GS 2. — 6. veljače 2024.

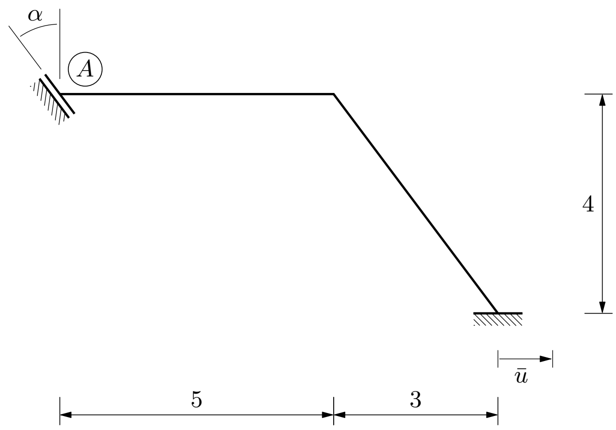
Zadatak 1.

Inženjerskom metodom pomakā nacrtajte dijagrame unutarnjih sila! Izračunajte duljinu pomaka točke A!

$\operatorname{tg}\alpha = {}^3/_4$

$EI = 162\,000\ \text{kNm}^2$

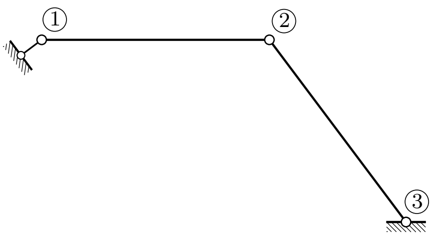
$\bar{u} = 2\ \text{mm}$



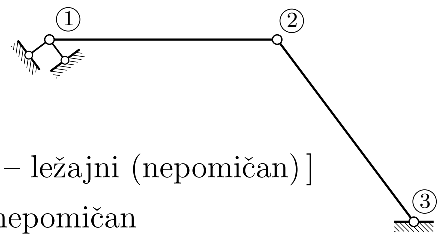
Lipe pred crkvom istom sam nagađao u ogromnim konturama crnih neodređenih masa što su se u tamnoj tamnini još tamnije od ostaloga valovlja tmine za sebe kočile i talasale.

K. S. Gjalski: *Notturno*

zglobna shema i kinematička analiza zglobne sheme (dodavanje spoj(ev)a s podlogom):



$s_{\min} = 2 \cdot 2 - 3 = 1$

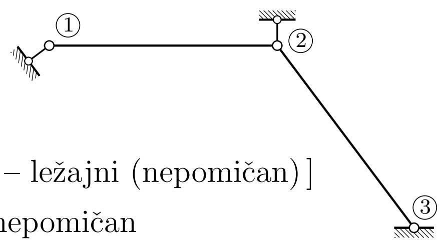


[čvor 3 – ležajni (nepomičan)]

čvor 1 nepomičan

→ čvor 2 nepomičan

ili



[čvor 3 – ležajni (nepomičan)]

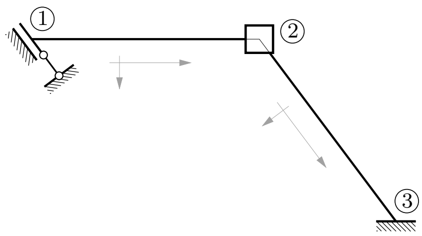
čvor 2 nepomičan

→ čvor 1 nepomičan

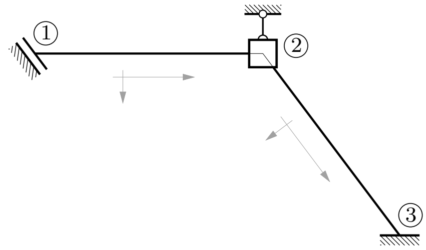
(spoj s podlogom u čvoru 2 može biti na bilo kojem pravcu osim osi štapa $\{2, 3\}$)

$\Rightarrow \quad s = s_{\min} = 1$

osnovni sistem za inženjersku metodu pomakā:

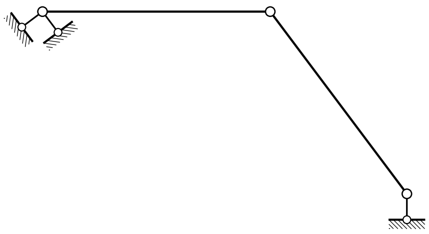


ili

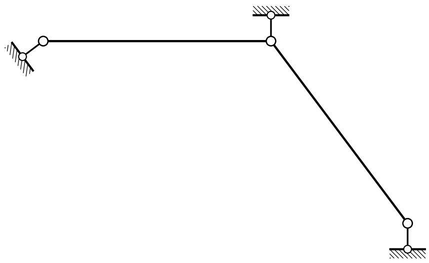


nepoznanice: φ_2 i δ_1 (ili u_1 ili w_1) nepoznanice: φ_2 i w_2 (ili u_2 ili . . .)

mehanizam za utjecaj prisilnoga pomaka ležaja:

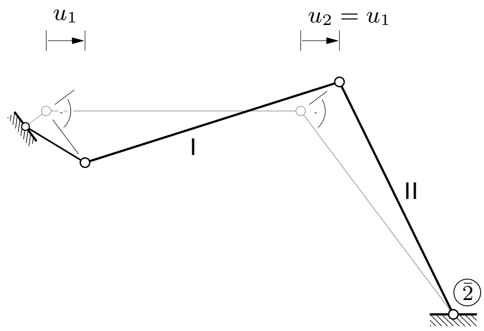


ili

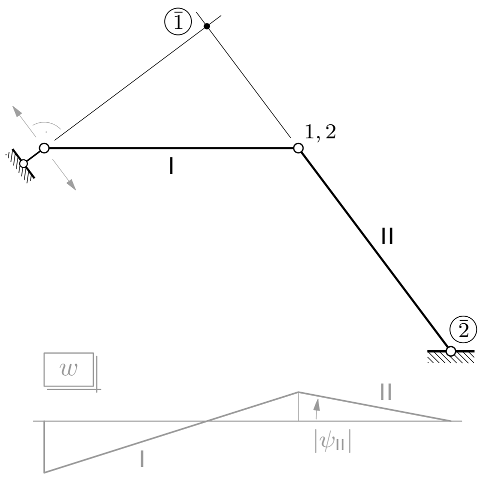


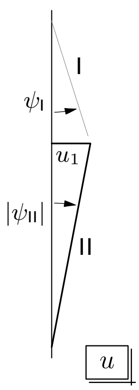
rješavanje pomoću lijevoga osnovnog sistema:

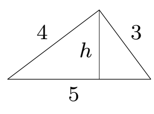
zôra radi, plan pomakā za neovisni translacijski pomak $\vec{u}_1$ (horizontalnu komponentu pomaka ležaja po kosom pravcu):



izrazi za kutove $\psi_{i,j}$ kao funkcije neovisnoga translacijskog pomaka u_1 (izvedeni pomoću dijagramā projekcijā pomakā na vertikalnu os):







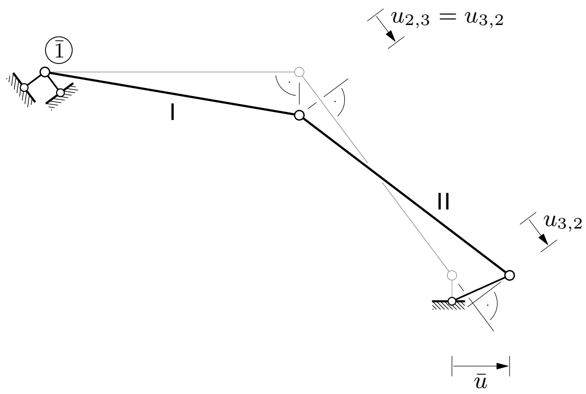
$$\frac{h}{3} = \frac{4}{5}$$

$$h = \frac{12}{5}$$

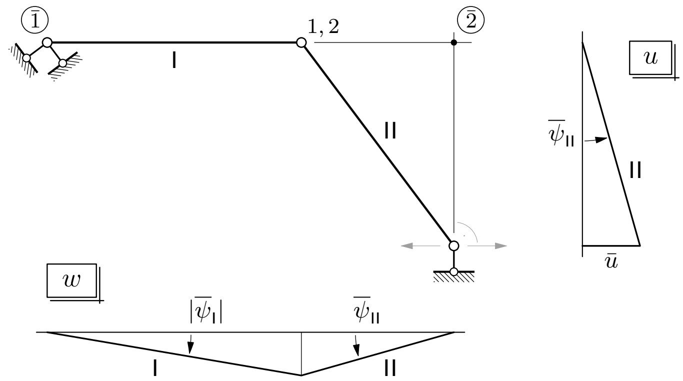
$$\psi_{\mathsf{I}} = \frac{u_1}{h} = \frac{5}{12}\,u_1 = \psi_{\{1,2\}}$$

$$\psi_{\mathsf{II}} = -\frac{u_1}{4} = \psi_{\{2,3\}}$$

(potpunosti radi, skiciran je i dijagram projekcija pomakā na vertikalnu os) zôra radi, plan pomakā za prisilni pomak ležaja:



izračunavanje kutova $\overline{\psi}_{i,j}$ pomoću dijagramā projekcijā pomakā na horizontalnu i na vertikalnu os:



$$\overline{\psi}_{\mathsf{II}} \;=\; \frac{\bar{u}}{4} \;=\; \overline{\psi}_{\{2,3\}}$$

$$\overline{\psi}_{\mathsf{II}} \cdot 3 \;=\; |\overline{\psi}_{\mathsf{I}}| \cdot 5$$

$$\overline{\psi}_{\mathsf{I}} \;=\; -\frac{3}{5}\,\overline{\psi}_{\mathsf{II}}$$

$$=\; -\frac{3}{20}\,\bar{u} \;=\; \overline{\psi}_{\{1,2\}}$$

vrijednosti momenata upetosti:

$$\overline{M}_{1,2} \;=\; \overline{M}_{2,1} \;=\; -6\,k_{\{1,2\}}\,\overline{\psi}_{\{1,2\}} \;=\; -6\,\frac{EI}{5}\left(-\frac{3}{20}\,\bar{u}\right)$$

$$=\; 6 \cdot \frac{162\,000}{5} \cdot 0{,}15 \cdot 0{,}002 \;=\; 58{,}32\,\mathrm{kNm}$$

$$\overline{M}_{2,3} \;=\; \overline{M}_{3,2} \;=\; -6\,k_{\{2,3\}}\,\overline{\psi}_{\{2,3\}} \;=\; -6\,\frac{EI}{5}\,\frac{\bar{u}}{4} \;=\; -6 \cdot \frac{162\,000}{5} \cdot 0{,}25 \cdot 0{,}002 \;=\; -97{,}2\,\mathrm{kNm}$$

