https://doi.org/10.31896/k.29.3
Original scientific paper
Accepted: 28 October 2025

MANDI ORLIĆ BACHLER

# On Fuss' Relations for Bicentric Polygons with an Odd Number of Vertices

**On Fuss' Relations for Bicentric Polygons with an Odd Number of Vertices**

**ABSTRACT**

This paper presents novel analytical forms of Fuss' relations for bicentric polygons with an odd number of sides and higher rotation numbers. The method is based on Poncelet's theorem and Radić's theorem and conjecture concerning the connection between Fuss' relations for different rotation numbers. Explicit analytical expressions are obtained for the bicentric triskaidecagon with $k = 2,4,6$ and for the bicentric pentadecagon with $k = 2$, while complete sets of relations are established for the bicentric heptadecagon $(k = 1,2,3,4,5,6,7,8)$ and enneadecagon $(k = 1,2,3,4,5,6,7,8,9)$. The proposed approach simplifies the derivation and enables a systematic extension of known Fuss' relations to higher-order bicentric polygons and new rotation numbers, confirming the validity of Radić's conjecture.

**Key words:** bicentric polygon, Fuss' relation, rotation number, triskaidecagon, pentadecagon, heptadecagon, enneadecagon

**MSC2020:** 51N20, 51M15, 37J45

**O Fussovim relacijama za bicentrične poligone s neparnim brojem vrhova**

**SAŽETAK**

U radu su izvedeni analitički oblici Fussovih relacija za bicentrične poligone s neparnim brojem stranica i višim brojevima rotacije. Metoda se temelji na Ponceletovom teoremu te Radićevom teoremu i slutnji u vezi s povezanošću Fussovih relacija za različite brojeve rotacije. Dobiveni su eksplicitni analitički izrazi za bicentrični trinaesterokut s rotacijskim brojem $k = 2,4,6$ i za bicentrični petnaesterokut s rotacijskim brojem $k = 2$, dok su uspostavljeni kompletni skupovi relacija za bicentrični sedamnaesterokut (za $k = 1,2,3,4,5,6,7,8$) i devetnaesterokut (za $k = 1,2,3,4,5,6,7,8,9$). Predloženi pristup pojednostavljuje izvođenje i omogućuje sustavno proširenje poznatih Fussovih relacija na bicentrične poligone višeg reda i nove brojeve rotacije, čime se potvrđuje valjanost Radićeve slutnje.

**Ključne riječi:** bicentrični poligon, Fussova relacija, rotacijski broj, trinaesterokut, petnaesterokut, sedamnaesterokut, devetnaesterokut

## 1 Introduction

A bicentric $n$-gon is a polygon with $n$ sides that is both tangential (it possesses an inscribed circle, or incircle) and cyclic (it possesses a circumscribed circle, or circumcircle). These two circles are nested, meaning that one lies entirely within the other.

Let $A_1,\dots,A_n$ be a bicentric $n$-gon with an incircle $C$ of radius $r$ centered at point $I$, and a circumcircle $K$ of radius $R$ centered at point $O$. Denote by $d$ the distance between their centers, where the circle $C$ lies inside the circle $K$. Let $T_1,\dots,T_n$ be the points of tangency of the sides (segments) $A_1A_2,\dots,A_nA_1$, respectively. The lengths $|A_iT_i|$, $i = 1,\dots,n$, are called the tangent lengths of the polygon $A_1,\dots,A_n$. If

$$\sum_{i=1}^{n} \arctan\left(\frac{|A_iT_i|}{r}\right) = k\pi \tag{1}$$

where $k$ is a positive integer satisfying $k \leq \frac{n-1}{2}$, the polygon $A_1,\dots,A_n$ is said to be $k$-circumscribed, and $k$ is called its rotation number [8].

The relation connecting the radii of the incircle and circumcircle, and the distance between their centers, is called a Fuss' relation. Throughout this paper, the notation $F_n^{(k)}(R,r,d) = 0$ will consistently denote the Fuss' relation for a bicentric $n$-gon with rotation number $k$, relating the radii $R$ and $r$ of the circumcircle and incircle and the distance $d$ between their centers.

The problem of determining the relation between the radii of the incircle and circumcircle and the distance between their centers for bicentric polygons is one of the classical topics of Euclidean geometry, originating from the results of Euler and Fuss. Euler first derived the corresponding relation for the triangle, while Nicolaus Fuss (1755–1826) obtained analogous relations for bicentric quadrilaterals, pentagons, hexagons, and heptagons. These expressions are now collectively known as Fuss' relations.

Later authors extended these relations to polygons of higher order, using various analytical, geometrical, and computational approaches. Particularly significant are the results of Mirko Radić, who developed a relatively simple and effective method for establishing Fuss' relations based on Poncelet's theorem. In papers [3]–[7], [9], Fuss' relations were derived for bicentric polygons up to $n = 11$, and for certain cases with $n = 13, 15, 17, 18$. However, these results do not include all possible values of the rotation number $k$. Specifically, for the bicentric triskaidecagon (13-gon), only the cases $k = 1,3,5$ were known, while for the bicentric pentadecagon (15-gon) the known cases were $k = 1,3,4,5,6$. The remaining cases $k = 2,4,6$ for $n = 13$ and $k = 2$ for $n = 15$ have not been obtained analytically so far, mainly due to the computational complexity of Radić's classical method, whose algebraic expressions grow exponentially with both $n$ and even values of $k$.

Recent investigations have further extended the study of bicentric and Poncelet-type polygons using computational and algebraic methods. For instance, Dragović and Radnović [2] analyzed Poncelet's porisms through elliptic and hyperelliptic function theory, establishing algebraic integrability and modular relations for multi-rotational polygons. New algebraic invariants associated with Poncelet–Jacobi bicentric polygons have been introduced and studied in the work of Roitman, Garcia, and Reznik [10], providing additional structural insight into the geometry underlying bicentric configurations. Compared with these works, the present paper emphasizes a purely analytic-geometric derivation of Fuss' relations, providing explicit polynomial forms for higher-order odd bicentric polygons. This complements rather than replaces the modern computational approaches, bridging classical Euclidean geometry with contemporary algebraic methods.

The present paper builds upon Radić's approach and provides new analytical forms of Fuss' relations for bicentric polygons with an odd number of sides and higher rotation numbers. The derivation is based on Poncelet's theorem, Radić's theorem on the connection between Fuss' relations, and Radić's conjecture on their equivalence for different rotation numbers. Detailed statements of these results and references to their proofs are given in Section 2.

The main contribution of this paper is the derivation of previously unknown Fuss' relations for:

- the bicentric triskaidecagon (13-gon) with the rotation numbers $k = 2,4,6$,
- the bicentric pentadecagon (15-gon) with the rotation number $k = 2$,
- the bicentric heptadecagon (17-gon) with the rotation numbers $k = 1,2,3,4,5,6,7,8$,
- and the bicentric enneadecagon (19-gon) with the rotation numbers $k = 1,2,3,4,5,6,7,8,9$.

In this way, the paper fills a previously unexplored part of the set of known bicentric polygons and demonstrates the efficiency of a generalized approach that combines Poncelet's theorem with Radić's results on the relationship between the Fuss' relations $F_n^{(k)}(R,r,d)$ and $F_{n/p}^{(q)}(R,r,d)$ where $p$ and $q$ are determined by the greatest common divisor of $n$ and $k$.

## 2 Theoretical background

This section provides a concise overview of the mathematical foundations on which the results of this paper are based. It includes the statements of Poncelet's theorem, Radić's theorem on the connection between Fuss' relations, and Radić's conjecture. Proofs of these theorems, and the derivation of the associated algorithms can be found in the cited works; therefore, only the relevant formulations and explanations are given here. Among them, Poncelet's theorem plays a central role in establishing the existence of bicentric polygons. This classical theorem is fundamental in the study of bicentric polygons and underlies the existence of closed polygons tangent to one circle and inscribed in another.

**Theorem 1 (Poncelet's theorem)** *Let $C$ and $K$ be two circles in a plane such that one lies entirely inside the other. Then exactly one of the following two statements holds:*

*a) There exists no bicentric $n$-gon whose incircle is $C$ and whose circumcircle is $K$.*

*b) There exist infinitely many bicentric n-gons whose incircle is C and circumcircle is K. For every point $A_1$ on the circle K, there exists a bicentric n-gon $A_1, A_2, \ldots, A_n$ whose incircle is C and circumcircle is K.*

A detailed proof of this classical theorem can be found in [1] and is not repeated here.

**Theorem 2** *Let n and k be positive integers, and let*

$$p = \gcd(n,k), \quad k = p \cdot q.$$

*Then the corresponding Fuss' relations satisfy*

$$F_n^{(k)}(R,r,d) = F_{n/p}^{(q)}(R,r,d),$$

*where n is the number of sides and k the rotation number.*

That is, analytically the Fuss' relation for a bicentric $n$-gon with rotation number $k$ is equivalent to the relation for a smaller bicentric $\frac{n}{p}$-gon and rotation number $q$. The proof of this theorem is given in [9], where a computational procedure based on tangent lengths and Poncelet's closure condition is also presented.

**Conjecture 1** *For an odd integer $n \geq 3$, the following equalities hold for the Fuss' relations of bicentric n-gons:*

$$F_n^{(i)}(R,r,d) = F_n^{(j)}(R,r,d)$$

$$\text{for all odd } i, j \text{ with } \gcd(i,n) = \gcd(j,n) = 1,$$

$$F_n^{(u)}(R,r,d) = F_n^{(v)}(R,r,d)$$

$$\text{for all even } u, v \text{ with } \gcd(u,n) = \gcd(v,n) = 1.$$

In other words, all Fuss' relations for rotation numbers that are coprime to $n$ are identical within the same parity class (odd or even). The Conjecture 1. has been verified analytically and numerically for numerous values up to $n = 18$, but it remains unproven in the general case. If eventually proven, this Conjecture 1. would reveal a fundamental structural symmetry among Fuss' relations for bicentric $n$-gons. Although Conjecture 1. has not yet been rigorously established in full generality, the results presented here rely on it as a practical working assumption. Its validity has been confirmed for all tested configurations through both analytical derivations and numerical verification, which supports the conjecture's applicability within the scope of the present study.

**Remark 1** *Radić's Conjecture 1. plays a fundamental role in simplifying the classification of Fuss' relations for bicentric n-gons with an odd number of sides [9]. It effectively reduces the number of distinct analytical forms that must be derived, since all relations within the same parity class (odd or even rotation numbers) are conjectured to coincide. In this paper, the Conjecture 1. is adopted as a working hypothesis for the higher-order cases ($n = 13, 15, 17, 19$) of this study, and the obtained results are in complete agreement with its predictions. Although the general proof of the Conjecture 1. remains open, the analytical and numerical confirmations presented here further support its validity and may provide further evidence supporting its eventual formal proof.*

## 3 Method

The analytical procedure for determining the coordinates of the vertices of a bicentric $n$-gon $A_1, A_2, \ldots, A_n$, when $n$ is an odd number and the rotation number $k = 1$, was originally presented in [3]. In this section, the method is summarized, further clarified, and extended to cases of higher rotation numbers $k > 1$. The procedure is based on Poncelet's theorem, which guarantees the existence of a closed polygon inscribed in one circle and tangent to another, and on Radić's Theorem 2. and Conjecture 1., which establish the relationships among the corresponding Fuss' relations $F_n^{(k)}(R,r,d) = 0$ for different values of $k$.

**Geometric setup.** According to Poncelet's theorem, the choice of initial point and orientation does not affect polygon closure. Therefore, without loss of generality, we consider a configuration symmetric with respect to the $x$-axis, which passes through the centers of the two circles. The incircle $C$ of radius $r$ is centered at the point $I(d,0)$, while the circumcircle $K$ of radius $R$ is centered at the origin $O(0,0)$. The parameter $d$ denotes the distance between their centers, and the inner circle $C$ lies strictly inside the outer circle $K$, that is,

$$R \geq d + r > 0.$$

This assumption guarantees that $C$ is completely contained within $K$ and will later justify excluding nonphysical factors in the closure condition.

The polygon vertices $A_1, \ldots, A_n$ lie on $K$, and the sides $A_iA_{i+1}$ are tangent to $C$.

Let $t_i$ denote the tangent to the circle $C$ at the point of tangency corresponding to the side $A_iA_{i+1}$, and let $T_i$ denote the tangency point on $C$. The coordinates of the first vertex $A_1$ are obtained as the intersection of the tangent $t_1$, drawn at the point $P(x_p, 0)$ on the circle $C$, with the circle $K$. The abscissa $x_p$ depends on the parity of the rotation number:

$$x_p = \begin{cases} d + r, & \text{if } k \text{ is odd,} \\ d - r, & \text{if } k \text{ is even.} \end{cases}$$

**Recursive construction of vertices.** Each subsequent vertex $A_i(x_i, y_i)$, for $i = 2, 3, \ldots, n$, is determined as the intersection of the tangent $t_i$ drawn from $A_{i-1}$ to the incircle $C$ and the circumcircle $K$. The equation of the tangent $t_i$ is obtained from two conditions:

1. the line $y = ax + b$ passes through the known vertex $A_{i-1}(x_{i-1}, y_{i-1})$, and
2. the line is tangent to the circle $C$.

