GS 2. — 28. kolovoza 2024.

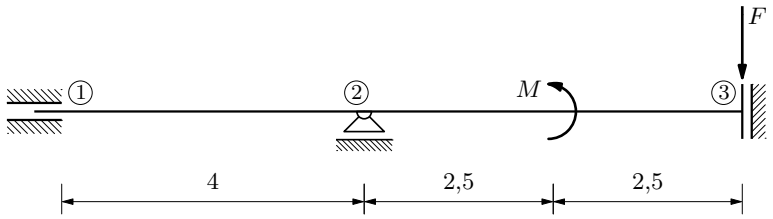
Zadatak 2.a.

Pomoću utjecajne linije nacrtane inženjerskom metodom pomakā izračunajte vrijednost reaktivnoga momenta u desnome ležaju!

$EI = \text{const.}$

$F = 125\ \text{kN}$

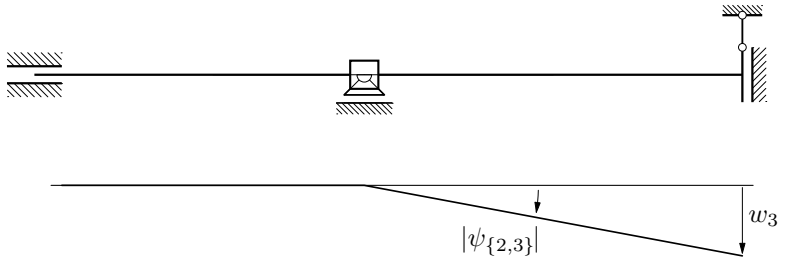
$M = 75\ \text{kNm}$



$$k_{1,2} = \frac{EI}{4}, \qquad k_{2,3} = \frac{EI}{5}$$

nepoznanice za inženjersku metodu pomakā: φ_2 & w_3

uz statičku kondenzaciju pomaka $\vec{w}_3$: φ_2



$$\psi_{\{2,3\}} = -\frac{w_3}{\ell_{\{2,3\}}} = -\frac{w_3}{5}$$

dvije nepoznanice — $\boldsymbol{\varphi_2}$ & $\boldsymbol{w_3}$:

izrazi za vrijednosti momenata na krajevima štapova:

$$M_{1,2} = 2\,k_{\{1,2\}}\,\varphi_2 = \frac{EI}{2}\,\varphi_2$$

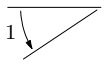
$$M_{2,1} = 4\,k_{\{1,2\}}\,\varphi_2 = EI\,\varphi_2$$

$$M_{2,3} = 4\,k_{\{2,3\}}\,\varphi_2 - 6\,k_{\{2,3\}}\,\psi_{\{2,3\}} + \overline{M}_{2,3} = \frac{4\,EI}{5}\,\varphi_2 + \frac{6\,EI}{25}\,w_3 + \overline{M}_{2,3}$$

$$M_{3,2} = 2\,k_{\{2,3\}}\,\varphi_2 - 6\,k_{\{2,3\}}\,\psi_{\{2,3\}} + \overline{M}_{3,2} = \frac{2\,EI}{5}\,\varphi_2 + \frac{6\,EI}{25}\,w_3 + \overline{M}_{3,2}$$

vrijednosti momenata upetosti za jedinični zaokret kraja 3 štapa $\{2,3\}$:

smisao vrtnje reaktivnoga momenta: 

jedinični kut: 

$$\overline{M}_{3,2} = 4\,k_{\{1,2\}} \cdot 1 = \frac{4\,EI}{5} \qquad \& \qquad \overline{M}_{2,3} = 2\,k_{\{2,3\}} \cdot 1 = \frac{2\,EI}{5}$$

jednadžba ravnoteže momenata u čvoru 2:

$$-M_{2,1} - M_{2,3} = 0 \qquad \Rightarrow \qquad M_{2,1} + M_{2,3} = 0$$

$$EI\,\varphi_2 + \frac{4\,EI}{5}\,\varphi_2 + \frac{6\,EI}{25}\,w_3 + \frac{2\,EI}{5} = 0$$

$$\frac{9\,EI}{5}\,\varphi_2 + \frac{6\,EI}{25}\,w_3 = -\frac{2\,EI}{5} \qquad \Rightarrow \qquad \frac{9}{5}\,\varphi_2 + \frac{6}{25}\,w_3 = -\frac{2}{5} \qquad (☠)$$

jednadžba virtualnih radova:

$$\big(M_{2,3} + M_{3,2}\big)\,\delta\psi_{\{2,3\}} = 0 \quad \forall\,\delta\psi_{\{2,3\}} \qquad \Rightarrow \qquad M_{2,3} + M_{3,2} = 0$$

$$\frac{6\,EI}{5}\,\varphi_2 + \frac{12\,EI}{25}\,w_3 = -\frac{6\,EI}{5} \qquad \Rightarrow \qquad \frac{6}{5}\,\varphi_2 + \frac{12}{25}\,w_3 = -\frac{6}{5} \qquad (🏴‍☠️)$$

Hmm, matrica sustava jednadžbi (☠) & (🏴‍☠️) nije simetrična?!