(izrazi za) vrijednosti momenata savijanja na krajevima štapova:

$$M_{1,2} \;=\; 2\,k_{\{1,2\}}\,\varphi_2 \;-\; 6\,k_{\{1,2\}}\,\psi_{\{1,2\}} \;+\; \overline{M}_{1,2}$$

$$=\; 2 \cdot 32\,400\,\varphi_2 \;-\; 6 \cdot 32\,400 \cdot \frac{5}{12}\,u_1 \;+\; 58{,}32 \;=\; 64\,800\,\varphi_2 \;-\; 81\,000\,u_1 \;+\; 58{,}32$$

$$M_{2,1} \;=\; 4\,k_{\{1,2\}}\,\varphi_2 \;-\; 6\,k_{\{1,2\}}\,\psi_{\{1,2\}} \;+\; \overline{M}_{2,1}$$

$$=\; 4 \cdot 32\,400\,\varphi_2 \;-\; 6 \cdot 32\,400 \cdot \frac{5}{12}\,u_1 \;+\; 58{,}32 \;=\; 129\,600\,\varphi_2 \;-\; 81\,000\,u_1 \;+\; 58{,}32$$

$$M_{2,3} = 4\,k_{\{2,3\}}\,\varphi_2 - 6\,k_{\{2,3\}}\,\psi_{\{2,3\}} + \overline{M}_{2,3}$$

$$= 4 \cdot 32\,400\,\varphi_2 - 6 \cdot 32\,400 \left(-\frac{u_1}{4}\right) - 97{,}2 = 129\,600\,\varphi_2 + 48\,600\,u_1 - 97{,}2$$

$$M_{3,2} = 2\,k_{\{2,3\}}\,\varphi_2 - 6\,k_{\{2,3\}}\,\psi_{\{2,3\}} + \overline{M}_{2,3}$$

$$= 2 \cdot 32\,400\,\varphi_2 - 6 \cdot 32\,400 \left(-\frac{u_1}{4}\right) - 97{,}2 = 64\,800\,\varphi_2 + 48\,600\,u_1 - 97{,}2$$

jednadžba ravnoteže momenata u čvoru 2:

$$-M_{2,1} + (-M_{2,3}) = 0 \quad \Big| \quad \times (-1)$$

$$\big[129\,600\,\varphi_2 - 81\,000\,u_1 + 58{,}32\big] + \big[129\,600\,\varphi_2 + 48\,600\,u_1 - 97{,}2\big] = 0$$

$$259\,200\,\varphi_2 - 32\,400\,u_1 - 38{,}88 = 0$$

jednadžba rada na virtualnim pomacima:

(„vodeći” je virtualni pomak pomak $\delta\vec{u}_1$ koji odgovara neovisnom translacijskom pomaku $\vec{u}_1$, pa je dijagram projekcija (virtualnih) pomaka na horizontalnu os jednak dijagramu prikazanom na najdonjoj slici na stranici 2, uz zamjene $u_1 \to \delta u_1$ i $\psi_i \to \delta\psi_i$, a veza je kutova $\delta\psi_{\{i,j\}}$ i orijentirane duljine δu_1 ista kao veza $\psi_{\{i,j\}}$ i u_1:

$$\delta\psi_{\{1,2\}} = {}^{5\,\delta u_1}/_{12} \qquad \& \qquad \delta\psi_{\{2,3\}} = {}^{-\delta u_1}/_{4}\ )$$

$$\big(M_{1,2} + M_{2,1}\big)\,\delta\psi_{\{1,2\}} + \big(M_{2,3} + M_{3,2}\big)\,\delta\psi_{\{2,3\}} = 0$$

$$\big(64\,800\,\varphi_2 - 81\,000\,u_1 + 58{,}32 + 129\,600\,\varphi_2 - 81\,000\,u_1 + 58{,}32\big) \cdot \left(\frac{5}{12}\,\delta u_1\right)$$
$$+ \big(129\,600\,\varphi_2 + 48\,600\,u_1 - 97{,}2 + 64\,800\,\varphi_2 + 48\,600\,u_1 - 97{,}2\big) \cdot \left(-\frac{\delta u_1}{4}\right) = 0$$

$$\big(32\,400\,\varphi_2 - 91\,800\,u_1 + 72{,}9\big)\,\delta u_1 = 0 \quad \forall\,\delta u_1$$

$$32\,400\,\varphi_2 - 91\,800\,u_1 + 72{,}9 = 0 \quad \Big| \quad \times (-1)$$

$$-32\,400\,\varphi_2 + 91\,800\,u_1 - 72{,}9 = 0$$

sustav jednadžbi i njegovo rješenje:

$$259\,200\,\varphi_2 - 32\,400\,u_1 = 38{,}88$$

$$-32\,400\,\varphi_2 + 91\,800\,u_1 = 97{,}2 \qquad \text{(sustav je simetričan)}$$

$$\varphi_2 = 0{,}000\,295\,385 \qquad \& \qquad u_1 = 0{,}001\,163\,08\ \mathrm{m}$$

vrijednosti momenata savijanja na krajevima štapova:

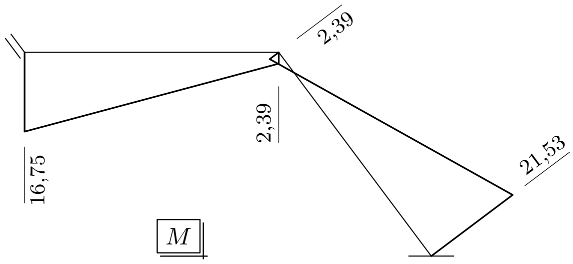
$$M_{1,2} = 64\,800 \cdot 0{,}000\,295\,385 - 81\,000 \cdot 0{,}001\,163\,08 + 58{,}32 = -16{,}748\,5\,\text{kNm}$$

$$M_{2,1} = 129\,600 \cdot 0{,}000\,295\,385 - 81\,000 \cdot 0{,}001\,163\,08 + 58{,}32 = 2{,}392\,42\,\text{kNm}$$

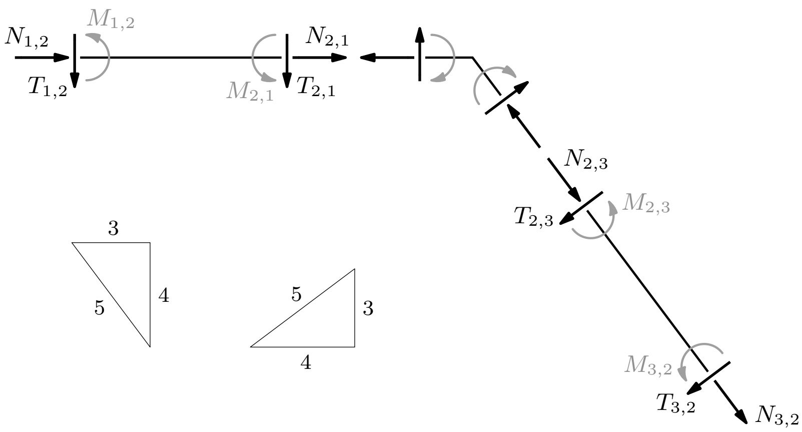
$$M_{2,3} = 129\,600 \cdot 0{,}000\,295\,385 + 48\,600 \cdot 0{,}001\,163\,08 - 97{,}2 = -2{,}392\,42\,\text{kNm}$$

$$M_{3,2} = 64\,800 \cdot 0{,}000\,295\,385 + 48\,600 \cdot 0{,}001\,163\,08 - 97{,}2 = -21{,}533\,4\,\text{kNm}$$

dijagram momenata savijanja:



vrijednosti poprečnih sila:



$$T_{1,2} \cdot 5 + M_{1,2} + M_{2,1} = 0 \qquad \text{(ravnoteža momenata u odnosu na kraj 2 štapa } \{1,2\})$$

$$T_{1,2} = -\frac{M_{1,2} + M_{2,1}}{5} = -\frac{-16{,}748\,5 + 2{,}392\,42}{5} = 2{,}871\,22\,\text{kN} = -T_{(\xi_{(1,2)}=0^+)}$$

$$T_{1,2} + T_{2,1} = 0 \qquad \text{(ravnoteža sila okomitih na os štapa } \{1,2\})$$

$$T_{2,1} = -T_{1,2} = -2{,}871\,22\,\text{kN} = T_{(\xi_{(1,2)}=5^-)}$$

$$T_{2,3} \cdot 5 + M_{2,3} + M_{3,2} = 0 \qquad \text{(ravnoteža momenata u odnosu na kraj 3 štapa } \{2,3\})$$

$$T_{2,3} = -\frac{M_{2,3} + M_{3,2}}{5} = -\frac{-2{,}392\,42 - 21{,}533\,4}{5} = 4{,}785\,16\,\text{kN} = -T_{(\xi_{(2,3)}=0^+)}$$

$$T_{2,3} + T_{3,2} = 0 \qquad \text{(ravnoteža sila okomitih na os štapa } \{2,3\})$$

$$T_{3,2} = -T_{2,3} = -4{,}785\,16\,\text{kN} = T_{(\xi_{(2,3)}=5^-)}$$

vrijednosti uzdužnih sila:

$$-T_{2,1} - \frac{3}{5}\,T_{2,3} - \frac{4}{5}\,N_{2,3} \;=\; 0 \qquad \text{(ravnoteža projekcija sila u čvoru 2 na os } z\text{)}$$

