)}_{=\Delta_a^\star}\underbrace{(l\xi+m\eta+n\zeta)}_{=g}=0$$

relating the coordinates of the pivot point P such that it yields a parallelian inconic $\mathcal{I}$ touching g. If we replace ξ, η, ζ with x, y, z, the first three factors are the equations of the sides of Δ's g-anticomplementary triangle $\Delta_a^\star$ and the fourth factor yields the equation of the line g. □

If g is the ideal line of the projectively closed Euclidean plane, the inconic $\mathcal{S}$ (4) described in Thm. 4 becomes the Steiner inellipse and the corresponding parallelian conics are then parabolas, cf. [1]. Therefore, $\mathcal{S}$ can be considered the g-Steiner inconic.

Here, we shall also remark that the g-Steiner deltoid $\mathcal{Q}$ is the image of g-Yff inconic $\sum_{\text{cyclic}} x^2-2yz=0$ under the g-isogonal transformation $x\to mnyz$, $y\to nlzx$, $z\to lmxy$. Fig. 5 shows the g-Steiner circumconic $\mathcal{S}'$ and its g-isogonal image $\mathcal{Q}$.

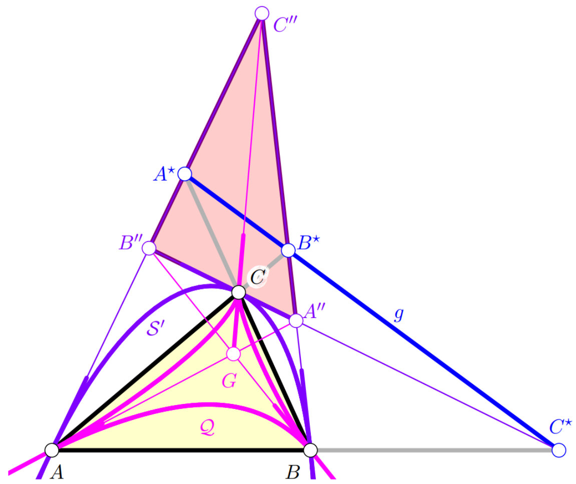

Figure 5: *The g-anticomplementary triangle $\Delta_a = A''B''C''$, the g-Steiner circumconic $\mathcal{S}'$, and its g-isogonal image $\mathcal{Q}$ (the g-Steiner deltoid).*

The intersections of subsequent tangents t_i to $\mathcal{P}$ at points P_i define (among others) the six points

$$\begin{aligned}&T_{23}:=t_2\cap t_3,\quad T_{36}:=t_3\cap t_6,\quad T_{61}:=t_6\cap t_1,\\ &T_{14}:=t_1\cap t_4,\quad T_{45}:=t_4\cap t_5,\quad T_{56}:=t_5\cap t_6,\end{aligned}\quad (7)$$

which lie on a single conic $\mathcal{T}$ with the equation

$$\mathcal{T}: \sum_{\text{cyclic}} lmn(m\eta+n\zeta)(3l\xi+m\eta+n\zeta)\eta^2\zeta^2x^2 = \sum_{\text{cyclic}} l\xi^2\left((2mn\eta\zeta)^2+\Big(\sum_{\text{cyclic}} lm\xi\eta((l\xi+m\eta)^2+6mn\eta\zeta)\Big)\right)yz. \quad (8)$$

We call $\mathcal{T}$ the *g-parallelian tangent conic* of P. It is obvious that $\mathcal{T}$ is the polar image of $\mathcal{I}^\star$ (*i.e.*, the dual of the conic $\mathcal{I}$ or the set of tangents of $\mathcal{I}$).

2.3 An elliptic sextic

In a way similar to the proof of Thm. 4, we can show that $\mathcal{T}$ is tangent to g if the pivot point P is chosen on the sextic

$$\mathcal{S}_{\mathcal{T}}: \sum_{\text{cyclic}} l^2m^2x^2y^2(lx-my)^2+2l^3mnx^3yz(lx-my-nz)= 6l^2m^2n^2x^2y^2z^2. \quad (9)$$

An example of such a sextic $\mathcal{S}_{\mathcal{T}}$ is displayed in Fig. 6. The sextic is shown together with a certain pivot point P, the corresponding parallelian conic $\mathcal{P}$, and the parallelian tangent conic $\mathcal{T}$ which touches g since the pivot P is chosen on the sextic $\mathcal{S}_{\mathcal{T}}$ in Fig. 7.