S jednadžbom virtualnoga rada može se malo petljati… Može se reći da smo prerano upotrijebili $\forall$ da uklonimo δ. Izrazimo li $\delta\psi_{\{2,3\}}$ kao funkciju δw_3 (na isti način kao što smo $\psi_{\{2,3\}}$ izrazili kao funkciju w_3), dobit ćemo

$$\big(M_{2,3} + M_{3,2}\big)\left(-\frac{\delta w_3}{5}\right) = 0 \quad \forall\,\delta w_3 \qquad \Rightarrow \qquad -\frac{1}{5}\big(M_{2,3} + M_{3,2}\big) = 0$$

$$-\frac{6\,EI}{5}\,\varphi_2 - \frac{12\,EI}{25}\,w_3 = \frac{6\,EI}{5} \quad \Big/ \quad \times(-1)$$

$$\frac{6\,EI}{5}\,\varphi_2 + \frac{12\,EI}{25}\,w_3 = -\frac{6\,EI}{5} \qquad \Rightarrow \qquad \frac{6}{25}\,\varphi_2 + \frac{12}{125}\,w_3 = -\frac{6}{25} \qquad (🏴‍☠️)$$

Matrica sustava jednadžbi (☠) & (🏴‍☠️) jest simetrična.

Sustavi (☠) & (🏴‍☠️) i (☠) & (🏴‍☠️) su ekvivalentni — imaju isto rješenje. Pomnožimo li neku jednadžbu sustava nekim brojem (različitim od nule), rješenje sustava se neće promijeniti — množenje (ili dijeljenje) jednadžbe brojem jedna je od elementarnih operacija Gaußova eliminacijskog postupka za rješavanje sustava linearnih jednadžbi.

rješenje sustava (☠) & (🏴‍☠️) (i, dakako, (☠) & (🏴‍☠️)):

$$\varphi_2 = \frac{1}{6} \qquad \& \qquad w_3 = -\frac{35}{12}$$

vrijednosti momenata na krajevima štapova:

$$M_{1,2} = \frac{EI}{2}\cdot\frac{1}{6} = \frac{EI}{12}$$

$$M_{2,1} = EI\cdot\frac{1}{6} = \frac{EI}{6}$$

$$M_{2,3} \;=\; \frac{4\,EI}{5}\cdot\frac{1}{6} \;-\; \frac{6\,EI}{25}\cdot\frac{35}{12} \;+\; \frac{2\,EI}{5} \;=\; -\frac{EI}{6}$$

$$M_{3,2} \;=\; \frac{2\,EI}{5}\cdot\frac{1}{6} \;-\; \frac{6\,EI}{25}\cdot\frac{35}{12} \;+\; \frac{4\,EI}{5} \;=\; \frac{EI}{6}$$

ili: (samo) **jedna nepoznanica — $\boldsymbol{\varphi_2}$**:

izrazi za vrijednosti momenata na krajevima štapova:

$$M_{1,2} \;=\; 2\,k_{\{1,2\}}\,\varphi_2 \;=\; \frac{EI}{2}\,\varphi_2$$

$$M_{2,1} \;=\; 4\,k_{\{1,2\}}\,\varphi_2 \;=\; EI\,\varphi_2$$

$$M^{\mathrm{c}}_{2,3} \;=\; k_{\{2,3\}}\,\varphi_2 \;+\overline{M}^{\mathrm{c}}_{2,3} \;=\; \frac{EI}{5}\,\varphi_2 \;+\overline{M}^{\mathrm{c}}_{2,3}$$

$$M^{\mathrm{c}}_{3,2} \;=\; -k_{\{2,3\}}\,\varphi_2 \;+\overline{M}^{\mathrm{c}}_{3,2} \;=\; -\frac{EI}{5}\,\varphi_2 \;+\overline{M}^{\mathrm{c}}_{3,2}$$

