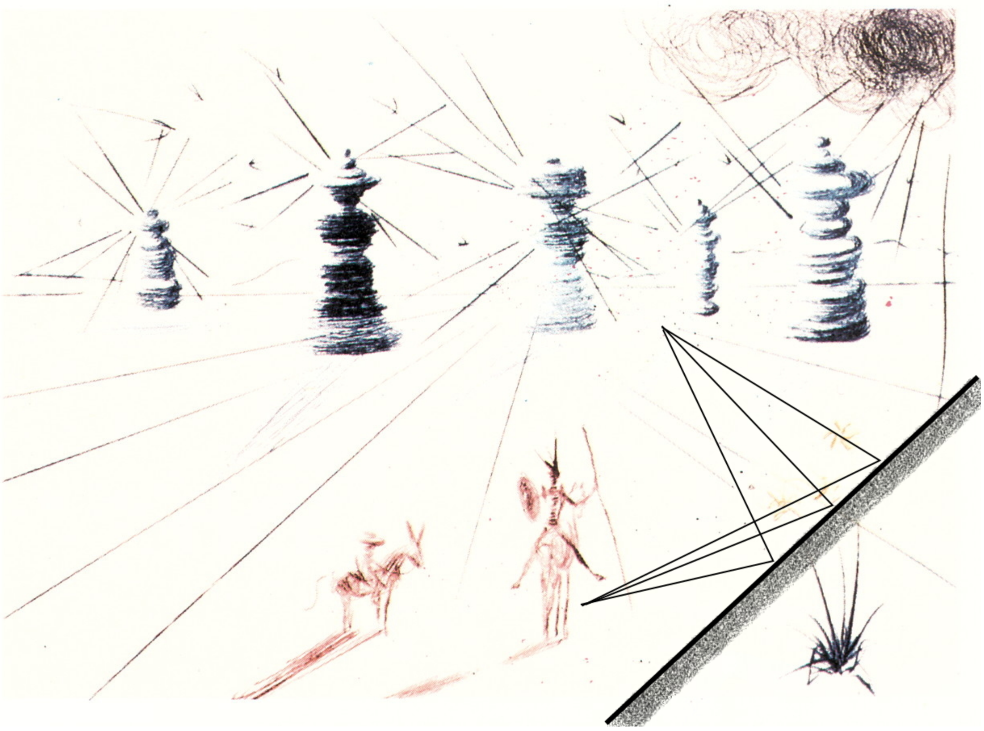
Newton–Raphsonov postupak

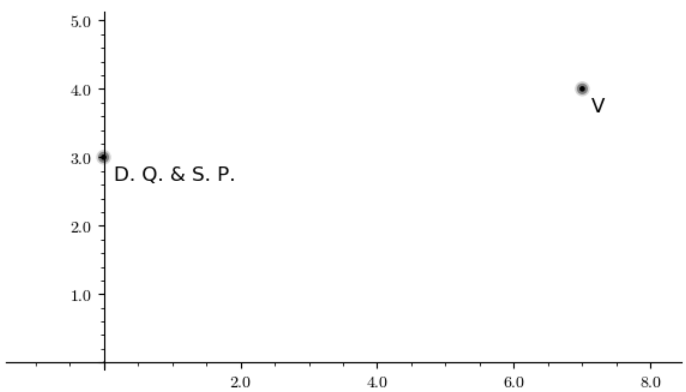
Don Quijote i vjetrenjače

Nađite točku u kojoj Don Quijote i Sancho Panza trebaju pristupiti obali rijeke da zbroj duljina puta do rijeke i puta od rijeke do vjetrenjača bude najmanji!



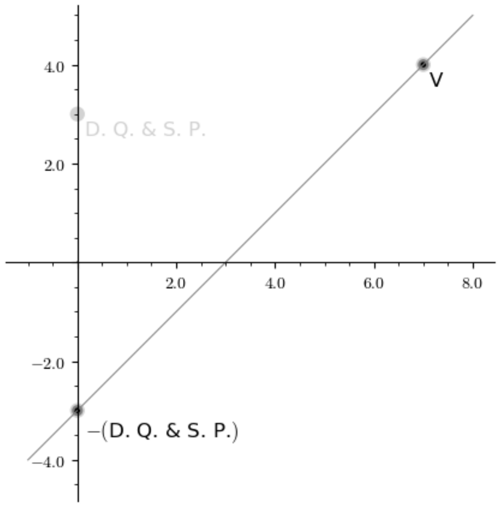
[Salvador Dalí]

U koordinatnom sustavu u kojem se obala rijeke poklapa s osi x koordinate su točke u kojoj su Don Quijote i Sancho Panza $Q(x_Q, y_Q) = Q(0, 3)$, dok su vjetrenjače u točki s koordinatama $V(x_V, y_V) = V(7, 4)$.