$$N_{2,3} \;=\; \frac{5}{4}\left(-T_{2,1} - \frac{3}{5}\,T_{2,3}\right) \;=\; -0{,}000\,147 \;\simeq\; 0 \;=\; N_{(\xi_{(2,3)}=0^+)}$$

$$N_{2,3} + N_{3,2} \;=\; 0 \qquad \text{(ravnoteža sila usporednih s osi štapa } \{2,3\}\text{)}$$

$$N_{3,2} \;=\; -N_{2,3} \;\simeq\; 0 \;=\; N_{(\xi_{(2,3)}=5^-)}$$

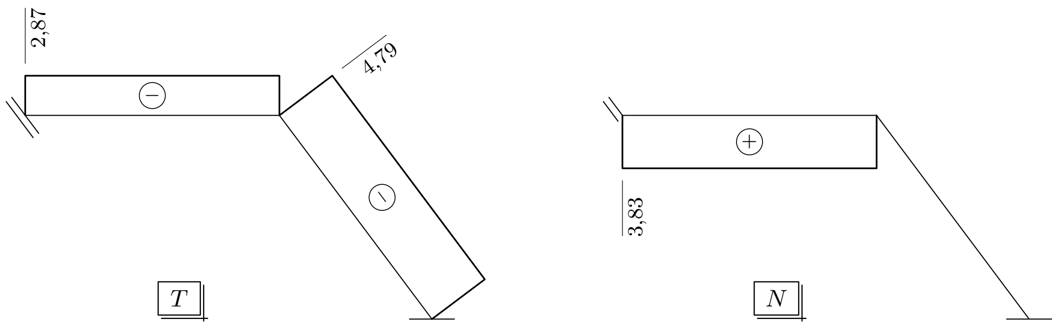
$$-N_{2,1} + \frac{4}{5}\,T_{2,3} - \frac{3}{5}\,N_{2,3} \;=\; 0 \qquad \text{(ravnoteža projekcija sila u čvoru 2 na os } x\text{)}$$

$$N_{2,1} \;=\; \frac{4}{5}\,T_{2,3} \;=\; 3{,}828\,131\ \text{kN} \;=\; N_{(\xi_{(2,1)}=5^-)}$$

$$N_{1,2} + N_{2,1} \;=\; 0 \qquad \text{(ravnoteža sila usporednih s osi štapa } \{1,2\}\text{)}$$

$$N_{1,2} \;=\; -N_{2,1} \;=\; -3{,}828\,131\ \text{kN} \;=\; -N_{(\xi_{(1,2)}=0^+)}$$

dijagrami poprečnih i uzdužnih sila:



duljina pomaka točke A:

geometrijski postupak:

nepoznanica u_1 (koja to više nije, jer smo je izračunali) duljina je horizontalne komponente pomaka točke A

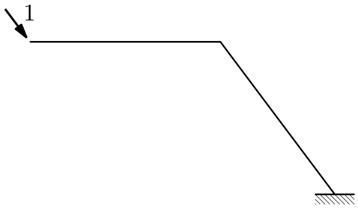
$$\frac{\delta_A}{u_1} \;=\; \frac{5}{4}$$

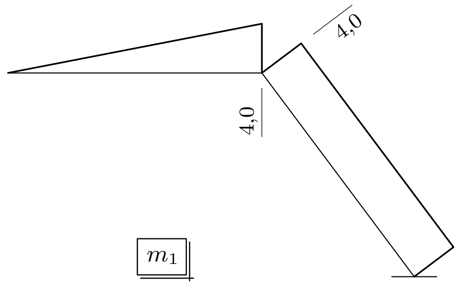
$$\delta_A \;=\; \frac{5}{4}\,u_1 \;=\; \frac{5}{4}\cdot 0{,}001\,163\,08$$

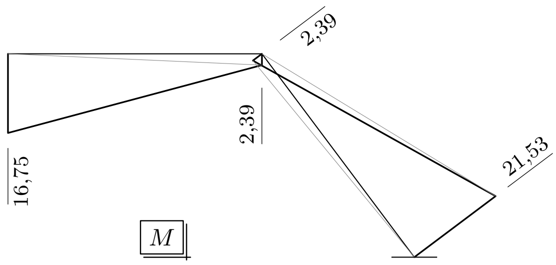
$$=\; 0{,}001\,938\,47\ \text{m}$$

$$=\; 1{,}94\ \text{mm}$$

metoda jedinične sile & redukcijski stavak:







redukcijski stavak: jedinična sila na bilo kojem osnovnom sistemu za metodu sila

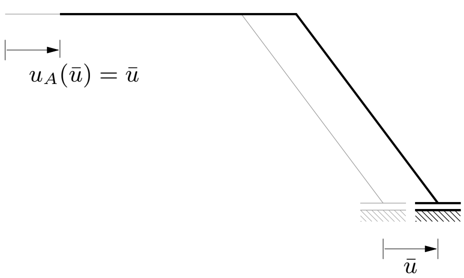
[domaća zabava: objasnite crtanje dijagrama m_1!]