From these conditions, the slope $a$ and the intercept $b$ are obtained by solving the system:

$$\begin{aligned} a x_{i-1} + b &= y_{i-1}, \\ r^2(1 + a^2) - (ad + b)^2 &= 0. \end{aligned} \tag{2}$$

The intersection of this tangent with the circumcircle $K$ gives the coordinates of the vertex $A_i$.

Since the polygon is symmetric with respect to the $x$-axis, the vertex $A_{(n+1)/2}$ lies on this axis, with the abscissa $x = -R$.

**Closure condition and derivation of the Fuss' relation.** In the geometric construction of a bicentric $n$-gon, we consider two opposite vertices, $A_1$ and $A_{(n+1)/2}$, on the circumcircle. From each of these vertices, tangents are drawn to an arbitrarily chosen vertex $A_i$ on the same circle. When the point $A_i$ is reached by successive tangent constructions starting from $A_1$, it is denoted by $A_i^{(1)}$. Likewise, when the same vertex is reached by tangents constructed in the opposite direction, starting from $A_{(n+1)/2}$, it is denoted by $A_i^{((n+1)/2)}$. The superscript therefore indicates the origin of the tangent sequence: $A_i^{(1)}$ corresponds to the forward traversal originating from $A_1$ (with successive index increments of $+1$), while $A_i^{((n+1)/2)}$ corresponds to the backward traversal originating from $A_{(n+1)/2}$ (with successive decrements of $-1$). All intermediate vertices generated along these two tangent sequences inherit the superscript of their respective starting vertex.

The closure condition of the bicentric $n$-gon requires that these two tangent sequences terminate at the same vertex on the circumcircle; in other words, the points obtained from the forward and backward constructions must coincide:

$$A_i^{(1)} \equiv A_i^{((n+1)/2)}.$$

This geometric requirement ensures that the polygon closes after $n$ sides and thus satisfies Poncelet's closure theorem.

By equating the corresponding abscissas (and equivalently the ordinates) of the two coinciding vertices, one obtains the analytical closure equation

$$\rho(R, r, d) \cdot F_n^{(k)}(R, r, d) = 0,$$

where $\rho(R, r, d)$ is a non-vanishing scalar factor, and $F_n^{(k)}(R, r, d) = 0$ represents the analytical form of the Fuss' relation for the given $n$ and rotation number $k$.

This formulation expresses the precise condition for polygonal closure and provides the foundation for deriving the explicit analytical expressions of the Fuss' relations discussed in the following sections.

**Extension to higher rotation numbers.** For polygons with rotation number $k > 1$, the construction proceeds analogously, but the initial tangent $t_1$ is drawn at the point $P(x_p, 0)$ chosen according to the parity of $k$. Each subsequent tangent $t_i$ is determined recursively using (2), while the closure condition is imposed after every $k$ steps, that is, after $k$ successive tangent mappings returning the polygon to the same orientation with respect to the incircle. The resulting equation in $(R, r, d)$ then yields the corresponding Fuss' relation $F_n^{(k)}(R, r, d) = 0$.

The bicentric $n$-gon constructed from the initial point $A_1(x_p, 0)$ thus represents the entire family of configurations satisfying Poncelet's closure condition. The same geometric construction, with the choice $x_p = d + r$ for odd $k$ and $x_p = d - r$ for even $k$, yields all corresponding Fuss' relations for different rotation numbers. In particular, this means that the same analytical form of Fuss' relation $F_n^{(k)}(R, r, d) = 0$ applies to all polygons with rotation numbers of the same parity (e.g., $k = 1, 3, 5, \ldots$ or $k = 2, 4, 6, \ldots$), as predicted by Conjecture 1.

This recursive–geometric procedure, combined with algebraic elimination of intermediate coordinates, enables the analytic derivation of Fuss' relations for various odd values of $n$ and rotation numbers $k$. The method thus provides a unified analytical approach applicable to all bicentric polygons that satisfy Poncelet's closure condition.

**Transition to the results.** The procedure described above enables the analytical determination of the closure condition for any bicentric $n$-gon satisfying Poncelet's theorem. By applying the method to specific values of $n$ and $k$, the corresponding Fuss' relations $F_n^{(k)}(R, r, d) = 0$ can be explicitly derived. In the following section, we present the results obtained for the bicentric 13-gon, 15-gon, 17-gon, and 19-gon, including several previously unknown relations corresponding to higher rotation numbers.

## 4 Results

Throughout this section we use the shorthand

$$p=\frac{R+d}{r},\qquad q=\frac{R-d}{r}. \tag{3}$$

For a bicentric $n$-gon the rotation number satisfies $k\in\{1,\ldots,\lfloor(n-1)/2\rfloor\}$. By Theorem 2. and Conjecture 1., the corresponding Fuss' relations $F_n^{(k)}(R,r,d)=0$ group into two parity classes (odd/even) that are identical within each class when $\gcd(k,n)=1$.

**Bicentric 13-gon.** For the bicentric 13-gon, the admissible rotation indices are $k\in\{1,2,3,4,5,6\}$. According to Theorem 2. and Conjecture 1., all Fuss relations for the bicentric 13-gon fall into two parity classes (odd and even):

$$F_{13}^{(1)}(R,r,d)=F_{13}^{(3)}(R,r,d)=F_{13}^{(5)}(R,r,d)=0, \tag{4}$$

$$F_{13}^{(2)}(R,r,d)=F_{13}^{(4)}(R,r,d)=F_{13}^{(6)}(R,r,d)=0. \tag{5}$$

The odd-class identity (4) was obtained in [3], here we derive the corresponding even-class relation (5).

**Theorem 3 (13-gon, even class)** *The bicentric 13-gon with rotation number $k=2$ satisfies*

$$F_{13}^{(2)}(R,r,d)=0,$$

*where the explicit polynomial form in $p,q$ is listed in Appendix A, Eq. (A.1).*

**Detailed proof.** We work in the symmetric coordinate setup from Section 3: the incircle

$$C:\ (x-d)^2+y^2=r^2\quad\text{and}\quad K:\ x^2+y^2=R^2,$$

with $C$ entirely inside $K$ and the $x$-axis joining the centers $O(0,0)$ and $I(d,0)$. For the even rotation number $k=2$ we start from the tangency point

$$P(d-r,0)\in C,$$

whose tangent to $C$ is the vertical line $x=d-r$. Its intersection with $K$ gives the first vertex

$$A_1=(x_1,y_1)=\left(d-r,\ \sqrt{R^2-(d-r)^2}\right),$$

where the sign of the square root is chosen so that $A_1$ lies above the $x$-axis. This choice only fixes orientation and does not affect the final closure condition.

*Tangent from a given vertex and the next intersection with $K$.* Let $A_{i-1}=(x_0,y_0)\in K$. A line $t_i: y=ax+b$ passes through $A_{i-1}$ iff $b=y_0-ax_0$. The condition that $t_i$ is tangent to $C$ is

$$\operatorname{dist}\big(I,(t_i)\big)^2=\frac{(ad+b)^2}{1+a^2}=r^2\iff r^2(1+a^2)-(ad+b)^2=0.$$

Substituting $b=y_0-ax_0$ yields a quadratic equation for the slope $a$:

$$\big(r^2-(d-x_0)^2\big)a^2-2(d-x_0)y_0\,a+\big(r^2-y_0^2\big)=0. \tag{6}$$

We pick the root that continues the forward traversal of the polygon (the other root corresponds to the opposite tangent).

Once $a$ is fixed, $t_i$ meets $K$ at two points whose $x$-coordinates are the roots of

$$(1+a^2)x^2+2abx+(b^2-R^2)=0.$$

Since one root is $x_0$, Vieta's formulas give the other root explicitly as

$$x_i=\frac{(a^2-1)x_0-2a\,y_0}{1+a^2},\qquad y_i=ax_i+b. \tag{7}$$

Equations (6)–(7) form the closed recurrence for successive vertices.

*Three steps from $A_1$.* Applying (6)–(7) first with $(x_0,y_0)=A_1$, and then again from $A_2$ to obtain $A_3^{(1)}$, we arrive (after clearing radicals by squaring and simplifying) at rational expressions in $(R,r,d)$. It is convenient to use the variables $p$ and $q$ introduced in (3). In these variables, the abscissa of $A_3^{(1)}$ takes the form

$$x_3^{(1)}=\frac{n_3^{(1)}}{d_3^{(1)}}, \tag{8}$$

with

$$\begin{aligned}n_3^{(1)}=&r\Big(-p^7(q-1)^4(q+1)^2\\&+p^6(q^2-1)^2(q^3+6q^2+3q+2)\\&+p^5q(q+1)^2(2q^4-5q^3-4q^2+7q-4)\\&-p^4q^2(q^5+10q^4-6q^3-12q^2+5q+2)\\&+p^3q^3(-4q^4+5q^3+6q^2+5q+8)\\&+p^2q^4(-q^3+2q^2+q-2)\\&+pq^5(2q^2-3q-4)+q^6(q+2)\Big),\end{aligned}$$

$$\begin{aligned}d_3^{(1)}=&2\Big(p^3(-(q-1)^2)(q+1)-p^2q(q+1)^2\\&+pq^2(q+1)+q^3\Big)^2.\end{aligned}$$

*Four steps from the symmetry point.* By symmetry with respect to the $x$-axis, the middle vertex of the 13-gon is

$A_{(13+1)/2} = A_7 = (-R,0)$.

Starting from $A_7$ and iterating (6)–(7) three times forward (to $A_6^{(7)}, A_5^{(7)}, A_4^{(7)}$) and once more to reach $A_3^{(7)}$, we obtain

$$x_3^{(7)} = \frac{n_3^{(7)}}{d_3^{(7)}}, \tag{9}$$

with

$$\begin{aligned} n_3^{(7)} =& r(p+q)\Big(p^{16}(q^2-1)^4(q^8-20q^6-26q^4-20q^2+1) \\ & -8p^{14}q^2(q^2-1)^4(q^6-5q^4-9q^2-3) \\ & +4p^{12}q^4(q^2-1)^2(7q^8+8q^6+50q^4+40q^2-9) \\ & -8p^{10}q^6(q^2-1)^2(7q^6+29q^4+49q^2+11) \\ & +2p^8q^8(35q^8+60q^6-158q^4-68q^2+99) \\ & -8p^6q^{10}(7q^6-3q^4-31q^2+11) \\ & +4p^4q^{12}(7q^4-18q^2-9)-8p^2q^{14}(q^2-3)+q^{16}\Big), \end{aligned}$$

$$\begin{aligned} d_3^{(7)} =& 2\Big(p^8(q^2-1)^2(q^4+6q^2+1) \\ & -4p^6q^2(q^2-1)^2(q^2+1) \\ & +p^4(6q^8-4q^6+6q^4)-4p^2(q^8+q^6)+q^8\Big)^2. \end{aligned}$$

*Poncelet closure and factorization.* Poncelet's closure for the bicentric 13-gon with $k=2$ requires that the two constructions agree at the same point on $K$, hence

$x_3^{(1)} = x_3^{(7)}$.

Clearing denominators in (8)–(9) and simplifying, we obtain a polynomial equation in $(R,r,d)$ which factors as

$$(d-r+R)\cdot\big(d^2-2rR-R^2\big)\cdot F_{13}^{(2)}(R,r,d)=0, \tag{10}$$

where $F_{13}^{(2)}(R,r,d)$ is the polynomial displayed in (A.1).

*Excluding extraneous factors.* Because $C$ lies strictly inside $K$, we have $R \geq d+r$ with $r>0$. The factor $d-r+R=0$ would imply $R=r-d \leq r+(-d) < r+d$, contradicting $R \geq d+r$ unless $r=0$ (degenerate). Moreover,

$$d^2-2rR-R^2 \leq d^2-2r(d+r)-(d+r)^2 = -\big(4dr+3r^2\big) < 0,$$

so $d^2-2rR-R^2=0$ is impossible under the nesting assumption. Hence the only admissible factor in (10) is

$F_{13}^{(2)}(R,r,d)=0$,

which is exactly the claimed Fuss' relation for the 13-gon with rotation number $k=2$.

This completes the detailed analytic derivation. □

**Graphical examples.** For illustration, we present a concrete bicentric configuration corresponding to the analytically and numerically verified configuration of the 13-gon with rotation number $k=2$. With $R=1$ and $d=0.1$, solving $F_{13}^{(2)}(R,r,d)=0$ yields

$r_{13,2} = 0.856554$,

while, according to the even-class relations (5), the corresponding radii for higher even rotation numbers are

$r_{13,4} = 0.561780, \qquad r_{13,6} = 0.119618.$

Figure 1 illustrates the configuration for $k=2$.