vrijednosti momenata upetosti za jedinični zaokret kraja 3 štapa $\{2,3\}$:

$$\overline{M}^{\mathrm{c}}_{3,2} \;=\; k_{\{1,2\}}\cdot 1 \;=\; \frac{EI}{5} \qquad \&\qquad \overline{M}^{\mathrm{c}}_{2,3} \;=\; -k_{\{2,3\}}\cdot 1 \;=\; -\frac{EI}{5}$$

jednadžba ravnoteže momenata u čvoru 2:

$$-M_{2,1} \;-\; M^{\mathrm{c}}_{2,3} \;=\; 0 \qquad \Rightarrow \qquad M_{2,1} \;+\; M^{\mathrm{c}}_{2,3} \;=\; 0$$

$$EI\,\varphi_2 \;+\; \frac{EI}{5}\,\varphi_2 \;-\; \frac{EI}{5} \;=\; 0 \qquad \Rightarrow \qquad \frac{6}{5}\,\varphi_2 \;=\; \frac{1}{5}$$

. . . i njezino rješenje:

$$\varphi_2 \;=\; \frac{1}{6}$$

vrijednosti momenata na krajevima štapova:

$$M_{1,2} \;=\; \frac{EI}{2}\cdot\frac{1}{6} \;=\; \frac{EI}{12}$$

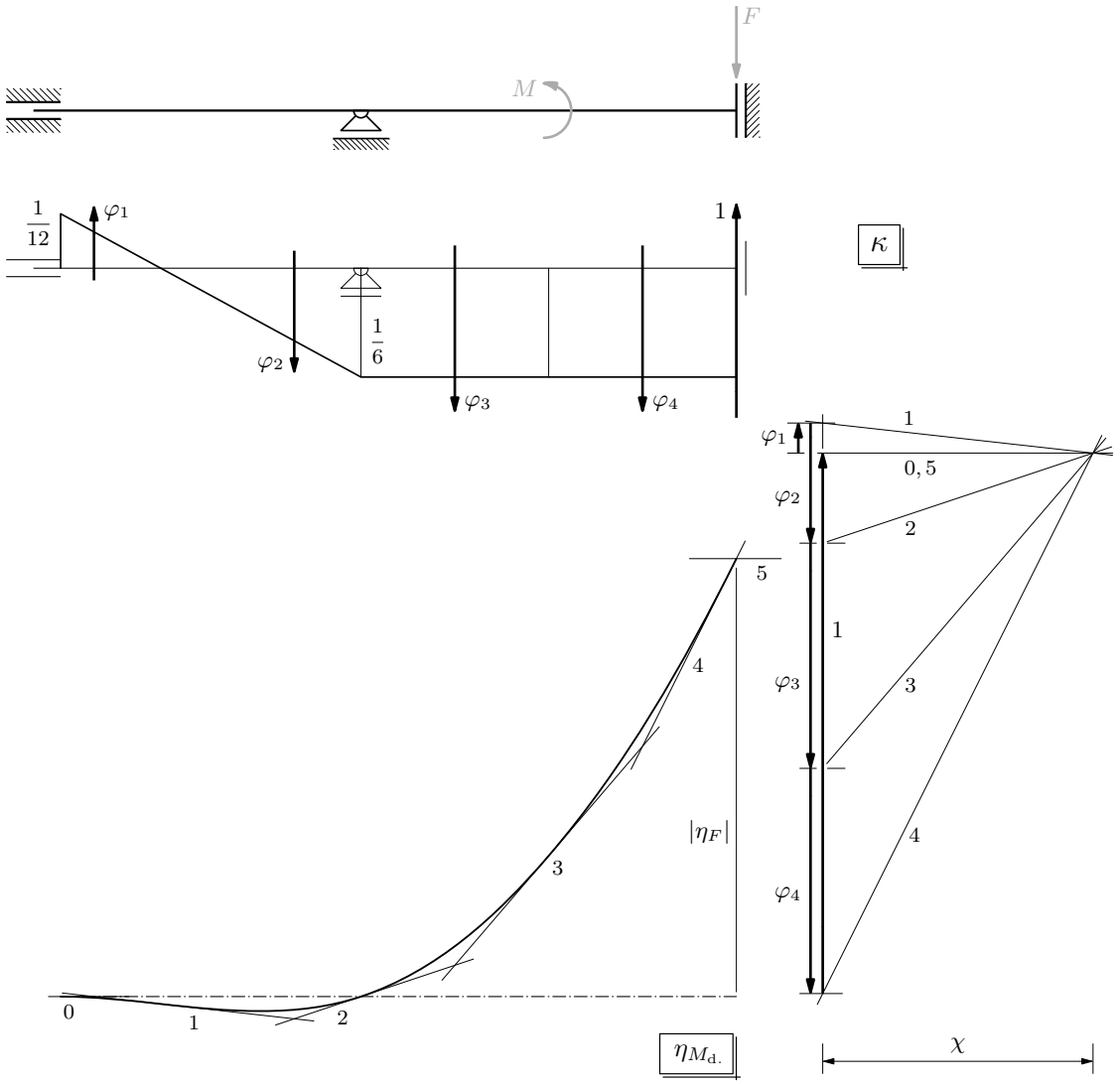
$$M_{2,1} \;=\; EI\cdot\frac{1}{6} \;=\; \frac{EI}{6}$$