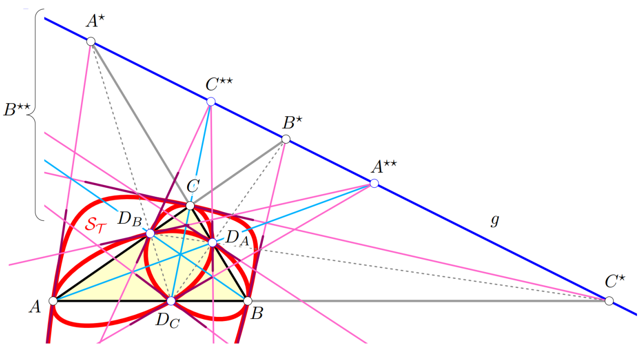

Figure 6: *The sextic $\mathcal{S}_{\mathcal{T}}$ as the locus of pivot points P whose g-parallelian tangent conic $\mathcal{T}$ touches g.*

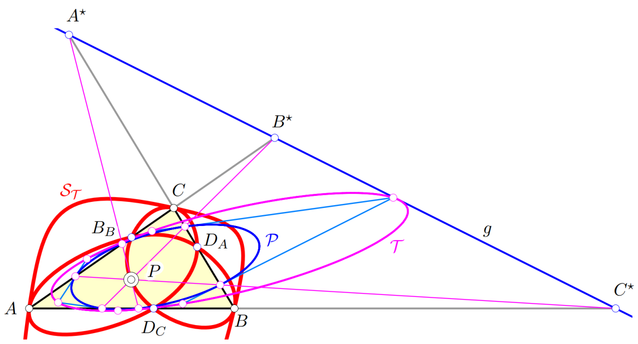

Figure 7: *The g-parallelian tangent conic $\mathcal{T}$ touches g, since it corresponds to a pivot point $P \in \mathcal{S}_{\mathcal{T}}$.*

The curve $\mathcal{S}_{\mathcal{T}}$ has three ordinary double points at the vertices of Δ. Further, it carries three tacnodes at the vertices D_A, D_B, D_C of the Cevian triangle of the triangle pole of g with respect to Δ. At any of these tacnodes, two linear branches emerge. For one of them, the node is a flat point (*i.e.*, it has a local expansion of the form $(t, t^4 + O(t^5))$, cf. [2, 4], only for the flat point branch). The lines $[A, A^\star]$ (cyclic) are the tangents at the flat points A (cyclic). $\mathcal{S}_{\mathcal{T}}$'s tangents to the linear branches at the ordinary double points $D_C \in [A, B]$ (cyclic) pass through the points $A^{\star\star} := [A, D_A] \cap g$ and $B^{\star\star} := [B, D_B] \cap g$ (cyclic). Moreover, any pair of ordinary double points on $\mathcal{S}_{\mathcal{T}}$ is collinear with a *star point*, *i.e.*, D_A, D_B, $C^\star$ (cyclic) are collinear (see Fig. 6).

2.4 The iteration of the porism

If $\mathbf{I}$, $\mathbf{P}$, and $\mathbf{T}$ denote the coefficient matrices of the conics $\mathcal{I}$, $\mathcal{P}$, and $\mathcal{T}$, we first note that they are regular, provided the proper choice of the pivot point P, *i.e.*, P not on any side of Δ or $\Delta_a^\star$ and also not on the g-Steiner circumconic $\mathcal{S}'$. Further, we can verify the following matrix identity

$$\mathbf{P}\mathbf{I}^{-1}\mathbf{P} = \lambda\mathbf{T}$$

with

$$\lambda = -\frac{\sum_{\text{cyclic}} l\xi}{4lmn \prod_{\text{cyclic}} l\xi + m\eta}$$

which depends on P and g solely. (The cyclic product is explained in nearly the same way as the cyclic sum.) This expresses what is clear from the construction: The conic $\mathcal{T}$ is the polar image of $\mathcal{I}$ with respect to $\mathcal{P}$. Moreover, from $\mathcal{I}$ and $\mathcal{P}$, we can construct the "next" circumscribed conic, say $\mathcal{U}$, by repeating the polarization, or in more simple terms, by once again intersecting neighboring tangents of $\mathcal{T}$ in order to obtain a further sextuple of conconic points. Hence, we can state:

Theorem 5 *The parallelian inconic $\mathcal{I}$ and the parallelian conic $\mathcal{P}$ span an exponential pencil of conics in which any pair of subsequent conics allows for a poristic family of hexagons.*

Proof. We have already found that $\lambda\mathbf{T} = \mathbf{P}\mathbf{I}^{-1}\mathbf{P}$ holds (with some λ depending on P and g). According to [13], the coefficient matrices $\mathbf{M}$ of the conics in the exponential pencil spanned by $\mathcal{I}$ and $\mathcal{P}$ are obtained from $\mathbf{I}$ and $\mathbf{P}$ as

$$\mathbf{M}(k) = \mathbf{P}(\mathbf{I}^{-1}\mathbf{P})^{k-1}, \quad k \in \mathbb{Z}.$$

For any integer k, $\mathbf{M}(k)$ and $\mathbf{M}(k+1)$ are the coefficient matrices of two conics that allow for the same kind of porism as $\mathcal{I}$ and $\mathcal{P}$ or $\mathcal{P}$ and $\mathcal{T}$ do:

We use the Cayley criterion [12, p. 432, Thm. 9.5.4] in order to show that the type of porism is preserved when tracing the discrete exponential pencil. If the coefficients of the power series $\sqrt{\det(\mathbf{P}\cdot t+\mathbf{I})} = a_0 + a_1t + a_2t^2 + a_3t^3 + \ldots$ fulfill $a_2 = 0$, $\mathcal{I}$ and $\mathcal{P}$ define a poristic triangle family; $a_3 = 0$ guarantees for a poristic family of interscribed quadrangles. If $\det\begin{pmatrix} a_2 & a_3 \\ a_3 & a_4 \end{pmatrix} = 0$ or $\det\begin{pmatrix} a_3 & a_4 \\ a_4 & a_5 \end{pmatrix} = 0$, then the conics $\mathcal{I}$ and $\mathcal{P}$ allow for poristic families of pentagons or hexagons. In any case, $\mathcal{I}$ serves as the inconic (touched by the sides of the polygons) and $\mathcal{P}$ is the circumconic (carrying the vertices of the polygons).

Now, we show that these conditions are valid for any pair of subsequent conics in the exponential pencil spanned by $\mathcal{I}$ and $\mathcal{P}$. For that purpose, we write down the discriminant of the Cayley function for $\mathbf{M}(k+1)$ and $\mathbf{M}(k)$ and find

$$\begin{aligned}&\det(\mathbf{M}(k+1)\cdot t+\mathbf{M}(k)) =\\ &\det(\mathbf{P}(\mathbf{I}^{-1}\mathbf{P})^{k}\cdot t+\mathbf{P}(\mathbf{I}^{-1}\mathbf{P})^{k-1}) =\\ &\det(\mathbf{P}\cdot t+\mathbf{P}(\mathbf{I}^{-1}\mathbf{P})^{-1})\det\left((\mathbf{I}^{-1}\mathbf{P})^{k}\right) =\\ &\det(\mathbf{P}\cdot t+\mathbf{I})\det\left((\mathbf{I}^{-1}\mathbf{P})^{k}\right),\end{aligned}$$

which shows that the above written power series is only multiplied by a constant factor $\det\left((\mathbf{I}^{-1}\mathbf{P})^{k}\right)$. The same is true for the coefficients, and since the determinants (used in the Cayley criterion) for the porisms are homogeneous in the power series' coefficients, they are vanishing independent of the choice of k. □

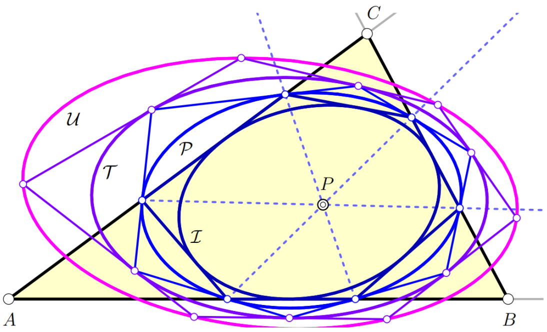

Figure 8: *The g-parallelian conic $\mathcal{P}$ and the g-parallelian inconic $\mathcal{I}$ constitute an exponential pencil of conics and set the basis for an infinite chain of nested poristic families of hexagons.*

Fig. 8 shows some conics out of the chain in the discrete exponential pencil spanned by $\mathcal{I}$ and $\mathcal{P}$. The conic $\mathcal{U}$ is the next in line: The coefficient matrix of its equation equals $\mathbf{U}=\mathbf{P}(\mathbf{I}^{-1}\mathbf{P})^2=\mathbf{T}\mathbf{I}^{-1}\mathbf{P}$, *i.e.*, $\mathcal{U}^\star$ ($\mathcal{U}$ considered as its set of tangents) is the polar image of $\mathcal{P}$ with regard to $\mathcal{T}$. The interscribed hexagons are also displayed. The order of the base conics $\mathcal{I}$ and $\mathcal{P}$ does not matter. Interchanging the matrices $\mathcal{I}$ and $\mathcal{P}$ in the usual parametrization of the exponential pencil as given in [13] means traversing the pencil in the opposite direction.

3 Tangent triangles

3.1 The triangle porism

There are two triples of tangents of the parallelian conic $\mathcal{P}$ that form two triangles $\Delta_U := U_1U_2U_3$ and $\Delta_V := V_1V_2V_3$ with a common circumconic $\mathcal{D}$. The vertices of the triangles are defined by

$$\begin{aligned}U_1 &:= t_3\cap t_5,\ U_2 := t_5\cap t_1,\ U_3 := t_1\cap t_3,\\ V_1 &:= t_2\cap t_4,\ V_2 := t_4\cap t_6,\ V_3 := t_6\cap t_2.\end{aligned}\tag{10}$$

Note that the triangle $U_1U_2U_3$ and $V_2V_3V_1$ (note the different orientation) are perspective to P, while the corresponding trilaterals are perspective to p given by (3).