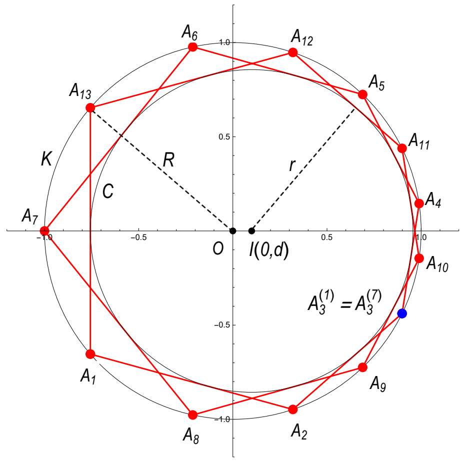


Figure 1: *Bicentric 13-gon for $R=1$, $d=0.1$, $r_{13,2}=0.856554$, and rotation number $k=2$. The incircle $C$ is tangent to all sides of the polygon, and the circumcircle $K$ passes through all vertices $A_1,\ldots,A_{13}$. The centers $O$ and $I$ are aligned on the $x$-axis. The polygon closes precisely after 13 tangents, confirming the analytical and numerical validity of the derived relation $F_{13}^{(2)}(R,r,d)=0$.*

**Bicentric 15-gon.** For the bicentric 15-gon, the admissible rotation indices are $k \in \{1,2,3,4,5,6,7\}$. According to Theorem 2. and Conjecture 1., all Fuss' relations for the bicentric 15-gon fall into four parity classes:

$$F_{15}^{(1)}(R,r,d) = F_{15}^{(7)}(R,r,d) = 0, \tag{11}$$

$$F_{15}^{(3)}(R,r,d) = F_{5}^{(1)}(R,r,d) = 0 \tag{12}$$

$$F_{15}^{(5)}(R,r,d) = F_{3}^{(1)}(R,r,d) = 0, \tag{13}$$

$$F_{15}^{(2)}(R,r,d) = F_{15}^{(4)}(R,r,d) = 0. \tag{14}$$

Identity (11) was obtained in [3], and the identities (12) and (13) were obtained in [9]. Here we derive the identity (14).

**Theorem 4** *For a bicentric 15-gon with circumcircle of radius R, incircle of radius r, and distance d between their centers, the corresponding Fuss' relations satisfy:*

$$F_{15}^{(2)}(R,r,d) = F_{15}^{(4)}(R,r,d) = 0.$$

*The explicit analytical forms of these relations are provided in Appendix A, Eq. (A.2).*

**Proof.** The proof follows the same analytic–geometric procedure as in Theorem 3. The coordinates of the vertices $A_i(x_i, y_i)$ are determined recursively from the tangent condition (2) for the bicentric configuration with $n = 15$. The closure condition is imposed by equating the abscissas (and, equivalently, the ordinates) of the vertices $A_3^{(1)}(x_3^{(1)}, y_3^{(1)})$ and $A_3^{(7)}(x_3^{(7)}, y_3^{(7)})$, obtained from tangents drawn from $A_1$ and $A_{(n+1)/2}$. Since both $k = 2$ and $k = 4$ correspond to even rotation numbers, the geometric construction and algebraic elimination are completely analogous to those in Theorem 3 (the 13-gon case), differing only in the degree of the resulting polynomial in $r$. After simplification, the equations $F_{15}^{(2)}(R,r,d) = 0$ and $F_{15}^{(4)}(R,r,d) = 0$ are obtained. □

**Bicentric 17-gon.** For the bicentric 17-gon, the admissible rotation indices are $k \in \{1,2,3,4,5,6,7,8\}$. According to Theorem 2. and Conjecture 1., all Fuss relations for the bicentric 17-gon can be grouped into two parity classes: one corresponding to odd rotation numbers ($k = 1,3,5,7$) and the other to even rotation numbers ($k = 2,4,6,8$). Each class satisfies a distinct analytical form of the Fuss' relation, which is common to all members of that class. In the following theorem, we present the proof that covers both cases, demonstrating the validity of the relations for odd and even rotation numbers.

**Theorem 5** *For a bicentric 17-gon with circumcircle of radius R, incircle of radius r, and distance d between their centers, the corresponding Fuss' relations satisfy:*

$$F_{17}^{(1)}(R,r,d) = F_{17}^{(3)}(R,r,d) = F_{17}^{(5)}(R,r,d) = F_{17}^{(7)}(R,r,d) = 0,$$

*for odd rotation numbers* $k = 1,3,5,7$*, and*

$$F_{17}^{(2)}(R,r,d) = F_{17}^{(4)}(R,r,d) = F_{17}^{(6)}(R,r,d) = F_{17}^{(8)}(R,r,d) = 0,$$

*for even rotation numbers* $k = 2,4,6,8$*. The explicit analytical forms of these relations are given in Appendix A as Eq. (A.3) for the odd rotation numbers (*$k = 1,3,5,7$*) and Eq. (A.4) for the even rotation numbers (*$k = 2,4,6,8$*).*

**Proof.** For the odd rotation numbers $k = 1,3,5,7$, the proof proceeds analogously to the case $k = 1$ discussed for the 13-gon and 15-gon in [3]. The first tangent $t_1$ is drawn at the point $P(r+d, 0)$ on the incircle $C$, and the subsequent tangents determine the vertices $A_i$ recursively using system (2). The closure condition is imposed by equating the abscissas of the vertices obtained from $A_1$ and $A_{(n+1)/2}$, ensuring that $A_i^{(1)} \equiv A_i^{(n+1)/2}$. Since all considered rotation numbers are odd, the geometric symmetry and analytic form of the procedure remain identical, leading to the family of relations listed above, which coincide according to Conjecture 1. □

**Bicentric 19-gon.** For the bicentric 19-gon, the admissible rotation indices are $k \in \{1,2,3,4,5,6,7,8,9\}$. According to Theorem 2. and Conjecture 1., all Fuss' relations for the bicentric 19-gon can be grouped into two parity classes: one corresponding to odd rotation numbers ($k = 1,3,5,7,9$) and the other to even rotation numbers ($k = 2,4,6,8$). Each class satisfies a distinct analytical form of the Fuss relation, which is common to all members of that class. In the following theorem, we present the proof that covers both cases, demonstrating the validity of the relations for odd and even rotation numbers.

**Theorem 6** *For a bicentric 19-gon with circumcircle of radius R, incircle of radius r, and distance d between their centers, the corresponding Fuss' relations satisfy:*

$$\begin{aligned} F_{19}^{(1)}(R,r,d) &= F_{19}^{(3)}(R,r,d) = F_{19}^{(5)}(R,r,d) \\ &= F_{19}^{(7)}(R,r,d) = F_{19}^{(9)}(R,r,d) = 0, \end{aligned}$$

*for odd rotation numbers* $k = 1,3,5,7,9$*, and*

$$\begin{aligned} F_{19}^{(2)}(R,r,d) &= F_{19}^{(4)}(R,r,d) = F_{19}^{(6)}(R,r,d) \\ &= F_{19}^{(8)}(R,r,d) = 0, \end{aligned}$$

*for even rotation numbers* $k = 2,4,6,8$*. The explicit analytical forms of these relations are given in Appendix A as Eq. (A.5) for the odd rotation numbers (*$k = 1,3,5,7,9$*) and Eq. (A.6) for the even rotation numbers (*$k = 2,4,6,8$*).*

**Proof.** For even rotation numbers, the proof follows the same analytic–geometric principle as in Theorem 3. The initial tangent is drawn at the point $P(d-r, 0)$ on the incircle $C$, corresponding to the even-$k$ configuration. Each subsequent vertex is determined using the tangent condition (2), while the closure condition is established by equating the abscissas of the vertices obtained from opposite sides of the symmetric configuration (for example, $A_3^{(1)}$ and $A_3^{(9)}$ for $n = 17$). The analytical elimination of the coordinates yields polynomial relations in $(R,r,d)$, producing the functions $F_{17}^{(2)}(R,r,d) = 0$, $F_{17}^{(4)}(R,r,d) = 0$, $F_{17}^{(6)}(R,r,d) = 0$, $F_{17}^{(8)}(R,r,d) = 0$, and analogously $F_{19}^{(2)}(R,r,d) = 0$, $F_{19}^{(4)}(R,r,d) = 0$,

$F_{19}^{(6)}(R,r,d)=0$, $F_{19}^{(8)}(R,r,d)=0$. For $k=2$, the construction reduces exactly to the case described in Theorem 3., while for higher even $k$ the algebraic form remains analogous but of higher order. □

**Discussion.** The results obtained for $n=15$, $n=17$, and $n=19$ confirm the general validity of the analytical method described in Section 3. All relations exhibit the expected symmetry with respect to the parity of the rotation number and reduce to simpler forms for specific $(n,k)$ pairs according to Radić's theorem 2. These findings, together with the detailed derivation for the 13-gon, complete the analytical description of Fuss' relations for bicentric polygons with an odd number of sides up to $n=19$.

The cases $n=13$ with $k=2,4,6$ and $n=15$ with $k=2$ had not been reported previously in the literature. The reason lies primarily in the rapidly increasing algebraic complexity of the closure condition $F_n^{(k)}(R,r,d)=0$ as both $n$ and $k$ increase, leading to polynomial equations of extremely high degree. The resulting polynomials reach very high algebraic degrees, which makes symbolic derivations increasingly intractable without computer algebra assistance. Earlier studies, such as [6]–[7], [9], were therefore limited to lower-degree cases due to these computational constraints. The present work extends those results by deriving explicit analytical expressions for the previously unresolved configurations.

Graphical representations of the bicentric 15-gon ($k=2,4$), the bicentric 17-gon ($k=1$–$8$), and the bicentric 19-gon ($k=1$–$9$) are provided in Appendix A.3. These figures illustrate the analytically derived bicentric configurations and visually confirm the polygonal closure predicted by the corresponding Fuss' relations. The numerical values of the incircle radii, computed for $R=1$ and $d=0.1$, are listed in Appendix A.2, providing quantitative verification of the analytical results.

## 5 Conclusion

New analytical forms of the Fuss' relations $F_n^{(k)}(R,r,d)=0$ have been derived for bicentric polygons with an odd number of sides and higher rotation numbers. Building upon Poncelet's theorem and Radić's theoretical framework, the proposed method allows the derivation of closed-form relations for previously unknown cases of the 13-gon, 15-gon, 17-gon, and 19-gon.

The proofs demonstrate how the analytic–geometric construction, combined with algebraic elimination, naturally yields the closure condition $\rho(R,r,d)F_n^{(k)}(R,r,d)=0$, from which the analytical expressions follow directly. One detailed proof (Theorem 4.1) was presented in full detail, while subsequent theorems were obtained analogously by applying the same geometric principle for different rotation numbers.

Numerical and graphical verifications confirmed the validity of all derived relations, illustrating complete consistency between analytical derivation and computational results. The unified approach developed here provides a clear, extensible framework for deriving Fuss' relations of higher order and may serve as a basis for future generalizations to other classes of bicentric and Poncelet-type polygons.

Beyond its theoretical interest, the presented framework has potential applications in computational geometry and design. The explicit analytical forms of Fuss' relations can be implemented in symbolic computation software to automatically generate bicentric configurations satisfying Poncelet's closure condition. Moreover, such relations may find use in computer-aided design of circular mechanisms, in the geometric modeling of conic envelopes, and in the study of discrete dynamical systems arising from Poncelet-type polygons. These aspects provide a promising direction for future interdisciplinary research.

**Acknowledgments.** This work is supported by Croatian Science Foundation under the project HRZZ-IP-2024-05-3882. The author thanks Prof. Zoran Kaliman, PhD, for his valuable suggestions, which greatly improved this manuscript.

## References

[1] DÖRRIE, H., *100 Great Problems of Elementary Mathematics*. Dover Publications, New York, 1965.

[2] DRAGOVIĆ, V., RADNOVIĆ, M., *Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics*. *Frontiers in Mathematics*, Birkhäuser, Basel, 2011, `https://doi.org/10.1007/978-3-0348-0015-0`

[3] ORLIĆ BACHLER, M., KALIMAN, Z., Fuss' relation for bicentric tridecagon and bicentric pentadecagon. *Math. Pannon.* **1**(1) (2024), 162–170, `https://doi.org/10.1556/314.2024.00015`

[4] ORLIĆ, M., KALIMAN, Z., Analytical derivation of Fuss' relations for bicentric hendecagon and dodecagon. *Acta Phys. Polon A* **128** (2015), 201–206, `https://doi.org/10.12693/APhysPolA.128.B-82`

[5] ORLIĆ, M., KALIMAN, Z. Mathematica in analytical derivation of Fuss' relation. In: *Proceedings of Aplimat 2007*, Bratislava, 2007, 457–462.

[6] RADIĆ, M., An improved method for establishing Fuss' relations for bicentric $n$-gons where $n \geq 4$ is an even integer. *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.* **519** (= 18) (2014), 145–170.

[7] RADIĆ, M., Certain relations between triangles and bicentric hexagons. *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.* **503** (= 16) (2009), 21–40.

[8] RADIĆ, M., Functions of triples of positive real numbers and their use in the study of bicentric polygons. *Beitr. Algebra Geom.* **54** (2013), 709–736, `https://doi.org/10.1007/s13366-012-0131-5`

[9] RADIĆ, M., ZATEZALO, A., About some kinds of bicentric polygons and concerning relations. *Math. Maced.* **4**(1) (2006), 47–73.