$$M_{2,3} \;=\; M^{\mathrm{c}}_{2,3} \;=\; \frac{EI}{5}\cdot\frac{1}{6} \;-\; \frac{EI}{5} \;=\; -\frac{EI}{6}$$

$$M_{3,2} \;=\; M^{\mathrm{c}}_{2,3} \;=\; -\frac{EI}{5}\cdot\frac{1}{6} \;+\; \frac{EI}{5} \;=\; \frac{EI}{6}$$

utjecajna linija:

mjerilo duljina: $\quad 1\,\mathrm{cm} \;::\; 1\,\mathrm{m}$



$$\varphi_1 \;=\; \frac{1}{2}\cdot\frac{1}{12}\cdot\left(\frac{1}{3}\cdot 4\right) \;=\; \frac{1}{18}\,, \qquad\qquad \varphi_2 \;=\; \frac{1}{2}\cdot\frac{1}{6}\cdot\left(\frac{2}{3}\cdot 4\right) \;=\; \frac{2}{9}$$

$$\varphi_3 \;=\; \varphi_4 \;=\; \frac{1}{6}\cdot\left(\frac{1}{2}\cdot 5\right) \;=\; \frac{5}{12}$$

provjera: $\quad -\dfrac{1}{18} \;+\; \dfrac{2}{9} \;+\; \dfrac{5}{12} \;+\; \dfrac{5}{12} \;-\; 1 \;=\; 0$ $\qquad$ [zašto?]

mjerilo kutova: $\quad 1\text{ cm} \;::\; \dfrac{5}{36}$

$$\widetilde{\varphi}_1 \;=\; \frac{2}{5} \;=\; 0{,}4\,\text{cm}, \qquad\qquad \widetilde{\varphi}_2 \;=\; \frac{8}{5} \;=\; 1{,}6\,\text{cm},$$

$$\widetilde{\varphi}_3 \;=\; \widetilde{\varphi}_4 \;=\; 3\,\text{cm}, \qquad\qquad \tilde{1} \;=\; \frac{36}{5} \;=\; 7{,}2\,\text{cm},$$

$$\chi \;=\; \frac{1}{2} \qquad \Rightarrow \qquad \widetilde{\chi} \;=\; 3{,}6\,\text{cm}$$

vrijednost momenta u desnome lažaju:

očitano: $|\widetilde{\eta}_F| \,=\, 58 \text{ i } {}^1\!/_3 \text{ mm} \,\simeq\, 5{,}83 \text{ cm}$

$$\chi \,=\, \frac{1}{2} \qquad \Rightarrow \qquad n \,=\, 2$$

$$|\eta_F| \,=\, \frac{m}{n}\,\tilde{\eta}_F \,=\, \frac{1}{2}\cdot 5{,}83 \,=\, 2{,}915, \qquad\qquad \eta_F \,=\, -2{,}915$$

nagib tangente na $\eta_{M_{\mathrm{d.}}}$ u hvatištu momenta:

$$\operatorname{tg}\alpha_M \,=\, \varphi_1 \,-\, \varphi_2 \,-\, \varphi_3 \,=\, \frac{1}{18} \,-\, \frac{2}{9} \,-\, \frac{5}{12} \,=\, -\frac{7}{12} \,=\, -0{,}583$$

$$M_{\mathrm{d.}} \,=\, F\,\eta_F \,+\, M\,(-\operatorname{tg}\alpha_M) \,=\, 125\cdot(-2{,}915) \,+\, 75\cdot\big(-(-0{,}583)\big) \,=\, -320{,}65\,\mathrm{kNm}$$

(smisao vrtnje je suprotan od pretpostavljenoga)