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A Generalization of Archimedean Circles on an Arbelos

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ABSTRACT

In this paper, we extend the classical notion of Archimedean circles, originally discovered by Archimedes in the arbelos, to the broader framework of the arbelos with overhang. By means of new constructions, we establish conditions under which circles in this generalized setting retain the characteristic radius property of Archimedean circles. Our results unify and extend previous findings, revealing deeper symmetries and structural invariants within these geometric figures.

Key words: Archimedean circles, Arbelos, Arbelos with Overhang

MSC2020: 51M04, 51N20

Poopćenje Arhimedovih kružnica na arbelosu

SAŽETAK

U ovom radu proširujemo klasičan pojam Arhimedovih kružnica, koji je izvorno otkrio Arhimed na arbelosu, na širi okvir arbelosa s produžetkom. Pomoću novih konstrukcija određujemo uvjete pod kojima kružnice u ovom poopćenom okruženju zadržavaju karakteristično svojstvo polumjera Arhimedovih kružnica. Naši rezultati ujedinjuju i proširuju prethodne rezultate, otkrivajući dublje simetrije i strukturne invarijante unutar ovih geometrijskih figura.

Ključne riječi: Arhimedove kružnice, arbelos, arbelos s produžetkom

1 Introduction

The *arbelos*, or "shoemaker's knife", is a classical figure in plane geometry, first studied by Archimedes. It consists of the region bounded by three mutually tangent semicircles with collinear diameters. One of the most remarkable aspects of the arbelos is the existence of an infinite family of circles that share a surprising property: they all have the same radius as a particular circle introduced by Archimedes. These circles are now known as *Archimedean circles* [2].

Archimedes proved that a specific circle constructed inside the arbelos—the so-called *Archimedes' circle*—has a radius equal to that of another circle tangent to the same boundaries (see Figure 1). Over the centuries, many additional Archimedean circles have been discovered, all exhibiting the same constancy in radius despite being derived from different constructions.

The study of Archimedean circles continues to fascinate geometers, both for the elegance of their construction and for the deeper geometric principles they reveal.

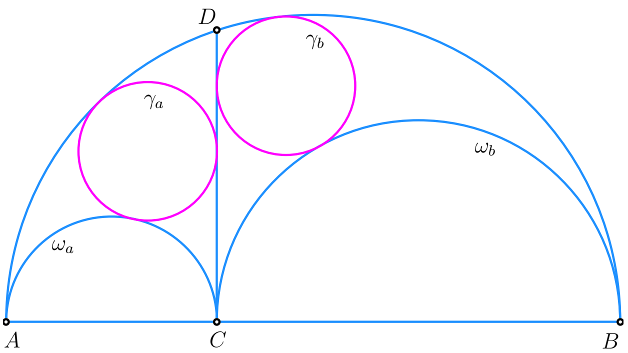

Figure 1: *The twin of the Archimedes' circle on the arbelos.*

A natural generalization of the classical arbelos, known as the *arbelos with overhang*, was introduced by H. Okumura in [7]. Furthermore, Okumura presented several results concerning Archimedean circles within this extended framework [3, 4, 5, 6]. An additional pair of Archimedean circles in the context of the arbelos was introduced by the present authors in [1].

Theorem 1 *(Archimedes' circle on the arbelos with overhang)* *Let AB be a segment, and construct the semicircle ω_c with diameter AB. Let C be a point on AB, and let the perpendicular from C to AB intersect ω_c at D. On rays CA and CB, choose points A' and B' respectively. Construct semicircles $\omega_{a'}$ and $\omega_{b'}$ with diameters CA' and CB'. This configuration is called the* arbelos with overhang. *Construct two circles: γ_a tangent to ω_c, $\omega_{a'}$, and CD; and γ_b tangent to ω_c, $\omega_{b'}$, and CD. Then γ_a and γ_b have equal radii if and only if $AA' = BB'$.*

The proof and some applications of Theorem 1 can be found in [6].