[10] ROITMAN, P., GARCIA, R., REZNIK, D. , New invariants of Poncelet-Jacobi bicentric polygons. *Arnold Math. J.* **7**(4) (2021), 619–637, `https://doi.org/10.1007/s40598-021-00188-6`



**Mandi Orlić Bachler**
orcid.org/0000-0002-9011-1892
e-mail: mandi.orlic@tvz.hr

University of Applied Sciences
Av. V. Holjevca 15, Zagreb, Croatia


## APENDIX A. Fuss' relations, numerical and graphical results

### A.1. Analytical forms of Fuss' relations

In all Fuss' relations presented below, the substitution $p = \frac{R+d}{r}$, $q = \frac{R-d}{r}$ is used. The following analytical expressions correspond to the bicentric polygons derived in Section 4.

$$
\begin{aligned}
F_{13}^{(2)}(R,r,d) =& (q-1)^{12}(q+1)^9 p^{21} - (q-1)^6 q (q+1)^{10} (3q^4+10q^2+3) p^{20} - 2(q-1)^6 q^2 (q+1)^9 (3q^4-4q^3+10q^2 \\
& -4q+3) p^{19} + 2(q-1)^6 q^3 (q+1)^4 (13q^8+36q^7+124q^6+156q^5+238q^4+156q^3+124q^2+36q+13) p^{18} \\
& + (q-1)^6 q^4 (q+1)^5 (9q^6+66q^5+71q^4+92q^3+71q^2+66q+9) p^{17} - (q-1)^6 q^5 (q+1)^4 (99q^6+118q^5+493q^4 \\
& +308q^3+493q^2+118q+99) p^{16} + 8(q-1)^2 q^6 (q+1)^5 (3q^8-8q^7+34q^6-32q^5+38q^4-32q^3+34q^2-8q+3) p^{15} \\
& + 8(q-1)^2 q^7 (q+1)^4 (27q^8-84q^7+166q^6-252q^5+318q^4-252q^3+166q^2-84q+27) p^{14} - 2(q-1)^2 q^8 (q+1)^5 (63q^6 \\
& -34q^5+169q^4-76q^3+169q^2-34q+63) p^{13} - 2(q-1)^2 q^9 (q+1)^4 (147q^6-370q^5+709q^4-684q^3+709q^2 \\
& -370q+147) p^{12} + 4(q-1)^2 q^{10} (63q^9+255q^8+536q^7+776q^6+930q^5+930q^4+776q^3+536q^2+255q+63) p^{11} \\
& + 4q^{11} (q^2-1)^2 (63q^6+38q^5+61q^4+188q^3+61q^2+38q+63) p^{10} - 2(q-1)^2 q^{12} (147q^7+449q^6+867q^5+1097q^4 \\
& +1097q^3+867q^2+449q+147) p^9 - 2q^{13} (q^2-1)^2 (63q^4+120q^3+74q^2+120q+63) p^8 + 8q^{14} (27q^7+q^6+5q^5-33q^4 \\
& -33q^3+5q^2+q+27) p^7 + 8q^{15}(q+1)^2 (3q^4+4q^3-16q^2+4q+3) p^6 + q^{16} (-99q^5+57q^4+26q^3+26q^2+57q-99) p^5 \\
& + q^{17}(q+1)^2 (9q^2+2q+9) p^4 + 2q^{18} (13q^3-5q^2-5q+13) p^3 - 6q^{19}(q+1)^2 p^2 - 3q^{20}(q+1) p + q^{21} = 0\,.
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
F_{15}^{(2)}(R,r,d) =& (q-1)^{10}(q+1)^{14} p^{24} + 4(q-1)^8 q (q+1)^7 (q^8-q^7+14q^6+q^5+34q^4+q^3+14q^2-q+1) p^{23} \\
& -4q^2 (q^2-1)^6 (q^{10}-8q^9-3q^8-64q^7+2q^6-112q^5+2q^4-64q^3-3q^2-8q+1) p^{22} - 4(q-1)^6 q^3 (q+1)^5 \\
& (9q^{10}+8q^9+37q^8+64q^7-46q^6+112q^5-46q^4+64q^3+37q^2+8q+9) p^{21} - 2(q-1)^6 q^4 (q+1)^4 (7q^{10} \\
& +116q^9+323q^8+608q^7+1078q^6+856q^5+1078q^4+608q^3+323q^2+116q+7) p^{20} + 4(q-1)^6 q^5 (q+1)^3 \\
& (35q^{10}+121q^9+211q^8+340q^7+458q^6+230q^5+458q^4+340q^3+211q^2+121q+35) p^{19} \\
& + 4(q-1)^6 q^6 (q+1)^4 (35q^8+132q^7+444q^6+412q^5+898q^4+412q^3+444q^2+132q+35) p^{18} \\
& - 4(q-1)^6 q^7 (q+1)^3 (75q^8+212q^7+444q^6+332q^5+818q^4+332q^3+444q^2+212q+75) p^{17}
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&-(q-1)^4q^8(q+1)^2\Big(465q^{10}+474q^9+1773q^8-168q^7-1726q^6-612q^5-1726q^4-168q^3+1773q^2\\
&+474q+465)\,p^{16}+8(q-1)^2q^9(q+1)^3\Big(45q^{10}-147q^9+309q^8-876q^7+1534q^6-1858q^5+1534q^4\\
&-876q^3+309q^2-147q+45)\,p^{15}+8q^{10}\left(q^2-1\right)^2\Big(111q^{10}-196q^9-163q^9-255q^8-726q^7-110q^6\\
&-110q^5-726q^4+243q^8-288q^7+30q^6+328q^5+30q^4-288q^3+243q^2-196q+111\Big)\,p^{14}\\
&-8(q-1)^2q^{11}\left(21q^{11}-175q^{10}-255q^3-163q^2-175q+21\right)p^{13}-4q^{12}\left(q^2-1\right)^2(273q^8-434q^7\\
&+504q^6-142q^5+302q^4-142q^3+504q^2-434q+273\Big)\,p^{12}-8(q-1)^2q^{13}(21q^9+238q^8+383q^7\\
&+237q^6+625q^5+625q^4+237q^3+383q^2+238q+21\Big)\,p^{11}+8q^{14}\left(q^2-1\right)^2\Big(111q^6-102q^5+177q^4\\
&-4q^3+177q^2-102q+111)\,p^{10}+8(q-1)^2q^{15}\Big(45q^7+147q^6+345q^5+247q^4+247q^3+345q^2+147q\\
&+45)\,p^9-q^{16}\left(q^2-1\right)^2\left(465q^4-52q^3+630q^2-52q+465\right)p^8-4(q-1)^2q^{17}\Big(75q^5+88q^4+253q^3\\
&+253q^2+88q+75)\,p^7+4q^{18}\left(q^2-1\right)^2\left(35q^2+16q+35\right)p^6+4(q-1)^2q^{19}\left(35q^3+19q^2+19q+35\right)p^5\\
&-2q^{20}(q+1)^2\left(7q^2-16q+7\right)p^4-4q^{21}\left(9q^3-8q^2-8q+9\right)p^3-4q^{22}(q+1)^2p^2+4q^{23}(q+1)p+q^{24}=0\,.
\end{aligned}
$$

$$
\begin{aligned}
F_{17}^{(1)}(R,r,d)=&(q-1)^{16}(q+1)^{20}p^{36}+4(q-1)^{17}q(q+1)^{12}\left(q^6+7q^4+7q^2+1\right)p^{35}-2(q-1)^{16}q^2(q+1)^{12}\\
&\left(5q^6+10q^5+35q^4+28q^3+35q^2+10q+5\right)p^{34}-4(q-1)^9q^3(q+1)^{12}(15q^{12}-60q^{11}+294q^{10}-588q^9\\
&+1377q^8-1656q^7+2260q^6-1656q^5+1377q^4-588q^3+294q^2-60q+15\Big)\,p^{33}+(q-1)^{10}q^4(q+1)^{12}(25q^{10}\\
&-274q^9+517q^8-1176q^7+1506q^6-2220q^5+1506q^4-1176q^3+517q^2-274q+25\Big)\,p^{32}+32(q-1)^9q^5(q+1)^{12}\\
&\left(13q^{10}-26q^9+149q^8-178q^7+438q^6-312q^5+438q^4-178q^3+149q^2-26q+13\right)p^{31}+16(q-1)^{10}q^6(q+1)^6\\
&\Big(9q^{14}+42q^{13}+234q^{12}+410q^{11}+892q^{10}+1214q^9+1937q^8+1788q^7+1937q^6+1214q^5+892q^4+410q^3+234q^2\\
&+42q+9)\,p^{30}-32(q-1)^9q^7(q+1)^6(55q^{14}+250q^{13}+820q^{12}+2074q^{11}+4234q^{10}+6542q^9+8843q^8+9660q^7\\
&+8843q^6+6542q^5+4234q^4+2074q^3+820q^2+250q+55\Big)\,p^{29}-4(q-1)^{10}q^8(q+1)^6(355q^{12}+480q^{11}+3066q^{10}\\
&+3376q^9+9725q^8+8176q^7+15180q^6+8176q^5+9725q^4+3376q^3+3066q^2+480q+355\Big)\,p^{28}\\
&+16(q-1)^9q^9(q+1)^6\Big(315q^{12}+1294q^{11}+4118q^{10}+8258q^9+15389q^8+19696q^7+22948q^6+19696q^5+15389q^4\\
&+8258q^3+4118q^2+1294q+315)\,p^{27}+8(q-1)^4q^{10}(q+1)^6\Big(749q^{16}-3476q^{15}+10942q^{14}-25756q^{13}+51400q^{12}\\
&-84404q^{11}+125282q^{10}-154652q^9+168022q^8-154652q^7+125282q^6-84404q^5+51400q^4-25756q^3+10942q^2\\
&-3476q+749)\,p^{26}-16(q-1)^5q^{11}(q+1)^6(637q^{14}-586q^{13}+2257q^{12}-6544q^{11}+8415q^{10}-15142q^9+18387q^8\\
&-15872q^7+18387q^6-15142q^5+8415q^4-6544q^3+2257q^2-586q+637\Big)\,p^{25}-4(q-1)^4q^{12}(q+1)^6(4095q^{14}\\
&-15830q^{13}+50625q^{12}-103812q^{11}+201091q^{10}-285930q^9+388285q^8-403320q^7+388285q^6-285930q^5\\
&+201091q^4-103812q^3+50625q^2-15830q+4095)\,p^{24}+32(q-1)^5q^{13}(q+1)^6(455q^{12}-1020q^{11}+2206q^{10}\\
&-5862q^9+7453q^8-9822q^7+11388q^6-9822q^5+7453q^4-5862q^3+2206q^2-1020q+455\Big)\,p^{23}\\
&+16(q-1)^4q^{14}(q+1)^6\Big(2015q^{12}-5812q^{11}+19561q^{10}-32800q^9+62365q^8-71980q^7+89910q^6-71980q^5\\
&+62365q^4-32800q^3+19561q^2-5812q+2015)\,p^{22}-32(q-1)^5q^{15}(q+1)^2(429q^{14}+110q^{13}-1298q^{12}\\
&-3398q^{11}-7160q^{10}-6470q^9-3235q^8-2500q^7-3235q^6-6470q^5-7160q^4-3398q^3-1298q^2+110q\\
&+429)\,p^{21}-2(q-1)^4q^{16}(q+1)^2(23881q^{14}+47718q^{13}+126203q^{12}+283116q^{11}+412761q^{10}+633018q^9\\
&+786787q^8+763368q^7+786787q^6+633018q^5+412761q^4+283116q^3+126203q^2+47718q+23881\Big)\,p^{20}\\
&+8(q-1)^5q^{17}(q+1)^2\Big(715q^{12}-1870q^{11}-11110q^{10}-15406q^9-22747q^8-20356q^7-17172q^6-20356q^5\\
&-22747q^4-15406q^3-11110q^2-1870q+715)\,p^{19}+4(q-1)^4q^{18}(q+1)^2(13585q^{12}+35200q^{11}+80542q^{10}\\
&+163952q^9+235663q^8+293392q^7+340868q^6+293392q^5+235663q^4+163952q^3+80542q^2+35200q\\
&+13585)\,p^{18}+8(q-1)^5q^{19}(q+1)^2\Big(715q^{10}+2134q^9+10483q^8+10464q^7+14658q^6+12564q^5+14658q^4
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
&+10464q^3+10483q^2+2134q+715)\,p^{17}-2(q-1)^4q^{20}(q+1)^2\,(23881q^{10}+66594q^9+141933q^8+241640q^7\\
&+335562q^6+337644q^5+335562q^4+241640q^3+141933q^2+66594q+23881)\,p^{16}\\
&-32q^{21}\,(429q^{15}-891q^{14}+1089q^{13}-1521q^{12}-57q^{11}+2541q^{10}-1567q^9-235q^8+235q^7+1567q^6\\
&-2541q^5+57q^4+1521q^3-1089q^2+891q-429)\,p^{15}+16q^{22}\,(q^2-1)^2\,(2015q^{10}+1130q^9+2360q^8\\
&-2282q^7+889q^6-8288q^5+889q^4-2282q^3+2360q^2+1130q+2015)\,p^{14}+16(q-1)^4q^{14}(q+1)^6\\
&(2015q^{12}-5812q^{11}+19561q^{10}-32800q^9+62365q^8-71980q^7+89910q^6-71980q^5+62365q^4-32800q^3\\
&+19561q^2-5812q+2015)\,p^{22}-32(q-1)^5q^{15}(q+1)^2\,(429q^{14}+110q^{13}-1298q^{12}-3398q^{11}-7160q^{10}\\
&-6470q^9-3235q^8-2500q^7-3235q^6-6470q^5-7160q^4-3398q^3-1298q^2+110q+429)\,p^{21}\\
&-2(q-1)^4q^{16}(q+1)^2\,(23881q^{14}+47718q^{13}+126203q^{12}+283116q^{11}+412761q^{10}+633018q^9+786787q^8\\
&+763368q^7+786787q^6+633018q^5+412761q^4+283116q^3+126203q^2+47718q+23881)\,p^{20}+8(q-1)^5q^{17}(q+1)^2\\
&(715q^{12}-1870q^{11}-11110q^{10}-15406q^9-22747q^8-20356q^7-17172q^6-20356q^5-22747q^4-15406q^3-11110q^2\\
&-1870q+715)\,p^{19}+4(q-1)^4q^{18}(q+1)^2\,(13585q^{12}+35200q^{11}+80542q^{10}+163952q^9+235663q^8+293392q^7\\
&+340868q^6+293392q^5+235663q^4+163952q^3+80542q^2+35200q+13585)\,p^{18}\\
&+8(q-1)^5q^{19}(q+1)^2\,(715q^{10}+2134q^9+10483q^8+10464q^7+14658q^6+12564q^5+14658q^4+10464q^3+10483q^2\\
&+2134q+715)\,p^{17}-2(q-1)^4q^{20}(q+1)^2\,(23881q^{10}+66594q^9+141933q^8+241640q^7+335562q^6+337644q^5\\
&+335562q^4+241640q^3+141933q^2+66594q+23881)\,p^{16}-32q^{21}\,(429q^{15}-891q^{14}+1089q^{13}-1521q^{12}\\
&-57q^{11}+2541q^{10}-1567q^9-235q^8+235q^7+1567q^6-2541q^5+57q^4+1521q^3-1089q^2+891q-429)\,p^{15}\\
&+16q^{22}\,(q^2-1)^2\,(2015q^{10}+1130q^9+2360q^8-2282q^7+889q^6-8288q^5+889q^4-2282q^3+2360q^2+1130q+2015)\,p^{14}\\
&+32q^{23}(q+1)^2\,(455q^{11}-1865q^{10}+4032q^9-6340q^8+8305q^7-9243q^6+9243q^5-8305q^4+6340q^3-4032q^2\\
&+1865q-455)\,p^{13}-4q^{24}\,(q^2-1)^2\,(4095q^8-204q^7+4704q^6-10900q^5+4034q^4-10900q^3+4704q^2-204q+4095)\,p^{12}\\
&-16q^{25}(q+1)^2\,(637q^9-2263q^8+4598q^7-6486q^6+7648q^5-7648q^4+6486q^3-4598q^2+2263q-637)\,p^{11}\\
&+8q^{26}\,(q^2-1)^2\,(749q^6-698q^5+925q^4-2200q^3+925q^2-698q+749)\,p^{10}+16q^{27}(q+1)^2\,(315q^7-865q^6+1605q^5\\
&-1999q^4+1999q^3-1605q^2+865q-315)\,p^9-4(q-1)^2q^{28}\,(355q^6+30q^5-559q^4-452q^3-559q^2+30q+355)\,p^8\\
&-32q^{29}\,(55q^7+3q^6+7q^5+57q^4-57q^3-7q^2-3q-55)\,p^7+16(q-1)^2q^{30}\,(9q^4-8q^3-41q^2-8q+9)\,p^6\\
&+32q^{31}\,(13q^5+7q^4-3q^3+3q^2-7q-13)\,p^5+5(q-1)^2q^{32}\,(5q^2-2q+5)\,p^4-20q^{33}\,(3q^3+q^2-q-3)\,p^3\\
&-10(q-1)^2q^{34}p^2+4(q-1)q^{35}p+q^{36}=0
\end{aligned}
$$