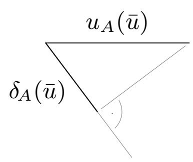
pomak neke točke zbroj je pomaka od utjecaja momenata savijanja i pomaka od utjecaja prisilnoga pomaka na osnovni sistem za metodu sila (ako prisilni pomak utječe na njega)

utjecaj momenata savijanja:

$$\begin{aligned}\delta_A(M) &= \sum \int \frac{m_1\,M}{EI}\,\mathrm{d}s \\ &= \frac{1}{EI}\left[\left(\frac{1}{2}\cdot 16{,}75\cdot 5\right)\left(\frac{1}{3}\cdot 4\right)(-1) + \left(\frac{1}{2}\cdot 2{,}39\cdot 5\right)\left(\frac{2}{3}\cdot 4\right)(-1)\right. \\ &\qquad \left. + \left(\frac{1}{2}\cdot 2{,}39\cdot 5\right)(4)\,(-1) + \left(\frac{1}{2}\cdot 21{,}53\cdot 5\right)(4)\right] = 0{,}000\,738\,477\end{aligned}$$

utjecaj prisilnoga pomaka na osnovni sistem za metodu sila:



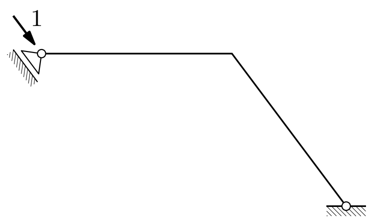


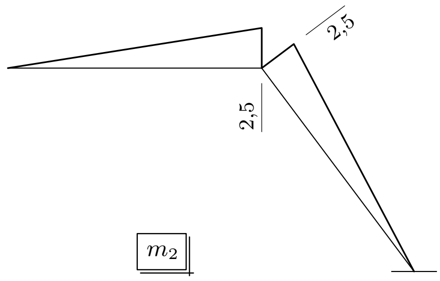
$$\frac{\delta_A(\bar{u})}{u_A(\bar{u})} = \frac{3}{5}$$

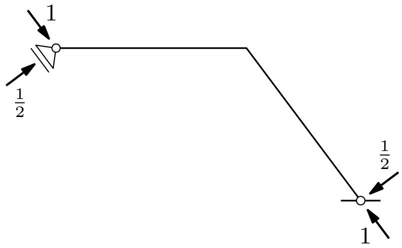
$$\delta_A(\bar{u}) = \frac{3}{5}\,u_A(\bar{u}) = \frac{3}{5}\,\bar{u} = 0{,}001\,2$$

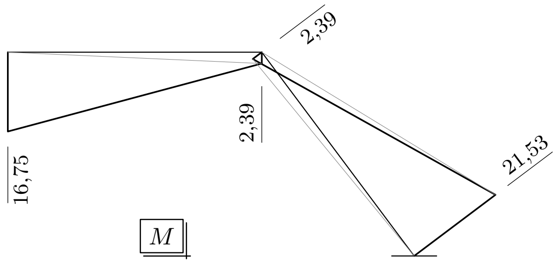
$$\delta_A = \delta_A(M) + \delta_A(\bar{u}) = 0{,}000\,738\,477 + 0{,}001\,2 = 0{,}001\,938\,48\ \mathrm{m} = 1{,}94\ \mathrm{mm}$$

ili:





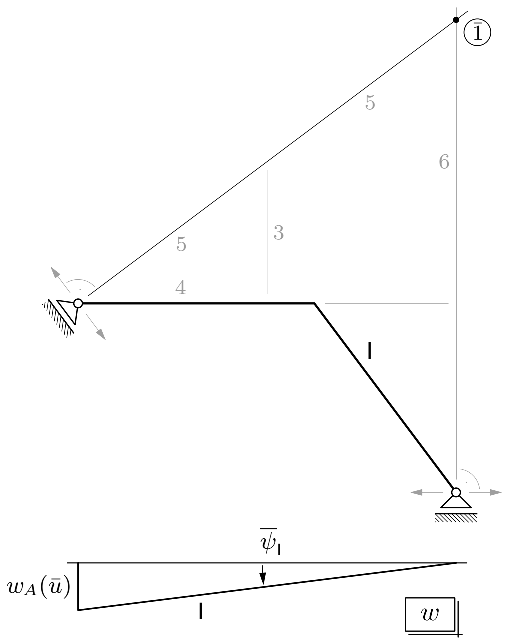


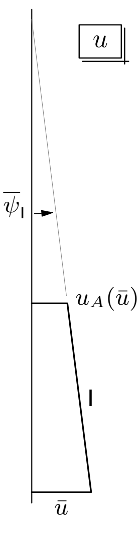


$$\delta_A(M) \;=\; \frac{1}{EI}\Bigg[\left(\frac{1}{2}\cdot 16{,}75\cdot 5\right)\left(\frac{1}{3}\cdot 2{,}5\right)(-1) \;+\; \left(\frac{1}{2}\cdot 2{,}39\cdot 5\right)\left(\frac{2}{3}\cdot 2{,}5\right)(-1)$$