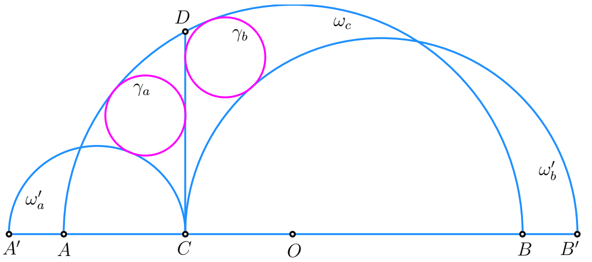

Figure 2: *The twin of the Archimedes' circle on the arbelos with overhang.*

In this paper, we generalize the concept of Archimedean circles to the arbelos with overhang and investigate constructions that preserve the equal-radius property under this extended configuration.

Theorem 2 *Let AB be a segment, and construct the semicircle ω_c with diameter AB. Let C be a point on the segment AB, and let the perpendicular line d from C to AB intersect ω_c at D. Suppose K and L are arbitrary points on the rays CA and CB, respectively. Construct the circles ω_k and ω_l centered at K and L and passing through C. Let A' and B' be the points dividing CK and CL in the same ratio, respectively. Construct the semicircles $\omega_{a'}$ and $\omega_{b'}$ with diameters CA' and CB'. Define $M(r_1)$ to be the circle tangent internally to both ω_k and ω_c and tangent externally to $\omega_{a'}$. Define $N(r_2)$ to be the circle tangent internally to both ω_l and ω_c and tangent externally to $\omega_{b'}$. If the distances from the centers of M and N to the line d are d_1 and d_2, respectively, then:*

i) $\dfrac{r_1}{d_1} = \dfrac{r_2}{d_2}$;

ii) if K tends to infinity, then $r_1 = \dfrac{a'b'}{a+b'}$;

iii) if L tends to infinity, then $r_2 = \dfrac{a'b'}{a'+b}$;

iv) if both K and L tend to infinity, then $r_1 = r_2$ if and only if $AA' = BB'$.

Remark 1 *If both K and L tend to infinity, then the semicircles $\omega_{a'}$ and $\omega_{b'}$ degenerate into the line CD. In this limiting case, we obtain $r_1 = r_2$ if and only if $AA' = BB'$, which means that the semicircles $\omega_c, \omega_{a'}, \omega_{b'}$ form an arbelos with overhang (see [6]). Hence, Theorem 2 naturally generalizes the arbelos with overhang.*

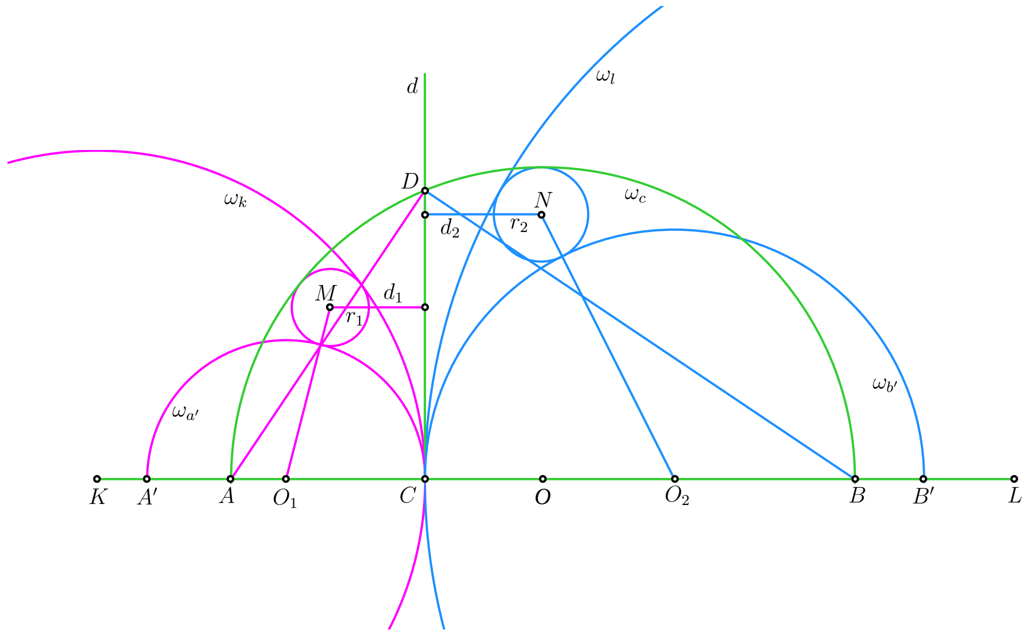

Figure 3: *A generalization of Archimedean circles on the arbelos with overhang.*