$$
\begin{aligned}
F_{17}^{(2)}(R,r,d)=&(q-1)^{20}(q+1)^{16}p^{36}-4(q-1)^{12}q(q+1)^{17}\,(q^6+7q^4+7q^2+1)\,p^{35}-2(q-1)^{12}q^2(q+1)^{16}\\
&(5q^6-10q^5+35q^4-28q^3+35q^2-10q+5)\,p^{34}+4(q-1)^{12}q^3(q+1)^9\,(15q^{12}+60q^{11}+294q^{10}+588q^9\\
&+1377q^8+1656q^7+2260q^6+1656q^5+1377q^4+588q^3+294q^2+60q+15)\,p^{33}+(q-1)^{12}q^4(q+1)^{10}\\
&(25q^{10}+274q^9+517q^8+1176q^7+1506q^6+2220q^5+1506q^4+1176q^3+517q^2+274q+25)\,p^{32}\\
&-32(q-1)^{12}q^5(q+1)^9\,(13q^{10}+26q^9+149q^8+178q^7+438q^6+312q^5+438q^4+178q^3+149q^2+26q\\
&+13)\,p^{31}+16(q-1)^6q^6(q+1)^{10}\,(9q^{14}-42q^{13}+234q^{12}-410q^{11}+892q^{10}-1214q^9+1937q^8-1788q^7\\
&+1937q^6-1214q^5+892q^4-410q^3+234q^2-42q+9)\,p^{30}+32(q-1)^6q^7(q+1)^9\,(55q^{14}-250q^{13}\\
&+820q^{12}-2074q^{11}+4234q^{10}-6542q^9+8843q^8-9660q^7+8843q^6-6542q^5+4234q^4-2074q^3+820q^2\\
&-250q+55)\,p^{29}-4(q-1)^6q^8(q+1)^{10}\,(355q^{12}-480q^{11}+3066q^{10}-3376q^9+9725q^8-8176q^7+15180q^6\\
&-8176q^5+9725q^4-3376q^3+3066q^2-480q+355)\,p^{28}-16(q-1)^6q^9(q+1)^9\,(315q^{12}-1294q^{11}+4118q^{10}
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
&-8258q^9+15389q^8-19696q^7+22948q^6-19696q^5+15389q^4-8258q^3+4118q^2-1294q+315) p^{27}\\
&+8(q-1)^6q^{10}(q+1)^4 (749q^{16}+3476q^{15}+10942q^{14}+25756q^{13}+51400q^{12}+84404q^{11}+125282q^{10}\\
&+154652q^9+168022q^8+154652q^7+125282q^6+84404q^5+51400q^4+25756q^3+10942q^2+3476q+749) p^{26}\\
&+16(q-1)^6q^{11}(q+1)^5 (637q^{14}+586q^{13}+2257q^{12}+6544q^{11}+8415q^{10}+15142q^9+18387q^8+15872q^7\\
&+18387q^6+15142q^5+8415q^4+6544q^3+2257q^2+586q+637) p^{25}-4(q-1)^6q^{12}(q+1)^4 (4095q^{14}\\
&+15830q^{13}+50625q^{12}+103812q^{11}+201091q^{10}+285930q^9+388285q^8+403320q^7+388285q^6\\
&+285930q^5+201091q^4+103812q^3+50625q^2+15830q+4095) p^{24}\\
&-32(q-1)^6q^{13}(q+1)^5 (455q^{12}+1020q^{11}+2206q^{10}+5862q^9+7453q^8+9822q^7+11388q^6+9822q^5\\
&+7453q^4+5862q^3+2206q^2+1020q+455) p^{23}+16(q-1)^6q^{14}(q+1)^4 (2015q^{12}+5812q^{11}+19561q^{10}\\
&+32800q^9+62365q^8+71980q^7+89910q^6+71980q^5+62365q^4+32800q^3+19561q^2+5812q+2015) p^{22}\\
&+32(q-1)^2q^{15}(q+1)^5 (429q^{14}-110q^{13}-1298q^{12}+3398q^{11}-7160q^{10}+6470q^9-3235q^8+2500q^7\\
&-3235q^6+6470q^5-7160q^4+3398q^3-1298q^2-110q+429) p^{21}-2(q-1)^2q^{16}(q+1)^4 (23881q^{14}\\
&-47718q^{13}+126203q^{12}-283116q^{11}+412761q^{10}-633018q^9+786787q^8-763368q^7+786787q^6\\
&-633018q^5+412761q^4-283116q^3+126203q^2-47718q+23881) p^{20}-8(q-1)^2q^{17}(q+1)^5 (715q^{12}\\
&+1870q^{11}-11110q^{10}+15406q^9-22747q^8+20356q^7-17172q^6+20356q^5-22747q^4+15406q^3\\
&-11110q^2+1870q+715) p^{19}+4(q-1)^2q^{18}(q+1)^4 (13585q^{12}-35200q^{11}+80542q^{10}-163952q^9\\
&+235663q^8-293392q^7+340868q^6-293392q^5+235663q^4-163952q^3+80542q^2-35200q+13585) p^{18}\\
&-8(q-1)^2q^{19}(q+1)^5 (715q^{10}-2134q^9+10483q^8-10464q^7+14658q^6-12564q^5+14658q^4-10464q^3\\
&+10483q^2-2134q+715) p^{17}-2(q-1)^2q^{20}(q+1)^4 (23881q^{10}-66594q^9+141933q^8-241640q^7\\
&+335562q^6-337644q^5+335562q^4-241640q^3+141933q^2-66594q+23881) p^{16}\\
&+32(q-1)^2q^{21} (429q^{13}+1749q^{12}+4158q^{11}+8088q^{10}+11961q^9+13293q^8+13058q^7+13058q^6\\
&+13293q^5+11961q^4+8088q^3+4158q^2+1749q+429) p^{15}+16q^{22} (q^2-1)^2 (2015q^{10}-1130q^9\\
&+2360q^8+2282q^7+889q^6+8288q^5+889q^4+2282q^3+2360q^2-1130q+2015) p^{14}\\
&-32(q-1)^2q^{23} (455q^{11}+1865q^{10}+4032q^9+6340q^8+8305q^7+9243q^6+9243q^5+8305q^4\\
&+6340q^3+4032q^2+1865q+455) p^{13}-4q^{24} (q^2-1)^2 (4095q^8+204q^7+4704q^6+10900q^5\\
&+4034q^4+10900q^3+4704q^2+204q+4095) p^{12}+16(q-1)^2q^{25} (637q^9+2263q^8+4598q^7+6486q^6\\
&+7648q^5+7648q^4+6486q^3+4598q^2+2263q+637) p^{11}+8q^{26} (q^2-1)^2 (749q^6+698q^5+925q^4\\
&+2200q^3+925q^2+698q+749) p^{10}-16(q-1)^2q^{27} (315q^7+865q^6+1605q^5+1999q^4+1999q^3\\
&+1605q^2+865q+315) p^9-4q^{28}(q+1)^2 (355q^6-30q^5-559q^4+452q^3-559q^2-30q+355) p^8\\
&+32q^{29} (55q^7-3q^6+7q^5-57q^4-57q^3+7q^2-3q+55) p^7+16q^{30}(q+1)^2 (9q^4+8q^3\\
&-41q^2+8q+9) p^6-32q^{31} (13q^5-7q^4-3q^3-3q^2-7q+13) p^5+5q^{32}(q+1)^2 (5q^2+2q+5) p^4\\
&+20q^{33} (3q^3-q^2-q+3) p^3-10q^{34}(q+1)^2p^2-4q^{35}(q+1)p+q^{36}=0.
\end{aligned}
$$

$$
\begin{aligned}
F^{(1)}_{19}(R,r,d) =&(q-1)^{25}(q+1)^{20}p^{45}-(q-1)^{16}q(q+1)^{20} (5q^8+60q^6+126q^4+60q^2+5) p^{44}-2(q-1)^{17}q^2(q+1)^{20}\\
&(5q^6-10q^5+35q^4-28q^3+35q^2-10q+5) p^{43}+2(q-1)^{16}q^3(q+1)^{12} (45q^{14}+210q^{13}+1143q^{12}+2724q^{11}\\
&+7437q^{10}+11022q^9+17999q^8+17144q^7+17999q^6+11022q^5+7437q^4+2724q^3+1143q^2+210q+45) p^{42}\\
&-(q-1)^{17}q^4(q+1)^{12} (5q^{12}-324q^{11}-838q^{10}-2484q^9-3349q^8-6408q^7-5972q^6-6408q^5-3349q^4\\
&-2484q^3-838q^2-324q+5) p^{41}-(q-1)^{16}q^5(q+1)^{12} (751q^{12}+1988q^{11}+12142q^{10}+18548q^9+51553q^8
\end{aligned}
\tag{A.5}
$$