The common circumconic $\mathcal{D}$ of the triangles Δ_U and Δ_V can be described by the homogeneous equation

$$\begin{aligned}\mathcal{D}:\ &\sum_{\text{cyclic}} l^2mn\xi\eta^2\zeta^2(m\eta+n\zeta)x^2+l\xi^2\Big(2l^2\xi^2(m^2\eta^2+n^2\zeta^2)+\\ &\sum_{\text{cyclic}} lm\xi\eta(l^2\xi^2+m^2\eta^2)+4l^2mn\xi^2\eta\zeta\Big)yz=0.\end{aligned}\tag{11}$$

Now, it is near to formulate the following result:

Theorem 6 *The pair $(\mathcal{P},\mathcal{D})$ of conics allows for a triangle porism.*

Proof. The existence of the triangle porism is clear by the same reasoning as used in the proof of Thm. 3. □

We can deduce some more porisms out of the previously described one:

Theorem 7 *The pair of conics $(\mathcal{P},\mathcal{D})$ allows for a hexagon porism, and $3n$-gon porisms with $n\in\mathbb{N}\setminus\{0\}$.*

Proof. Like in the proof of Thm. 5, we use the Cayley criterion [12, p. 432, Thm. 9.5.4] in order to show the existence of porism with $3n$-gons inconic $\mathcal{P}$ and circumconic $\mathcal{D}$. For that purpose, we extract the coefficient matrices $\mathbf{D}$ and $\mathbf{P}$ from the equations of $\mathcal{D}$ and $\mathcal{P}$, and expand the function $\sqrt{\det(t\cdot\mathbf{D}+\mathbf{P})}$ in a power series $S=a_0+a_1t+a_2t^2+\dots$. The criterion for the existence of a triangle porism is $a_2=0$ (which is clearly fulfilled). According to CAYLEY, the criteria for the existence of poristic families of hexagons, nonagons, ... equal

$$\det\begin{pmatrix}a_3 & a_4\\ a_4 & a_5\end{pmatrix}=0,\quad \det\begin{pmatrix}a_2 & a_3 & a_4 & a_5\\ a_3 & a_4 & a_5 & a_6\\ a_4 & a_5 & a_6 & a_7\\ a_5 & a_6 & a_7 & a_8\end{pmatrix}=0,\dots$$

which also turns out to be satisfied, as do the further criteria. □

Poristic families of quadrangles or other polygons with a vertex number that is not a multiple of 3 interscribed between $\mathcal{D}$ and $\mathcal{P}$ cannot occur for admissible choices of P (see also Tab. 2).

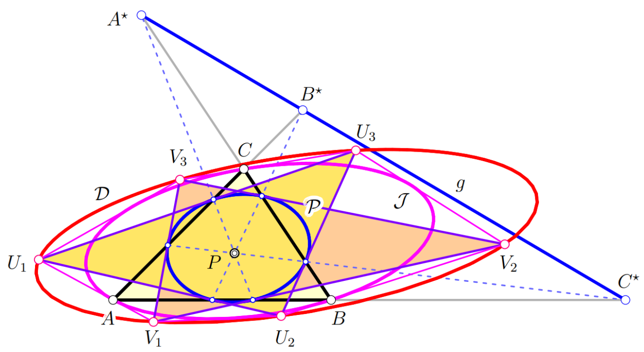

Figure 9: *The conic $\mathcal{D}$ through the vertices of the tangent triangles Δ_U and Δ_V of $\mathcal{P}$ and the inconic $\mathcal{J}$ of the hexagon $U_1V_1U_2V_2U_3V_3$ both give rise to two independent families of porisms.*

Here, we shall state explicitly that (in general) the existence of a poristic family of triangles does not necessarily imply the existence of a poristic family of hexagons, and *vice versa*.

The union of the two triangles Δ_U and Δ_V can be viewed as a degenerate hexagon. In that respect, $\mathcal{P}$ and $\mathcal{D}$ can already serve as a base of the pencil of conics allowing for a poristic family of hexagons. Indeed, the hexagon $H_2 := U_1V_1U_2V_2U_3V_3$ (alternately chosen vertices of Δ_U and Δ_V) is tangent to a single conic:

Theorem 8 *The hexagon H_2 has an inconic $\mathcal{J}$ with the trilinear equation*

$$\mathcal{J}: \sum_{\text{cyclic}} \frac{l\xi^2}{x} = 0 \tag{12}$$

which is at the same time a circumconic of the base triangle Δ.