$$+\; \left(\frac{1}{2}\cdot 2{,}39\cdot 5\right)\left(\frac{2}{3}\cdot 2{,}5\right)(-1) \;+\; \left(\frac{1}{2}\cdot 21{,}53\cdot 5\right)\left(\frac{1}{3}\cdot 2{,}5\right)\Bigg]$$

$$= \; -0{,}000\,061\,471\,2$$





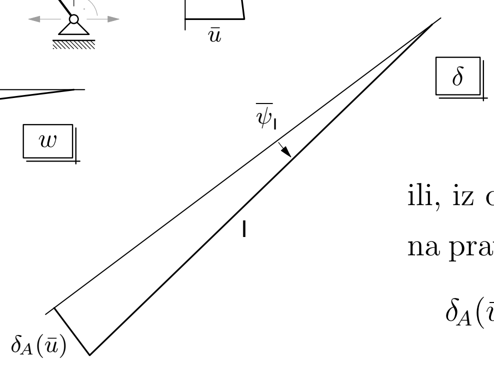
$$\overline{\psi}_{\mathsf{I}} \;=\; \frac{\bar{u}}{10}$$

$$u_A(\bar{u}) \;=\; \overline{\psi}_{\mathsf{I}}\cdot 6 \;=\; 0{,}001\,2$$

$$w_A(\bar{u}) \;=\; \overline{\psi}_{\mathsf{I}}\cdot 8 \;=\; 0{,}001\,6$$

$$\delta_A(\bar{u}) \;=\; \sqrt{u_A^2(\bar{u}) \;+\; w_A^2(\bar{u})}$$

$$= \; 0{,}002$$

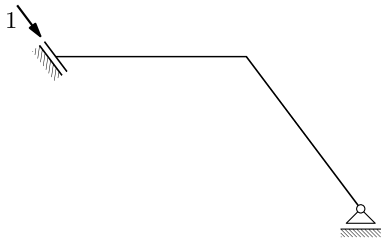


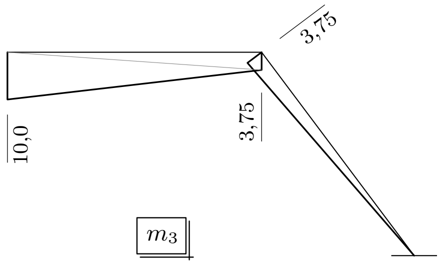
ili, iz dijagrama projekcija na pravac pomaka točke A:

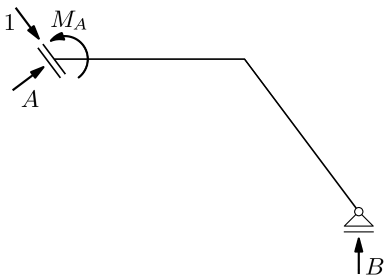
$$\delta_A(\bar{u}) \;=\; \overline{\psi}_{\mathsf{I}}\cdot 10 \;=\; 0{,}002$$

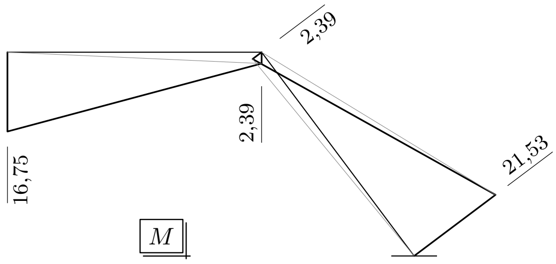
$$\delta_A \;=\; \delta_A(M) \;+\; \delta_A(\bar{u}) \;=\; -0{,}000\,061\,471\,2 \;+\; 0{,}002 \;=\; 0{,}001\,938\,53\,\mathrm{m} \;=\; 1{,}94\,\mathrm{mm}$$

ili, osnovni sistem za metodu sila na koji prisilni pomak ne utječe:









rješavanje osnovnoga sistema (za metodu sila) na koji djeluje jedinična sila:

$$\sum F_x \;=\; 0 \;: \qquad 1^h \;+\; A^h \;=\; 0$$

$$A^h \;=\; -1^h \;=\; -\frac{3}{5}$$

$$\frac{A}{A^h} \;=\; \frac{5}{4} \quad\Rightarrow\quad A \;=\; \frac{5}{4}\,A^h \;=\; -\frac{3}{4} \;=\; 0{,}75$$

$$\sum F_z \;=\; 0 \;: \qquad 1^v \;-\; A^v \;-\; B \;=\; 0$$

$$\frac{A^v}{A^h} \;=\; \frac{3}{4} \quad\Rightarrow\quad A^v \;=\; \frac{3}{4}\,A^h \;=\; \frac{3}{4}\left(-\frac{3}{5}\right) \;=\; -\frac{9}{20}$$

$$B \;=\; 1^v \;-\; A^v \;=\; \frac{4}{5} - \left(-\frac{9}{20}\right) \;=\; \frac{5}{4} \;=\; 1{,}25$$