$$+50120q^7+82052q^6+50120q^5+51553q^4+18548q^3+12142q^2+1988q+751\Big)p^{40}+4(q-1)^9q^6(q+1)^{12}\big(165q^{18}$$
$$-750q^{17}+3945q^{16}-9320q^{15}+25672q^{14}-45304q^{13}+89152q^{12}-114072q^{11}+159594q^{10}-152628q^9+159594q^8$$
$$-114072q^7+89152q^6-45304q^5+25672q^4-9320q^3+3945q^2-750q+165\Big)p^{39}+4(q-1)^{10}q^7(q+1)^{12}\Big(955q^{16}$$
$$-4120q^{15}+15380q^{14}-44792q^{13}+100556q^{12}-179080q^{11}+282220q^{10}-357736q^9+389618q^8-357736q^7$$
$$+282220q^6-179080q^5+100556q^4-44792q^3+15380q^2-4120q+955\Big)p^{38}-(q-1)^9q^8(q+1)^{12}\Big(5605q^{16}-12720q^{15}$$
$$+79144q^{14}-134992q^{13}+435372q^{12}-552944q^{11}+1155224q^{10}-1068816q^9+1564510q^8-1068816q^7+1155224q^6$$
$$-552944q^5+435372q^4-134992q^3+79144q^2-12720q+5605\Big)p^{37}-(q-1)^{10}q^9(q+1)^{12}\big(13015q^{14}-57594q^{13}$$
$$+197245q^{12}-469636q^{11}+977279q^{10}-1469734q^9+2023725q^8-2150072q^7+2023725q^6-1469734q^5+977279q^4$$
$$-469636q^3+197245q^2-57594q+13015\big)p^{36}+2(q-1)^9q^{10}(q+1)^6\big(13623q^{20}+58692q^{19}+224578q^{18}+644388q^{17}$$
$$+1537387q^{16}+3161936q^{15}+5726616q^{14}+8736400q^{13}+12079710q^{12}+14399416q^{11}+15362124q^{10}+14399416q^9$$
$$+12079710q^8+8736400q^7+5726616q^6+3161936q^5+1537387q^4+644388q^3+224578q^2+58692q+13623\Big)p^{35}$$
$$+2(q-1)^{10}q^{11}(q+1)^6\Big(15105q^{18}+33250q^{17}+68825q^{16}+227120q^{15}+458932q^{14}+936248q^{13}+1612484q^{12}$$
$$+2079824q^{11}+2366638q^{10}+2622156q^9+2366638q^8+2079824q^7+1612484q^6+936248q^5+458932q^4+227120q^3$$
$$+68825q^2+33250q+15105\big)p^{34}-(q-1)^9q^{12}(q+1)^6$$
$$(92055q^{18}+383630q^{17}+1452895q^{16}+3794064q^{15}+8841036q^{14}$$
$$+16031112q^{13}+27191804q^{12}+36730864q^{11}+46308290q^{10}+48125652q^9+46308290q^8+36730864q^7+27191804q^6$$
$$+16031112q^5+8841036q^4+3794064q^3+1452895q^2+383630q+92055\Big)p^{33}-(q-1)^{10}q^{13}(q+1)^6\Big(43605q^{16}$$
$$+167220q^{15}+342208q^{14}+1128700q^{13}+2176220q^{12}+3799508q^{11}+5324928q^{10}+6750204q^9+6541086q^8$$
$$+6750204q^7+5324928q^6+3799508q^5+2176220q^4+1128700q^3+342208q^2+167220q+43605\Big)p^{32}$$
$$+16(q-1)^9q^{14}(q+1)^6\Big(14535q^{16}+53048q^{15}+204500q^{14}+460152q^{13}+1051356q^{12}+1636168q^{11}+2599852q^{10}$$
$$+3002376q^9+3472314q^8+3002376q^7+2599852q^6+1636168q^5+1051356q^4+460152q^3+204500q^2+53048q+14535\Big)p^{31}$$
$$+16(q-1)^4q^{15}(q+1)^6\Big(969q^{20}+9664q^{19}-54654q^{18}+152816q^{17}-329771q^{16}+491984q^{15}-652264q^{14}+928624q^{13}$$
$$-1334686q^{12}+1710096q^{11}-1911092q^{10}+1710096q^9-1334686q^8+928624q^7-652264q^6+491984q^5-329771q^4$$
$$+152816q^3-54654q^2+9664q+969\big)p^{30}-2(q-1)^5q^{16}(q+1)^6\Big(227715q^{18}-235650q^{17}+1366235q^{16}-2686160q^{15}$$
$$+5032460q^{14}-10636024q^{13}+13987228q^{12}-19540144q^{11}+23852538q^{10}-22605324q^9+23852538q^8-19540144q^7$$
$$+13987228q^6-10636024q^5+5032460q^4-2686160q^3+1366235q^2-235650q+227715\Big)p^{29}+2(q-1)^4q^{17}(q+1)^6$$
$$\Big(53295q^{18}-333170q^{17}+1258535q^{16}-2648080q^{15}+5210748q^{14}-8023224q^{13}+12000716q^{12}-15961456q^{11}+20634466q^{10}$$
$$-21500076q^9+20634466q^8-15961456q^7+12000716q^6-8023224q^5+5210748q^4-2648080q^3+1258535q^2-333170q$$
$$+53295\big)p^{28}+4(q-1)^5q^{18}(q+1)^6\Big(176035q^{16}-304880q^{15}+1143728q^{14}-2592944q^{13}+4381796q^{12}-7836752q^{11}$$
$$+10347280q^{10}-12071952q^9+13810290q^8-12071952q^7+10347280q^6-7836752q^5+4381796q^4-2592944q^3+1143728q^2$$
$$-304880q+176035\big)p^{27}-4(q-1)^4q^{19}(q+1)^6\Big(85595q^{16}-305012q^{15}+1099080q^{14}-2008892q^{13}+3983924q^{12}$$
$$-5691028q^{11}+8686584q^{10}-9857724q^9+11570274q^8-9857724q^7+8686584q^6-5691028q^5+3983924q^4-2008892q^3$$
$$+1099080q^2-305012q+85595\big)p^{26}+4(q-1)^5q^{18}(q+1)^6\Big(176035q^{16}-304880q^{15}+1143728q^{14}-2592944q^{13}+4381796q^{12}$$
$$-7836752q^{11}+10347280q^{10}-12071952q^9+13810290q^8-12071952q^7+10347280q^6-7836752q^5+4381796q^4-2592944q^3$$
$$+1143728q^2-304880q+176035\big)p^{27}-4(q-1)^4q^{19}(q+1)^6\Big(85595q^{16}-305012q^{15}+1099080q^{14}-2008892q^{13}$$
$$+3983924q^{12}-5691028q^{11}+8686584q^{10}-9857724q^9+11570274q^8-9857724q^7+8686584q^6-5691028q^5+3983924q^4$$
$$-2008892q^3+1099080q^2-305012q+85595\big)p^{26}-2(q-1)^5q^{20}(q+1)^6\big(432497q^{14}-1004458q^{13}+2946827q^{12}$$
$$-6570372q^{11}+10501177q^{10}-15673590q^9+19962699q^8-20001720q^7+19962699q^6-15673590q^5+10501177q^4-6570372q^3$$
$$+2946827q^2-1004458q+432497\big)p^{25}+2(q-1)^4q^{21}(q+1)^2\Big(314925q^{18}+507650q^{17}+1705925q^{16}+3630000q^{15}$$
$$+5794180q^{14}+9676216q^{13}+12644116q^{12}+17459152q^{11}+21057910q^{10}+20357004q^9+21057910q^8+17459152q^7$$

$$
\begin{aligned}
&+12644116q^6+9676216q^5+5794180q^4+3630000q^3+1705925q^2+507650q+314925) p^{24}+8(q-1)^5q^{22}(q+1)^2 (104975q^{16}\\
&+135200q^{15}+206700q^{14}+150960q^{13}-264852q^{12}+53296q^{11}+490644q^{10}+1067520q^9+1730826q^8+1067520q^7\\
&+490644q^6+53296q^5-264852q^4+150960q^3+206700q^2+135200q+104975) p^{23}-8(q-1)^4q^{23}(q+1)^2 (104975q^{16}\\
&+240500q^{15}+641004q^{14}+1312380q^{13}+2086220q^{12}+3309396q^{11}+4310420q^{10}+5043900q^9+5682762q^8+5043900q^7\\
&+4310420q^6+3309396q^5+2086220q^4+1312380q^3+641004q^2+240500q+104975) p^{22}-2(q-1)^5q^{24}(q+1)^2 (314925q^{14}\\
&+372814q^{13}+338975q^{12}+679468q^{11}+320917q^{10}+1684114q^9+3313695q^8+3251688q^7+3313695q^6+1684114q^5+320917q^4\\
&+679468q^3+338975q^2+372814q+314925) p^{21}+2(q-1)^4q^{25}(q+1)^2 (432497q^{14}+1132638q^{13}+2780427q^{12}\\
&+5405644q^{11}+8464185q^{10}+11947394q^9+15022667q^8+15142056q^7+15022667q^6+11947394q^5+8464185q^4\\
&+5405644q^3+2780427q^2+1132638q+432497) p^{20}+4(q-1)^5q^{26}(q+1)^2 (85595q^{12}+127940q^{11}+82070q^{10}+330420q^9\\
&+439669q^8+712072q^7+982836q^6+712072q^5+439669q^4+330420q^3+82070q^2+127940q+85595) p^{19}\\
&-4(q-1)^4q^{27}(q+1)^2 (176035q^{12}+465360q^{11}+1110750q^{10}+1955664q^9+2999117q^8+3584480q^7\\
&+4196964q^6+3584480q^5+2999117q^4+1955664q^3+1110750q^2+465360q+176035) p^{18}\\
&-2q^{28}(q+1)^2 (53295q^{15}-114525q^{14}-130109q^{13}+897919q^{12}-1732589q^{11}+1609255q^{10}-682809q^9\\
&+161411q^8-161411q^7+682809q^6-1609255q^5+1732589q^4-897919q^3+130109q^2+114525q-53295) p^{17}\\
&+2q^{29} (q^2-1)^2 (227715q^{12}+92816q^{11}+441422q^{10}-130352q^9+327981q^8-1085280q^7+253444q^6-1085280q^5\\
&+327981q^4-130352q^3+441422q^2+92816q+227715) p^{16}-16q^{30}(q+1)^2 (969q^{13}-11381q^{12}+42210q^{11}\\
&-82450q^{10}+111859q^9-107151q^8+75852q^7-75852q^6+107151q^5-111859q^4+82450q^3-42210q^2+11381q\\
&-969) p^{15}-16q^{31} (q^2-1)^2 (14535q^{10}-450q^9+27455q^8-32640q^7+26298q^6-69692q^5+26298q^4-32640q^3\\
&+27455q^2-450q+14535) p^{14}+q^{32}(q+1)^2 (43605q^{11}-194675q^{10}+474095q^9-701177q^8+798466q^7-841550q^6\\
&+841550q^5-798466q^4+701177q^3-474095q^2+194675q-43605) p^{13}+q^{33} (q^2-1)^2 (92055q^8-54340q^7\\
&+168844q^6-291580q^5+182330q^4-291580q^3+168844q^2-54340q+92055) p^{12}-2q^{34}(q+1)^2 (15105q^9\\
&-48033q^8+98500q^7-132452q^6+143158q^5-143158q^4+132452q^3-98500q^2+48033q-15105) p^{11}\\
&-2q^{35} (q^2-1)^2 (13623q^6-15782q^5+23569q^4-39460q^3+23569q^2-15782q+13623) p^{10}+q^{36} (13015q^9\\
&-4095q^8+5820q^7+10900q^6-11774q^5+11774q^4-10900q^3-5820q^2+4095q-13015) p^9+(q-1)^2q^{37} (5605q^6\\
&+2370q^5-3685q^4-836q^3-3685q^2+2370q+5605) p^8-4q^{38} (955q^7+255q^6+79q^5+651q^4-651q^3-79q^2\\
&-255q-955) p^7-4(q-1)^2q^{39} (165q^4+76q^3-206q^2+76q+165) p^6+q^{40} (751q^5+349q^4-66q^3+66q^2-349q\\
&-751) p^5+5(q-1)^2q^{41} (q^2+14q+1) p^4-10q^{42} (9q^3+q^2-q-9) p^3+10(q-1)^2q^{43}p^2+5(q-1)q^{44}p-q^{45}=0
\end{aligned}
$$