Proof. We use the trilinear representation (10) of the vertices of Δ_U and Δ_V in order to compute the trilinear coordinates of the sides s_i of the hexagon. In order to find an equation of the inconic $\mathcal{J}$ of H_2, we compute the kernel of the 6×6 matrix whose columns (or rows) are the Veronese images of the trilinear coordinates of the six lines s_i (see [12, p. 241]). This kernel is one-dimensional (provided that the six lines are tangent to single conic) and a base vector of the kernel yields the coefficients of the equation

$$\mathcal{J}^\star: \sum_{\text{cyclic}} l^2\xi^4x^2 - 2mn\eta^2\zeta^2yz = 0$$

of a line conic (quadratic set of lines) containing the six sides s_i of H_2. The corresponding point conic $\mathcal{J}$ is the dual to $\mathcal{J}^\star$, and thus, the respective matrices are related by $\mathbf{J} = \mathbf{J}^{\star-1}$ (cf. [12, p. 273]). Hence, $\mathcal{J}$ is the inconic given in (12) and it is clearly seen that the vertices of Δ are contained in $\mathcal{J}$. □

We can also show:

Theorem 9 *The pair of conics $(\mathcal{J}, I)$ allows for poristic families of $3n$-gons.*

Proof. We use the Cayley criterion (cf. [12, p. 432]) in order to verify the contents of this theorem. □

With the help of the Cayley criterion (cf. [12, p. 432]), we can show that the pair $(\mathcal{D}, \mathcal{J})$ allows for a poristic family of hexagons and even dodecagons.

The conics $\mathcal{D}$ and I span a pencil of the third kind. The pivot point P equals the common pole and the common polar line is already found as the common polar line of I and $\mathcal{P}$ with the trilinear representation (3).

As a consequence of Thms. 6, 7, 8, and 9 we can state a result on infinitely many poristic families of triangles and hexagons:

Theorem 10
1. The pair of conics $(\mathcal{D}, \mathcal{P})$ spans an exponential pencil of conics and any pair of subsequent conics in the pencil allows for a poristic family of $3n$-gons.
2. The pair of conics $(\mathcal{D}, \mathcal{J})$ spans an exponential pencil of conics and any pair of subsequent conics in the pencil allows for a poristic family of hexagons.
3. The pair of conics $(\mathcal{J}, I)$ spans an exponential pencil of conics and any pair of subsequent conics in the pencil allows for a poristic family of $3n$-gons.

4 Special assumptions and cases

In this section, we shall discuss the previously mentioned porisms in case of special choices of P and/or g. It is clear that the choice $g = a : b : c$ leads to the elementary geometric parallelians and the related porisms which are studied in detail in [1]. Another even more special and in some sense simpler case is obtained if $P = 1:1:1$ and $g = 1:1:1$. In terms of elementary triangle geometry, P is the incenter and g the anti-orthic axis. Moreover, g is the triangle polar of P (with respect to the base triangle Δ).

4.1 g antiorthic axis, $P = X_1$

The choice of $P = 1:1:1 = g$ yields

$$\mathcal{J}: \sum_{\text{cyclic}} xy = 0 \quad \text{and} \quad I: \sum_{\text{cyclic}} x^2 - 2yz = 0. \tag{13}$$

Conics with equations of that particular form belong to the family of Yff conics (cf. [19, 22]). The porism between I

and $\mathcal{J}$ are already studied even for various finite projective planes (see [22]) and turned out to be *Universal Porisms* in the sense of N. WILDBERGER [30]. In any case, $\mathcal{I}$ and $\mathcal{J}$ as given in (13) admit $3n$-gon porisms according to Thm. 9. The conics $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{D}$ are not contained in the exponential pencil spanned by $\mathcal{J}$ and $\mathcal{I}$.

4.2 g is the triangle polar line of P with respect to Δ

P is given by the homogeneous coordinates $\xi : \eta : \zeta \neq 0 : 0 : 0$, $\xi\eta\zeta \neq 0$, and P not on any side of the g-anticomplementary triangle, then P's triangle polar line is given by $\eta\zeta : \zeta\xi : \xi\eta$. The latter together with P comprises the pair of common pole and polar line of any two out of the five g-parallelian conics we have seen so far. This special assumption does not change the porisms that we have discovered in the previous sections. The Cayley criterion makes clear that no porisms other than such with triangle and hexagon families will occur.

The same holds true if we choose $P = \xi : \eta : \zeta = g$.

4.3 g ideal, pivot P a triangle center

If the line g is chosen as the ideal line of the plane of Δ, *i.e.*, $g = a : b : c$, then we deal with the case of Euclidean parallelians as described in [1]. However, in this case the centers of the conics $\mathcal{P}$, $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{D}$ coincide with known triangle centers if we choose a triangle center for the pivot point. We do not aim at a complete list, some centers can be read off from Tab. 1. The numbers X_i given to the centers correspond to the list of triangle centers in [17, 18].