Budući da se obala rijeke poklapa s osi x, koordinate su tražene točke $(x, 0)$, pa je nepoznanica samo x.

„grafičko” rješenje:



$$y = \frac{3+4}{7}x - 3 = x - 3$$

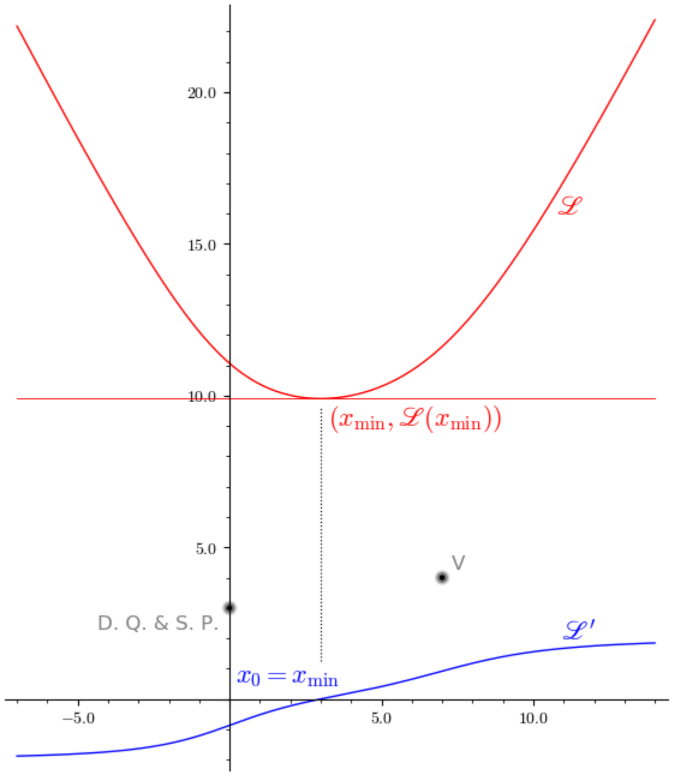
$$x - 3 = 0$$

$$x = 3$$

$$\mathscr{L}(x) \simeq 9{,}899\,494\,936\,611\,665$$

funkcija minimum koje tražimo

$$\mathscr{L} : x \mapsto \sqrt{(x-x_Q)^2 + y_Q^2} + \sqrt{(x-x_V)^2 + y_V^2}$$
$$= \sqrt{x^2 + 3^2} + \sqrt{(x-7)^2 + 4^2} = \sqrt{x^2 + 9} + \sqrt{(x-7)^2 + 16}$$



uvjet minimuma:

$$\mathscr{L}'(x) = 0$$

$$\mathscr{L}' : x \mapsto \frac{x - x_Q}{\sqrt{(x-x_Q)^2 + y_Q^2}} + \frac{x - x_V}{\sqrt{(x-x_V)^2 + y_V^2}} = \frac{x}{\sqrt{x^2+9}} + \frac{x-7}{\sqrt{(x-7)^2+16}}$$

$$\frac{x}{\sqrt{x^2+9}} + \frac{x-7}{\sqrt{(x-7)^2+16}} = 0$$

za Newton–Raphsonov postupak potrebna je i $\big(\mathscr{L}'\big)' = \mathscr{L}''$:

$$\begin{aligned}\mathscr{L}'' : x \mapsto\ & -\frac{(x-x_Q)^2}{\left(\sqrt{(x-x_Q)^2+y_Q^2}\right)^3} - \frac{(x-x_V)^2}{\left(\sqrt{(x-x_V)^2+y_V^2}\right)^3} \\ & + \frac{1}{\sqrt{(x-x_Q)^2+y_Q^2}} + \frac{1}{\sqrt{(x-x_V)^2+y_V^2}} \\ & = -\frac{x^2}{\left(\sqrt{x^2+9}\right)^3} - \frac{(x-7)^2}{\left(\sqrt{(x-7)^2+16}\right)^3} + \frac{1}{\sqrt{x^2+9}} + \frac{1}{\sqrt{(x-7)^2+16}}\end{aligned}$$

tangenta na graf funkcije $\mathscr{L}'$ u točki $x^{(k)}$ i njezina nul–točka:

$$\bar{f}^{(k)} : x \mapsto \mathscr{L}''\big(x^{(k)}\big)\, x + \mathscr{L}'\big(x^{(k)}\big) - \mathscr{L}''\big(x^{(k)}\big)\, x^{(k)}$$

$$\bar{x}_0^{(k)} = x^{(k)} - \frac{\mathscr{L}'\big(x^{(k)}\big)}{\mathscr{L}''\big(x^{(k)}\big)}$$

uvjet prekida iteracije:

$$\mathscr{L}'(x) < \tau$$

iteracija s uvjetom $\tau = 10^{-6}$ za pretpostavku $x^{(0)} = 5$:

(ispis pseudorealnih brojeva sa 6 značajnih znamenaka)

k	$x^{(k)}$	$\mathscr{L}\big(x^{(k)}\big)$	$\mathscr{L}'\big(x^{(k)}\big)$	$\mathscr{L}''\big(x^{(k)}\big)$
0	5,0	10,303 1	0,410 279	0,224 282
1	3,170 70	9,902 48	0,034 864 0	0,202 444
2	2,998 48	9,899 50	−0,000 312 581	0,206 278
3	3,0*	9,899 49	$-2{,}964\,70 \cdot 10^{-8}$	

* sa 16 značajnih znamenaka 2,999 999 856 249 473
(točnost prikaza pseudorealnih brojeva na računalu)

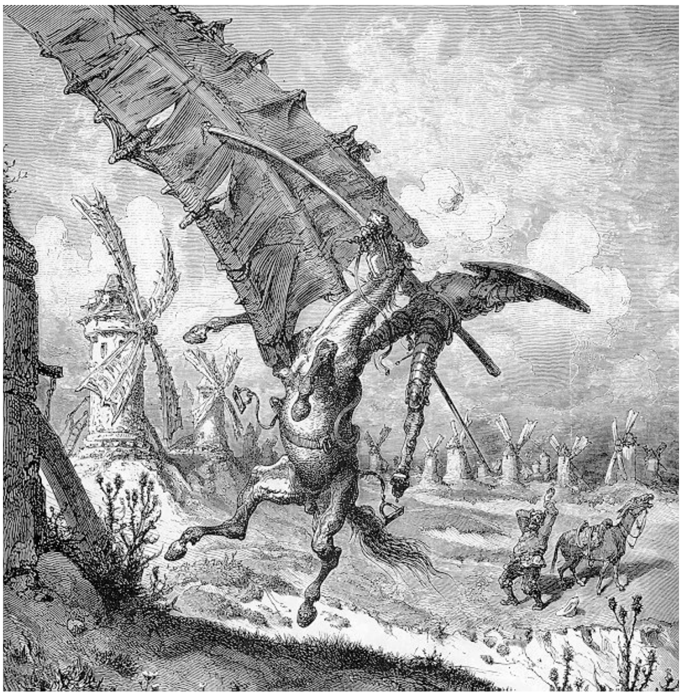
iteracija s uvjetom $\tau = 10^{-6}$ za pretpostavku $x^{(0)} = 0$:

k	$x^{(k)}$	$\mathscr{L}(x^{(k)})$	$\mathscr{L}'(x^{(k)})$	$\mathscr{L}''(x^{(k)})$
0	0,0	11,062 3	−0,868 243	0,363 865
1	2,386 17	9,939 59	−0,133 086	0,230 058
2	2,964 66	9,899 62	−0,007 304 00	0,207 177
3	2,999 92	9,899 49	−0,000 016 636 1	0,206 242
4	3,0*	9,899 49	$-8{,}387\,64 \cdot 10^{-11}$	

* sa 16 značajnih znamenaka 2,999 999 999 593 306

iteracija (s nevažno kojim uvjetom) za pretpostavku $x^{(0)} = 12$:

k	$x^{(k)}$	$\mathscr{L}(x^{(k)})$	$\mathscr{L}'(x^{(k)})$	$\mathscr{L}''(x^{(k)})$
0	12,0	18,772 4	1,751 0 11	0,065 701 5
1	−14,651 0	36,972 4	−1,963 03	0,004 189 87
2	453,868	900,764	1,999 94	$2{,}755\,35 \cdot 10^{-7}$
3	$-7{,}257\,93 \cdot 10^{6}$	$1{,}451\,59 \cdot 10^{7}$	−2,0	$6{,}535\,39 \cdot 10^{-20}$
4	$3{,}060\,26 \cdot 10^{19}$	$6{,}118\,05 \cdot 10^{19}$	2,0	0,0
5	∞			



[Gustave Doré]