$$
\begin{aligned}
F_{19}^{(2)}(R,r,d) =&(q-1)^{20}(q+1)^{25}p^{45}+(q-1)^{20}q(q+1)^{16} (5q^8+60q^6+126q^4+60q^2+5) p^{44}\\
&-2(q-1)^{20}q^2(q+1)^{17} (5q^6+10q^5+35q^4+28q^3+35q^2+10q+5) p^{43}-2(q-1)^{12}q^3(q+1)^{16} (45q^{14}-210q^{13}\\
&+1143q^{12}-2724q^{11}+7437q^{10}-11022q^9+17999q^8-17144q^7+17999q^6-11022q^5+7437q^4-2724q^3+1143q^2\\
&-210q+45) p^{42}-(q-1)^{12}q^4(q+1)^{17} (5q^{12}+324q^{11}-838q^{10}+2484q^9-3349q^8+6408q^7-5972q^6+6408q^5\\
&-3349q^4+2484q^3-838q^2+324q+5) p^{41}+(q-1)^{12}q^5(q+1)^{16} (751q^{12}-1988q^{11}+12142q^{10}-18548q^9\\
&+51553q^8-50120q^7+82052q^6-50120q^5+51553q^4-18548q^3+12142q^2-1988q+751) p^{40}\\
&+4(q-1)^{12}q^6(q+1)^9 (165q^{18}+750q^{17}+3945q^{16}+9320q^{15}+25672q^{14}+45304q^{13}+89152q^{12}+114072q^{11}\\
&+159594q^{10}+152628q^9+159594q^8+114072q^7+89152q^6+45304q^5+25672q^4+9320q^3+3945q^2+750q+165) p^{39}\\
&-4(q-1)^{12}q^7(q+1)^{10} (955q^{16}+4120q^{15}+15380q^{14}+44792q^{13}+100556q^{12}+179080q^{11}+282220q^{10}+357736q^9\\
&+389618q^8+357736q^7+282220q^6+179080q^5+100556q^4+44792q^3+15380q^2+4120q+955) p^{38}\\
&-(q-1)^{12}q^8(q+1)^9 (5605q^{16}+12720q^{15}+79144q^{14}+134992q^{13}+435372q^{12}+552944q^{11}+1155224q^{10}+1068816q^9
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
&+1564510q^{8}+1068816q^{7}+1155224q^{6}+552944q^{5}+435372q^{4}+134992q^{3}+79144q^{2}+12720q+5605\Big)\,p^{37}\\
&+(q-1)^{12}q^{9}(q+1)^{10}\,(13015q^{14}+57594q^{13}+197245q^{12}+469636q^{11}+977279q^{10}+1469734q^{9}+2023725q^{8}+2150072q^{7}\\
&+2023725q^{6}+1469734q^{5}+977279q^{4}+469636q^{3}+197245q^{2}+57594q+13015\Big)\,p^{36}+2(q-1)^{6}q^{10}(q+1)^{9}\,(13623q^{20}\\
&-58692q^{19}+224578q^{18}-644388q^{17}+1537387q^{16}-3161936q^{15}+5726616q^{14}-8736400q^{13}+12079710q^{12}\\
&-14399416q^{11}+15362124q^{10}-14399416q^{9}+12079710q^{8}-8736400q^{7}+5726616q^{6}-3161936q^{5}+1537387q^{4}\\
&-644388q^{3}+224578q^{2}-58692q+13623)\,p^{35}-2(q-1)^{6}q^{11}(q+1)^{10}\,\Big(15105q^{18}-33250q^{17}+68825q^{16}-227120q^{15}\\
&+458932q^{14}-936248q^{13}+1612484q^{12}-2079824q^{11}+2366638q^{10}-2622156q^{9}+2366638q^{8}-2079824q^{7}+1612484q^{6}\\
&-936248q^{5}+458932q^{4}-227120q^{3}+68825q^{2}-33250q+15105\Big)\,p^{34}-(q-1)^{6}q^{12}(q+1)^{9}\,(92055q^{18}-383630q^{17}\\
&+1452895q^{16}-3794064q^{15}+8841036q^{14}-16031112q^{13}+27191804q^{12}-36730864q^{11}+46308290q^{10}\\
&-48125652q^{9}+46308290q^{8}-36730864q^{7}+27191804q^{6}-16031112q^{5}+8841036q^{4}-3794064q^{3}+1452895q^{2}\\
&-383630q+92055)\,p^{33}+(q-1)^{6}q^{13}(q+1)^{10}\,\Big(43605q^{16}-167220q^{15}+342208q^{14}-1128700q^{13}\\
&+2176220q^{12}-3799508q^{11}+5324928q^{10}-6750204q^{9}+6541086q^{8}-6750204q^{7}+5324928q^{6}-3799508q^{5}+2176220q^{4}\\
&-1128700q^{3}+342208q^{2}-167220q+43605)\,p^{32}+16(q-1)^{6}q^{14}(q+1)^{9}\,\Big(14535q^{16}-53048q^{15}+204500q^{14}\\
&+460152q^{13}+1051356q^{12}-1636168q^{11}+2599852q^{10}-3002376q^{9}+3472314q^{8}-3002376q^{7}+2599852q^{6}-1636168q^{5}\\
&+1051356q^{4}-460152q^{3}+204500q^{2}-53048q+14535)\,p^{31}-16(q-1)^{6}q^{15}(q+1)^{4}\,(969q^{20}-9664q^{19}-54654q^{18}-152816q^{17}\\
&-329771q^{16}-491984q^{15}-652264q^{14}-928624q^{13}-1334686q^{12}-1710096q^{11}-1911092q^{10}-1710096q^{9}-1334686q^{8}\\
&-928624q^{7}-652264q^{6}-491984q^{5}-329771q^{4}-152816q^{3}-54654q^{2}-9664q+969\Big)\,p^{30}\\
&-2(q-1)^{6}q^{16}(q+1)^{5}\,\Big(227715q^{18}+235650q^{17}+1366235q^{16}+2686160q^{15}+5032460q^{14}+10636024q^{13}+13987228q^{12}\\
&+19540144q^{11}+23852538q^{10}+22605324q^{9}+23852538q^{8}+19540144q^{7}+13987228q^{6}+10636024q^{5}+5032460q^{4}+2686160q^{3}\\
&+1366235q^{2}+235650q+227715)\,p^{29}-2(q-1)^{6}q^{17}(q+1)^{4}\,\Big(53295q^{18}+333170q^{17}+1258535q^{16}+2648080q^{15}+5210748q^{14}\\
&+8023224q^{13}+12000716q^{12}+15961456q^{11}+20634466q^{10}+21500076q^{9}+20634466q^{8}+15961456q^{7}+12000716q^{6}\\
&+8023224q^{5}+5210748q^{4}+2648080q^{3}+1258535q^{2}+333170q+53295\Big)\,p^{28}+4(q-1)^{6}q^{18}(q+1)^{5}\,\Big(176035q^{16}\\
&+304880q^{15}+1143728q^{14}+2592944q^{13}+4381796q^{12}+7836752q^{11}+10347280q^{10}+12071952q^{9}+13810290q^{8}\\
&+12071952q^{7}+10347280q^{6}+7836752q^{5}+4381796q^{4}+2592944q^{3}+1143728q^{2}+304880q+176035\Big)\,p^{27}\\
&+4(q-1)^{6}q^{19}(q+1)^{4}\,\Big(85595q^{16}+305012q^{15}+1099080q^{14}+2008892q^{13}+3983924q^{12}+5691028q^{11}+8686584q^{10}+9857724q^{9}\\
&+11570274q^{8}+9857724q^{7}+8686584q^{6}+5691028q^{5}+3983924q^{4}+2008892q^{3}+1099080q^{2}+305012q+85595\Big)\,p^{26}\\
&-2(q-1)^{6}q^{20}(q+1)^{5}\,(432497q^{14}+1004458q^{13}+2946827q^{12}+6570372q^{11}+10501177q^{10}+15673590q^{9}+19962699q^{8}\\
&+20001720q^{7}+19962699q^{6}+15673590q^{5}+10501177q^{4}+6570372q^{3}+2946827q^{2}+1004458q+432497\Big)\,p^{25}\\
&-2(q-1)^{2}q^{21}(q+1)^{4}\,\Big(314925q^{18}-507650q^{17}+1705925q^{16}-3630000q^{15}+5794180q^{14}-9676216q^{13}+12644116q^{12}\\
&-17459152q^{11}+21057910q^{10}-20357004q^{9}+21057910q^{8}-17459152q^{7}+12644116q^{6}-9676216q^{5}+5794180q^{4}-3630000q^{3}\\
&+1705925q^{2}-507650q+314925)\,p^{24}+8(q-1)^{2}q^{22}(q+1)^{5}\,\Big(104975q^{16}-135200q^{15}+206700q^{14}-150960q^{13}-264852q^{12}\\
&-53296q^{11}+490644q^{10}-1067520q^{9}+1730826q^{8}-1067520q^{7}+490644q^{6}-53296q^{5}-264852q^{4}-150960q^{3}+206700q^{2}\\
&-135200q+104975)\,p^{23}+8(q-1)^{2}q^{23}(q+1)^{4}\,\Big(104975q^{16}-240500q^{15}+641004q^{14}-1312380q^{13}+2086220q^{12}\\
&-3309396q^{11}+4310420q^{10}-5043900q^{9}+5682762q^{8}-5043900q^{7}+4310420q^{6}-3309396q^{5}+2086220q^{4}-1312380q^{3}\\
&+641004q^{2}-240500q+104975)\,p^{22}-2(q-1)^{2}q^{24}(q+1)^{5}\,(314925q^{14}-372814q^{13}+338975q^{12}-679468q^{11}+320917q^{10}\\
&-1684114q^{9}+3313695q^{8}-3251688q^{7}+3313695q^{6}-1684114q^{5}+320917q^{4}-679468q^{3}+338975q^{2}-372814q+314925\Big)\,p^{21}\\
&-2(q-1)^{2}q^{25}(q+1)^{4}\,(432497q^{14}-1132638q^{13}+2780427q^{12}-5405644q^{11}+8464185q^{10}-11947394q^{9}+15022667q^{8}\\
&-15142056q^{7}+15022667q^{6}-11947394q^{5}+8464185q^{4}-5405644q^{3}+2780427q^{2}-1132638q+432497\Big)\,p^{20}\\
&+4(q-1)^{2}q^{26}(q+1)^{5}\,\Big(85595q^{12}-127940q^{11}+82070q^{10}-330420q^{9}+439669q^{8}-712072q^{7}+982836q^{6}-712072q^{5}\\
&+439669q^{4}-330420q^{3}+82070q^{2}-127940q+85595)\,p^{19}+4(q-1)^{2}q^{27}(q+1)^{4}\,(176035q^{12}-465360q^{11}+1110750q^{10}\\
&-1955664q^{9}+2999117q^{8}-3584480q^{7}+4196964q^{6}-3584480q^{5}+2999117q^{4}-1955664q^{3}+1110750q^{2}-465360q+176035\Big)\,p^{18}
\end{aligned}
$$

$$
\begin{aligned}
&-2(q-1)^2q^{28}\Big(53295q^{15}+114525q^{14}-130109q^{13}-897919q^{12}-1732589q^{11}-1609255q^{10}-682809q^{9}-161411q^{8}-161411q^{7}\\
&-682809q^{6}-1609255q^{5}-1732589q^{4}-897919q^{3}-130109q^{2}+114525q+53295\Big)p^{17}-2q^{29}\left(q^2-1\right)^2\left(227715q^{12}-92816q^{11}\right.\\
&\left.+441422q^{10}+130352q^{9}+327981q^{8}+1085280q^{7}+253444q^{6}+1085280q^{5}+327981q^{4}+130352q^{3}+441422q^{2}-92816q+227715\right)p^{16}\\
&-16(q-1)^2q^{30}\Big(969q^{13}+11381q^{12}+42210q^{11}+82450q^{10}+111859q^{9}+107151q^{8}+75852q^{7}+75852q^{6}+107151q^{5}+111859q^{4}\\
&+82450q^{3}+42210q^{2}+11381q+969\Big)p^{15}+16q^{31}\left(q^2-1\right)^2\Big(14535q^{10}+450q^{9}+27455q^{8}+32640q^{7}+26298q^{6}+69692q^{5}\\
&+26298q^{4}+32640q^{3}+27455q^{2}+450q+14535\Big)p^{14}+(q-1)^2q^{32}\left(43605q^{11}+194675q^{10}+474095q^{9}+701177q^{8}+798466q^{7}\right.\\
&\left.+841550q^{6}+841550q^{5}+798466q^{4}+701177q^{3}+474095q^{2}+194675q+43605\right)p^{13}-q^{33}\left(q^2-1\right)^2\left(92055q^{8}+54340q^{7}\right.\\
&\left.+168844q^{6}+291580q^{5}+182330q^{4}+291580q^{3}+168844q^{2}+54340q+92055\right)p^{12}-2(q-1)^2q^{34}\left(15105q^{9}+48033q^{8}+98500q^{7}\right.\\
&\left.+132452q^{6}+143158q^{5}+143158q^{4}+132452q^{3}+98500q^{2}+48033q+15105\right)p^{11}+2q^{35}\left(q^2-1\right)^2\Big(13623q^{6}+15782q^{5}\\
&+23569q^{4}+39460q^{3}+23569q^{2}+15782q+13623\Big)p^{10}+q^{36}\Big(13015q^{9}+4095q^{8}+5820q^{7}-10900q^{6}-11774q^{5}-11774q^{4}-10900q^{3}\\
&+5820q^{2}+4095q+13015\Big)p^{9}-q^{37}(q+1)^2\left(5605q^{6}-2370q^{5}-3685q^{4}+836q^{3}-3685q^{2}-2370q+5605\right)p^{8}-4q^{38}\Big(955q^{7}-255q^{6}\\
&+79q^{5}-651q^{4}-651q^{3}+79q^{2}-255q+955\Big)p^{7}+4q^{39}(q+1)^2\left(165q^{4}-76q^{3}-206q^{2}-76q+165\right)p^{6}\\
&+q^{40}\left(751q^{5}-349q^{4}-66q^{3}-66q^{2}-349q+751\right)p^{5}-5q^{41}(q+1)^2\left(q^2-14q+1\right)p^{4}-10q^{42}\left(9q^3-q^2-q+9\right)p^{3}\\
&-10q^{43}(q+1)^2p^2+5q^{44}(q+1)p+q^{45}=0.
\end{aligned}
$$

### A.2. Numerical results and verification

All numerical computations were carried out in *Wolfram Mathematica* with 20-digit precision. In this paper, the values are rounded to six significant digits. Unless otherwise stated, parameters were fixed to