P	$C(\mathcal{P})$	$C(\mathcal{T})$	$C(\mathcal{I})$	$C(\mathcal{J})$	$C(\mathcal{D})$
X_1	X_{1001}	?	X_{1125}	X_3	?
X_2	X_2	X_2	X_2	X_2	X_2
X_3	X_{182}	?	X_{140}	X_{1147}	?
X_4	X_{10002}	?	X_5	X_{6523}	?
X_5	X_{10003}	?	X_{3628}	X_{6663}	?
X_6	X_{182}	?	X_{3589}	X_{206}	?
X_7	X_{10004}	?	X_{142}	X_{17113}	?
X_8	X_{10005}	?	X_{10}	X_{6552}	?
X_9	X_{1001}	?	X_{6666}	X_{6600}	?
X_{10}	X_{3842}	?	X_{3634}	X_{4075}	?
X_{11}	X_{1006}	?	X_{6667}	X_{64440}	?
X_{12}	?	?	X_{6668}	?	?
X_{19}	?	?	X_{40530}	X_{15259}	?
X_{20}	X_{47381}	?	X_3	?	?

Table 1: *The centers of the Euclidean parallelian conics $\mathcal{P}$, $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{D}$ depending on certain choices of the pivot point P as a triangle center. The line g equals the ideal line $\omega = a : b : c$ in all cases and the numbers of the centers equal those in [17, 18].*

The question marks indicate yet unnamed triangle centers serving as centers of various g-parallelian conics.

5 Porisms of arbitrary types between different pairings of conics

In the following, we shall derive the conditions on the pivot point P such that poristic families of triangles, quadrangles, pentagons, and hexagons occur between any pair of conics out of the five conics $\mathcal{P}$, $\mathcal{T}$, $\mathcal{I}$, $\mathcal{D}$, and $\mathcal{J}$. It turns out that these conditions are certain algebraic loci in the plane of Δ. In the majority of the cases, these loci are elliptic cubics and elliptic sextics. in some cases, the degrees of the loci are even higher, and then, we simply write $\mathcal{C}^d$ for a degree d curve.

In Table 2, we list the sets of possible pivot points for certain types of porisms. Some of these set occur frequently: We define $\mathbb{P}^{2\star} = \mathbb{P}^2 \setminus \{\Delta^\star, \Delta^\star_a\}$. The cubic curves that show up regularly belong to a certain class of (triangle) cubics. Their equations are of similar shapes and depend on two (homogeneous) parameters:

$$\mathcal{C}^3_{\alpha,\beta} = \alpha\left(\sum_{\text{cyclic}} lm(lx+my)xy\right) + \beta\, lmnxyz = 0, \quad \alpha : \beta \neq 0 : 0.$$

We note that in this linear one-parameter family of cubics we find three degenerate curves. These are $\mathcal{C}^3_{0,1} = \Delta^\star$, $\mathcal{C}^3_{1,2} = \Delta^\star_a$, and $\mathcal{C}^3_{1,3} = \mathcal{S}' \cap g$. The family also contains the rational cubic $\mathcal{C}_{1,-6}$ with an isolated node at $l^{-1} : m^{-1} : n^{-1}$ with the complex conjugate tangents

$$lx + \varepsilon my + \varepsilon^2 nz = 0 \quad \text{and} \quad lx + \varepsilon^2 my + \varepsilon nz = 0,$$

where ε is a complex (not real) cube root of 1.

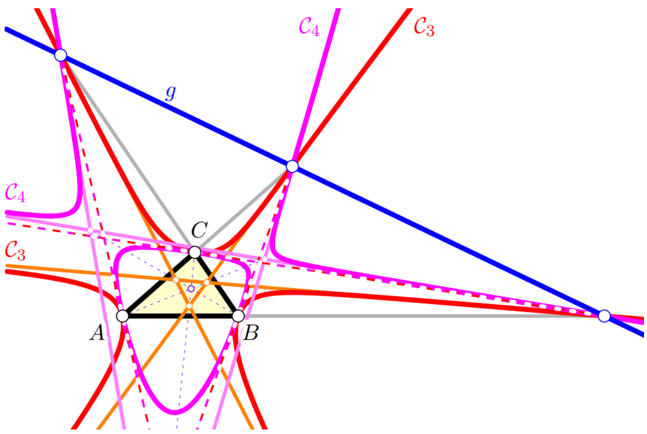

Figure 10: *Two examples of cubics $\mathcal{C}_3$, $\mathcal{C}_4$ housing pivot points for parallelian conics allowing for triangle and quadrangle porisms between $\mathcal{J}$ and $\mathcal{P}$.*

Fig. 10 shows two of the cubics on which a pivot point P is to be chosen such that a triangle or quadrangle porism between two particular conics comes into being.