2 Proof of Theorem 2

Proof. (See Figure 3.) Assume $a < b$ and denote

$$CA' = 2a, \qquad CB' = 2b, \qquad AC = 2a', \qquad AB = 2b',$$

$$KA = k, \qquad LB = l.$$

Let O_1, O_2, and O be the midpoints of CA', CB', and AB, respectively. In Cartesian coordinates, place

$$O(b'-a',0), \qquad O_1(-a,0), \qquad O_2(b,0), \qquad K(-k-2a,0),$$

$$L(l+2b,0), \qquad M(-d_1,y), \qquad N(d_2,z).$$

Since A' and B' divide CK and BL in the same ratio, the relation

$$\frac{k}{a} = \frac{l}{b} \tag{1}$$

holds.

We now consider the following distance relations:

$$KM^2 = (KC - r_1)^2$$
$$\text{that is, } (-d_1 + k + 2a)^2 + y^2 = (k + 2a - r_1)^2, \tag{2}$$
$$MO_1^2 = (a + r_1)^2$$
$$\text{that is, } (-d_1 + a)^2 + y^2 = (a + r_1)^2, \tag{3}$$
$$MO^2 = (a' + b' - r_1)^2$$
$$\text{that is, } (b' - a' + d_1)^2 + y^2 = (a' + b' - r_1)^2. \tag{4}$$

Subtracting (3) from (2) gives

$$(a+k)d_1 = r_1(3a+k). \tag{5}$$

Subtracting (3) from (4) gives

$$(a - a' + b')d_1 + r_1(a + a' + b') = 2a'b'. \tag{6}$$

Solving equations (5) and (6) for r_1 yields

$$r_1 = \frac{a'b'(3a+k)}{2a(a-a'+2b') + (a+b')k}. \tag{7}$$

From (5), the quotient r_1/d_1 is

$$\frac{r_1}{d_1} = \frac{a+k}{3a+k} = \frac{1+\frac{k}{a}}{3+\frac{k}{a}}. \tag{8}$$

A completely analogous computation with point L gives

$$\frac{r_2}{d_2} = \frac{1+\frac{l}{b}}{3+\frac{l}{b}}. \tag{9}$$

Using relations (1), (8), and (9), we obtain the equality

$$\frac{r_1}{d_1} = \frac{r_2}{d_2},$$

which completes the proof of part (i) of Theorem 2.

We now examine the limit as k tends to infinity. Using (7), the expression for r_1 becomes

$$\begin{aligned} r_1 &= \lim_{k\to\infty} \frac{3aa'b' + a'b'k}{2a(a-a'+2b') + (a+b')k} \\ &= \lim_{k\to\infty} \frac{\frac{3aa'b'}{k} + a'b'}{\frac{2a(a-a'+2b')}{k} + a + b'} \\ &= \frac{a'b'}{a+b'}. \end{aligned}$$

Thus part (ii) of Theorem 2 is established.

A similar limiting argument with l tending to infinity gives

$$r_2 = \frac{a'b'}{a'+b}.$$

Finally, if both K and L tend to infinity, then the expressions obtained above satisfy

$$r_1 = \frac{a'b'}{a+b'}, \qquad r_2 = \frac{a'b'}{a'+b}.$$

The radii r_1 and r_2 are equal exactly in the case where

$$\frac{a'b'}{a+b'} = \frac{a'b'}{a'+b}.$$

Since $a'b'$ is nonzero, the equality of the two fractions occurs exactly when

$$a + b' = a' + b,$$

or equivalently when

$$AA' = BB'.$$

This completes the proof of part (iv) of Theorem 2. □

3 Conclusion

In this paper, we have extended the classical theory of Archimedean circles to the generalized setting of the arbelos with overhang. By introducing suitable constructions, we demonstrated that circles in this configuration retain the defining equal-radius property, analogous to that of the original Archimedean circles.

Our generalizations preserve the elegance and structural harmony of the classical case while also uncovering new symmetries and invariants inherent in the modified figure. These results enrich the study of the arbelos and its extensions, illustrating how classical geometric phenomena persist under broader transformations.

We hope this work encourages further exploration of Archimedean-type configurations in other generalized geometries, thereby contributing to the continuing dialogue between classical and modern geometry.

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