$R=1, \quad d=0.1,$

and the values of $r_{n,k}$ were obtained as numerical roots of the corresponding Fuss' relations.

| $n$ | $k$ | **Fuss' relation** | $r_{n,k}$ | **Figure** |
|---|---|---|---|---|
| 13 | 2 | (A.1) | 0.856554 | Fig. 1 |
| 13 | 4 | (A.1) | 0.561780 | Fig. 2 |
| 13 | 6 | (A.1) | 0.119618 | Fig. 2 |
| 15 | 2 | (A.2) | 0.876400 | Fig. 3 |
| 15 | 4 | (A.2) | 0.660078 | Fig. 3 |
| 17 | 1 | (A.3) | 0.899948 | Fig. 4 |
| 17 | 2 | (A.4) | 0.887300 | Fig. 5 |
| 17 | 3 | (A.3) | 0.827988 | Fig. 4 |
| 17 | 4 | (A.4) | 0.726898 | Fig.5 |
| 17 | 5 | (A.3) | 0.595549 | Fig. 4 |
| 17 | 6 | (A.4) | 0.441564 | Fig. 5 |
| 17 | 7 | (A.3) | 0.271442 | Fig. 4 |
| 17 | 8 | (A.4) | 0.091569 | Fig. 5 |
| 19 | 1 | (A.5) | 0.899986 | Fig. 6 |
| 19 | 2 | (A.6) | 0.893235 | Fig. 7 |
| 19 | 3 | (A.5) | 0.851932 | Fig. 6 |
| 19 | 4 | (A.6) | 0.773686 | Fig. 7 |
| 19 | 5 | (A.5) | 0.667936 | Fig. 6 |
| 19 | 6 | (A.6) | 0.541099 | Fig. 7 |
| 19 | 7 | (A.5) | 0.398100 | Fig. 6 |
| 19 | 8 | (A.6) | 0.243524 | Fig. 7 |
| 19 | 9 | (A.5) | 0.081955 | Fig. 6 |

Table 1: *Summary of numerically verified values of $r_{n,k}$ for $R=1$ and $d=0.1$.*

### A.3. Graphical results

Figures below illustrate the geometric configurations corresponding to the computed values of $r_{n,k}$. All polygons are symmetric with respect to the $x$-axis and are simultaneously tangent to the incircle $C$ and inscribed in the circumcircle $K$. Each figure visually confirms the bicentric property of the polygon and the validity of the derived Fuss' relation for the corresponding rotation number $k$.

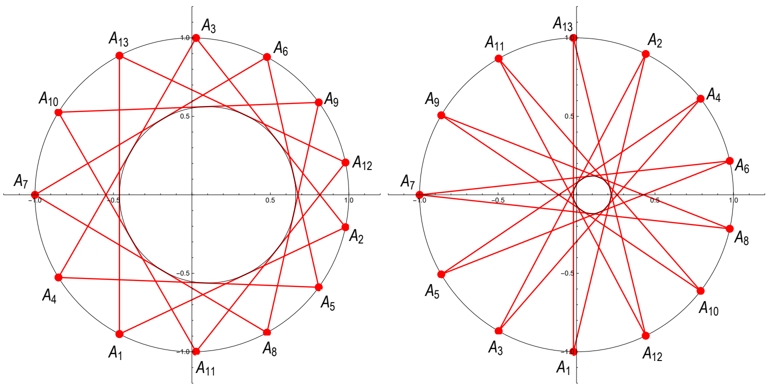


Figure 2: *Symmetrical bicentric 13-gon for $R=1$ and $d=0.1$: (left) $r_{13,4}=0.561780$ ($k=4$); (right) $r_{13,6}=0.119618$ ($k=6$). Each configuration satisfies the corresponding even-class Fuss' relation $F_{13}^{(k)}(R,r,d)=0$.*

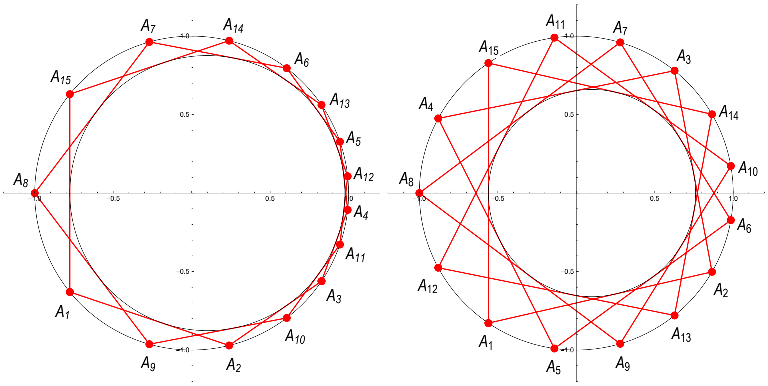


Figure 3: *Symmetrical bicentric 15-gon for $R=1$ and $d=0.1$: (left) $r_{15,2}=0.876400$ ($k=2$); (right) $r_{15,4}=0.660078$ ($k=4$). Each configuration satisfies the corresponding even-class Fuss' relation $F_{15}^{(k)}(R,r,d)=0$.*

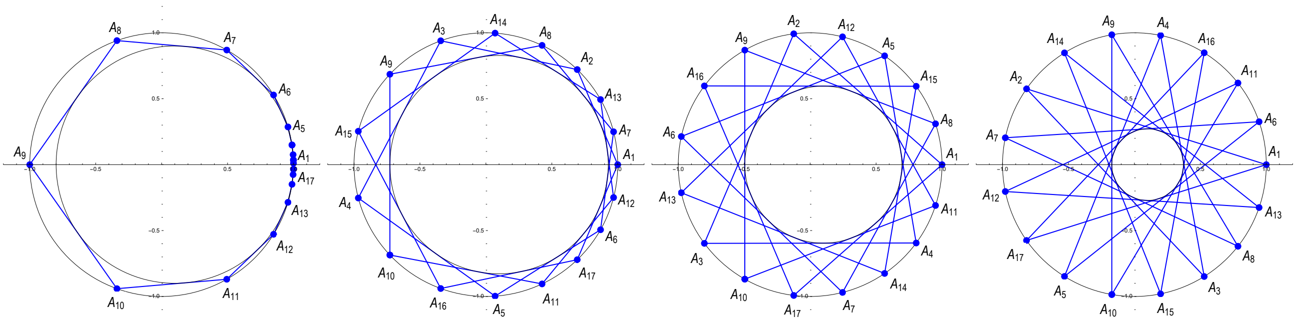

Figure 4: *Symmetrical bicentric 17-gon for $R=1$ and $d=0.1$: (first) $r_{17,1}=0.899948$ ($k=1$); (second) $r_{17,3}=0.827988$ ($k=3$); (third) $r_{17,5}=0.595549$ ($k=5$); (fourth) $r_{17,7}=0.271442$ ($k=7$). Each configuration satisfies the corresponding odd-class Fuss' relation $F_{17}^{(k)}(R,r,d)=0$.*

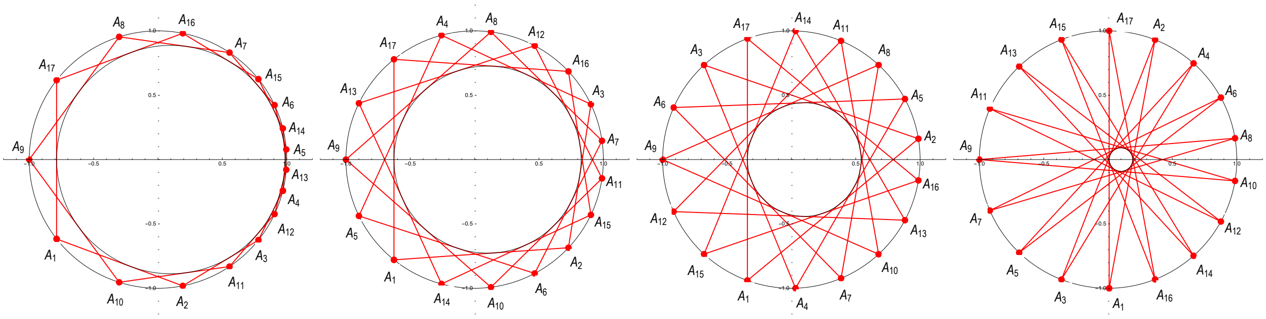

Figure 5: *Symmetrical bicentric 17-gon for* $R = 1$ *and* $d = 0.1$*: (first)* $r_{17,2} = 0.887300$ *(*$k = 2$*); (second)* $r_{17,4} = 0.726898$ *(*$k = 4$*); (third)* $r_{17,6} = 0.441564$ *(*$k = 6$*); (fourth)* $r_{17,8} = 0.0915694$ *(*$k = 8$*). Each configuration satisfies the corresponding even-class Fuss' relation* $F_{17}^{(k)}(R,r,d) = 0$.

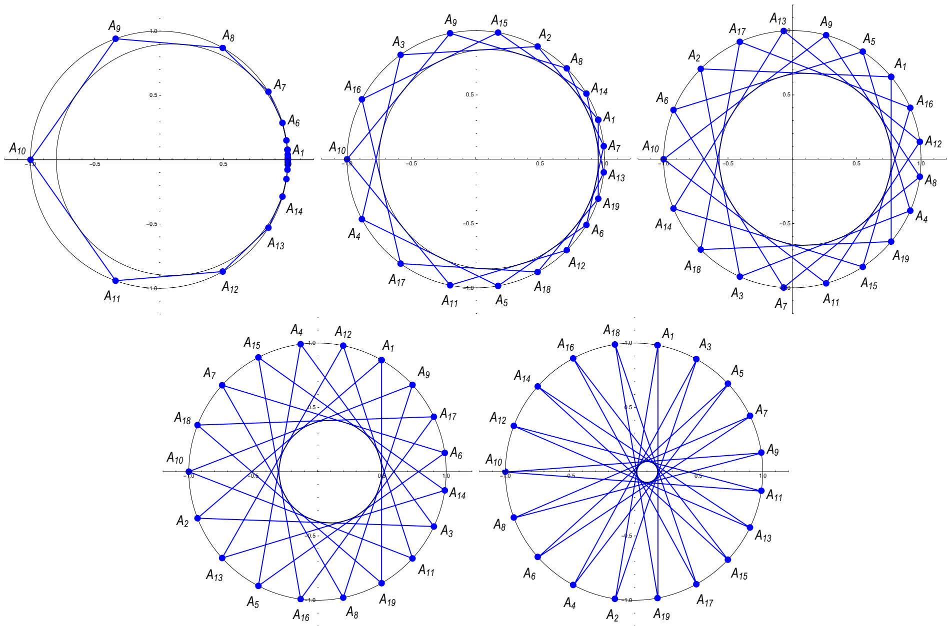

Figure 6: *Symmetrical bicentric 19-gon for* $R = 1$ *and* $d = 0.1$*: (first)* $r_{19,1} = 0.899986$ *(*$k = 1$*); (second)* $r_{19,3} = 0.851932$ *(*$k = 3$*); (third)* $r_{19,5} = 0.667936$ *(*$k = 5$*); (fourth)* $r_{19,7} = 0.398100$ *(*$k = 7$*); (fifth)* $r_{19,9} = 0.0819548$ *(*$k = 9$*). Each configuration satisfies the corresponding odd-class Fuss' relation* $F_{19}^{(k)}(R,r,d) = 0$.

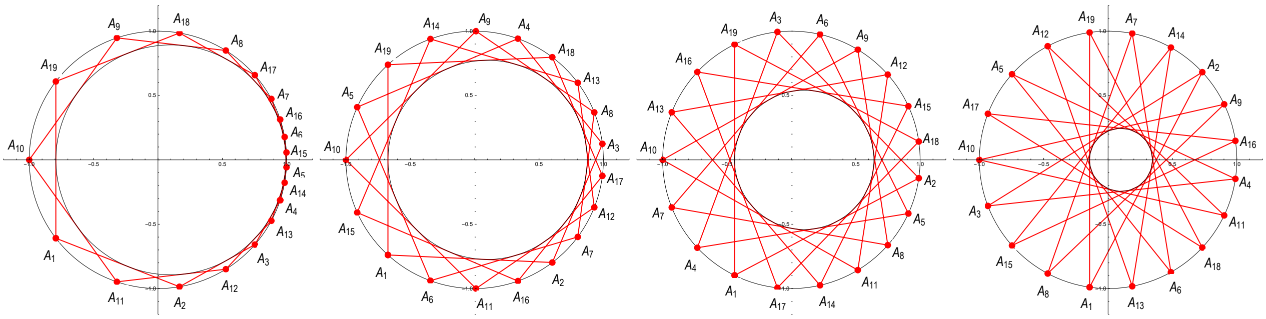

Figure 7: *Symmetrical bicentric 19-gon for* $R = 1$ *and* $d = 0.1$*: (first)* $r_{19,2} = 0.893235$ *(*$k = 2$*); (second)* $r_{19,4} = 0.773686$ *(*$k = 4$*); (third)* $r_{19,6} = 0.541099$ *(*$k = 6$*); (fourth)* $r_{19,8} = 0.243524$ *(*$k = 8$*). Each configuration satisfies the corresponding even-class Fuss' relation* $F_{19}^{(k)}(R,r,d) = 0$.