The loci of pivot points possibly allowing for triangle, quadrangle, pentagon, and hexagon porisms are in some cases sextic curves of the same type as the curve (9) appearing in Section 2.3. These curves form a linear four-parameter family with the equations

$$\mathcal{C}^6_{\alpha,\beta,\gamma,\delta} = \alpha\left(\sum_{\text{cyclic}} l^2m^2(l^2x^2+m^2y^2)x^2y^2\right) +\beta\left(\sum_{\text{cyclic}} l^4mnx^4yz\right)+\gamma\left(\sum_{\text{cyclic}} l^3mn(my+nz)x^3yz\right) +\delta l^2m^2n^2x^2y^2z^2=0, \quad \alpha:\beta:\gamma:\delta \neq 0:0:0:0.$$

	triangles	quadrangles	pentagons	hexagons
$(\mathcal{T},\mathcal{P})$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathbb{P}^{2\star}$
$(\mathcal{P},\mathcal{T})$	$\mathcal{C}^{12}$	$\mathcal{C}^3_{1,0}$, $\mathcal{C}^3_{1,0}$, $\mathcal{C}^{12}$	$\mathcal{C}^{36}$	$\mathcal{C}^6_{1,2,10,30}$, $\mathcal{C}^6_{1,2,2,-2}$, 2 $\mathcal{C}^{12}$
$(\mathcal{P},\mathcal{I})$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathcal{P}^{2\star}$
$(\mathcal{I},\mathcal{P})$	$\mathcal{C}^{12}$	$\mathcal{C}^3_{1,1}$, $\mathcal{C}^3_{1,4}$, $\mathcal{C}^{12}$	$\mathcal{C}^{36}$	$\mathcal{C}^6_{1,2,2,2}$, $\mathcal{C}^6_{1,2,2,10}$, 3 $\mathcal{C}^{12}$
$(\mathcal{D},\mathcal{P})$	$\mathcal{P}^{2\star}$	$\emptyset$	$\emptyset$	$\mathcal{P}^{2\star}$
$(\mathcal{P},\mathcal{D})$	$\mathcal{C}^3_{1,1}$, $\mathcal{C}^3_{1,4}$, $\mathcal{C}^6_{1,2,-2,,-10}$	$\mathcal{C}^{12}$, $\mathcal{C}^6_{1,2,2,-2}$	2 $\mathcal{C}^9$, $\mathcal{C}^{18}$	$\mathcal{C}^3_{1,0}$, $\mathcal{C}^3_{1,4}$, $\mathcal{C}^6_{1,2,-2,-10}$, $\mathcal{C}^{12}$, $\mathcal{C}^{24}$
$(\mathcal{P},\mathcal{J})$	$\mathcal{C}^6_{1,2,-4,-5}$	$\mathcal{C}^3_{1,-1}$, $\mathcal{C}^6_{1,2,-2,-7}$	$\mathcal{C}^{18}$	$\mathcal{C}^3_{1,-5}$, $\mathcal{C}^6_{1,2,-4,-5}$, $\mathcal{C}^{12}$
$(\mathcal{J},\mathcal{P})$	$\mathcal{C}^3_{4,3}$	$\mathcal{C}^3_{10,21}$	$\mathcal{C}^9$	$\mathcal{C}^3_{4,3}$, $\mathcal{C}^3_{16,39}$, $\mathcal{C}^6_{64,128,692.1257}$
$(\mathcal{T},\mathcal{I})$	$\mathcal{C}^3_{1,4}$	$\mathcal{C}^3_{3,14}$	$\mathcal{C}^3_{1,5}$, $\mathcal{C}^6_{1,2,13,32}$	$\mathcal{C}^3_{1,4}$, $\mathcal{C}^3_{1,6}$, $\mathcal{C}^9$
$(\mathcal{I},\mathcal{T})$	$\mathcal{C}^3_{1,0}$, $\mathcal{C}^3_{1,4}$, $\mathcal{C}^{12}$, $\mathcal{C}^6_{1,2,14,30}$	$\mathcal{C}^{12}$, $\mathcal{C}^{24}$	2 $\mathcal{C}^9$, $\mathcal{C}^{18}$, $\mathcal{C}^{36}$	?
$(\mathcal{T},\mathcal{D})$	$\mathcal{C}^6_{1,2,-4,-5}$	$\mathcal{C}^3_{1,-1}$, $\mathcal{C}^6_{1,2,-2,-7}$	$\mathcal{C}^{18}$	$\mathcal{C}^3_{1-54}$, $\mathcal{C}^6_{1,2,-4,-5}$, $\mathcal{C}^{12}$
$(\mathcal{D},\mathcal{T})$	$\mathcal{C}^3_{4,3}$	$\mathcal{C}^3_{10,21}$	$\mathcal{C}^9$	$\mathcal{C}^3_{4,3}$, $\mathcal{C}^3_{16,39}$, $\mathcal{C}^6_{64,128,692,1257}$
$(\mathcal{T},\mathcal{J})$	$\emptyset$	$\mathcal{C}^3_{1,1}$, $\mathcal{C}^3_{1,-7}$	$\mathcal{C}^3_{1,-1}$, $\mathcal{C}^3_{1,5}$, $\mathcal{C}^6_{1,2,-2,-7}$	$\mathcal{C}^3_{1,-3}$, $\mathcal{C}^9$
$(\mathcal{J},\mathcal{T})$	$\mathcal{C}^3_{1,0}$, $\mathcal{C}^3_{5,12}$, $\mathcal{C}^6_{3,6,64,66}$	$\mathcal{C}^6_{1,2,-6,-18}$, $\mathcal{C}^{12}$	$\mathcal{C}^9$, $\mathcal{C}^{18}$	$\mathcal{C}^3_{1,0}$, $\mathcal{C}^3_{5,12}$, $\mathcal{C}^3_{3,6,64,66}$, $\mathcal{C}^{12}$, $\mathcal{C}^{24}$
$(\mathcal{I},\mathcal{D})$	$\mathcal{C}^{24}$	$\mathcal{C}^{12}$, $\mathcal{C}^{24}$	$\mathcal{C}^{96}$	?
$(\mathcal{D},\mathcal{I})$	$\emptyset$	$\mathcal{C}^3_{1,4}$	$\mathcal{C}^6_{4,8,35,70}$	$\mathcal{C}^6_{1,2,13,32}$
$(\mathcal{J},\mathcal{I})$	$\mathbb{P}^{2\star}$	$\emptyset$	$\emptyset$	$\mathbb{P}^{2\star}$
$(\mathcal{I},\mathcal{J})$	$\mathcal{C}^3_{1,0}$, $\mathcal{C}^3_{1,4}$, $\mathcal{C}^6_{1,2,-2,-10}$	$\mathcal{C}^3_{1,\pm\sqrt{8}}$, $\mathcal{C}^6_{1,2,\pm\sqrt{8},\pm\sqrt{8}}$	2 $\mathcal{C}^9$, $\mathcal{C}^{18}$	$\mathcal{C}^3_{1,0}$, $\mathcal{C}^3_{1,4}$, $\mathcal{C}^6_{1,2,-2,-10}$ $\mathcal{C}^{12}$, $\mathcal{C}^{24}$
$(\mathcal{D},\mathcal{J})$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathbb{P}^{2\star}$
$(\mathcal{J},\mathcal{D})$	$\mathcal{C}^{12}$	$\mathcal{C}^3_{1,0}$, $\mathcal{C}^3_{1,4}$, $\mathcal{C}^{12}$	$\mathcal{C}^{36}$	$\mathcal{C}^6_{1,2,2,-2}$, $\mathcal{C}^6_{1,2,10,30}$, 3 $\mathcal{C}^{12}$

Table 2: *The porisms and chains of porism between various pairs of g-parallelian conics.*

The sextic curves of this type *that occur as loci of pivot points P* have six double points and are of genus 1. This is not the case for arbitrary choices of $\alpha:\beta:\gamma:\delta$. Three of the six double points are ordinary nodes, the remaining three are tacnodes. The latter carry a flat point on one of the linear branches at the node. For details on tangents, see Sec. 2.3.

The order of the conics in the pairs $(\mathcal{A},\mathcal{B})$ matters: CAYLEY's criterion [12, p. 432, Thm. 9.5.4] uses the coefficients of the power series of $\sqrt{\det(t\cdot\mathbf{A}+\mathbf{B})}$ and it is easily verified that $\det(t\cdot\mathbf{B}+\mathbf{A})=\det(\mathbf{BA}^{-1}\cdot\mathbf{AB}^{-1}(t\cdot\mathbf{B}+\mathbf{A}))=\det(\mathbf{BA}^{-1})\cdot\det(t\cdot\mathbf{A}+\mathbf{AB}^{-1}\mathbf{A})\neq\det(t\cdot\mathbf{A}+\mathbf{B})$. The first conic, here $\mathcal{A}$, is assumed to be the 'circumconic'. In many cases, this is also justified by the fact that the circumconic contains six points obtained as the intersection of certain tangents of the 'inconic'. This might be of importance in finite geometries or in geometries over algebraically non-closed fields.

6 Conclusion and final remarks

The porisms constructed in the previous sections are objects of projective geometry. Their embeddings in metric (Euclidean or non-Euclidean) geometries can be studied from the projective point of view by prescribing an elliptic or a hyperbolic involution on g. This yields the cases of Euclidean and pseudo-Euclidean parallelians and the related conics. The counterparts in elliptic and hyperbolic geometry are not that near. The algebraic curves given in Tab. 2 are not studied in detail and it is questionable whether these pivot loci allow for an investigation of the corresponding porisms. The rather high algebraic degrees may not be helpful.

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