$$\sum M_{/A} \;=\; 0 \;: \qquad M_A \;+\; B\cdot 8 \;=\; 0$$

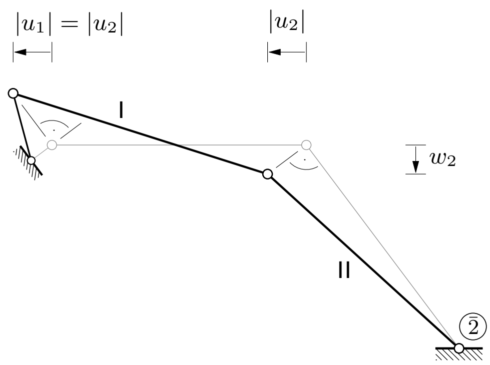
$$M_A \;=\; -B\cdot 8 \;=\; -\frac{5}{4}\cdot 8 \;=\; -10$$

vrijednost momenta savijanja u spoju štapova: $M_{(x=5)} \;=\; B\cdot 3 \;=\; \frac{15}{4} \;=\; 3{,}75$

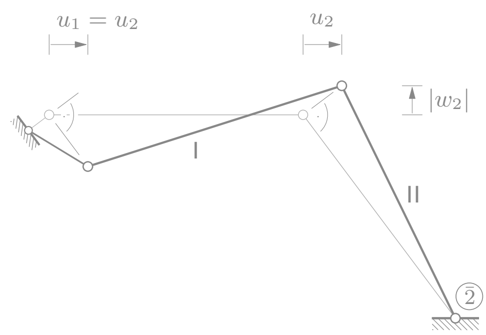
$$\begin{aligned}\delta_A \;=\; \delta_A(M) \;=\; \frac{1}{EI}\Bigg[&\left(\frac{1}{2}\cdot 16{,}75\cdot 5\right)\left(\frac{2}{3}\cdot 10{,}0+\frac{1}{3}\cdot 3{,}75\right)\\ &+\left(\frac{1}{2}\cdot 2{,}39\cdot 5\right)\left(\frac{1}{3}\cdot 10{,}0+\frac{2}{3}\cdot 3{,}75\right)\\ &+\left(\frac{1}{2}\cdot 2{,}39\cdot 5\right)\left(\frac{2}{3}\cdot 3{,}75\right)+\left(\frac{1}{2}\cdot 21{,}53\cdot 5\right)\left(\frac{1}{3}\cdot 3{,}75\right)(-1)\Bigg]\\ =\;& 0{,}001\,938\,40\,\mathrm{m} \;=\; 1{,}94\,\mathrm{mm}\end{aligned}$$

rješavanje pomoću desnoga osnovnog sistema za inženjersku metodu pomakā (najgornja slika na stranici 2):

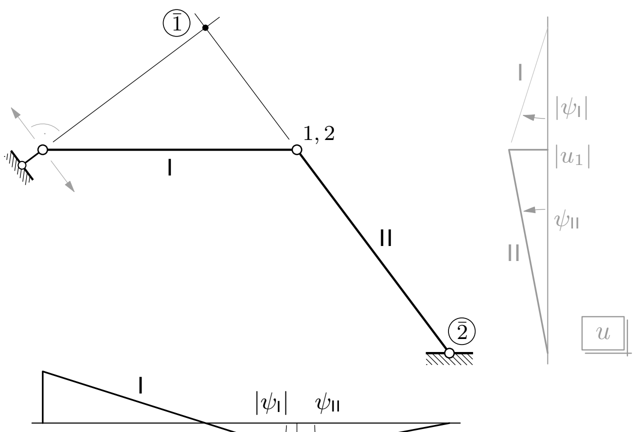
zôra radi, plan pomakā za neovisni translacijski pomak $\vec{w}_2$:

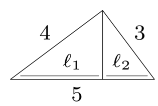


uzeli smo, kao što je uobičajeno, da je pomak $\vec{w}_2$ prema dolje; uzmemo li da je taj pomak prema gore i još da je (za crtež) $w_2 = {}^{3\,u_2}/_4 = {}^{3\,u_1}/_4$, plan pomakā (kao ni dijagrami projekcija pomakā) neće se razlikovati od plana pomakā (i dijagrama projekcija pomakā) za pomak $\vec{u}_1$ prikazanoga na stranici 2:



izrazi za kutove $\psi_{i,j}$ kao funkcije neovisnoga translacijskog pomaka w_2 (izvedeni pomoću dijagramā projekcijā pomakā na vertikalnu os):





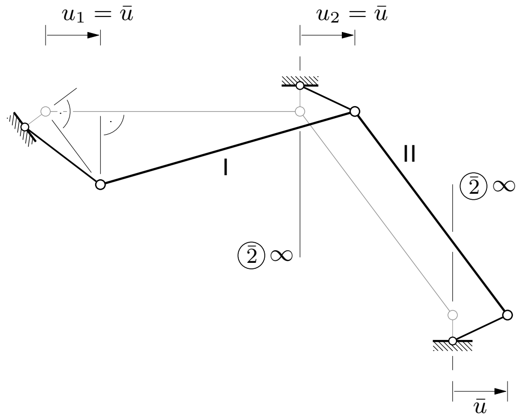
$$\frac{\ell_2}{3} = \frac{3}{5}$$

$$\ell_2 = \frac{9}{5}$$

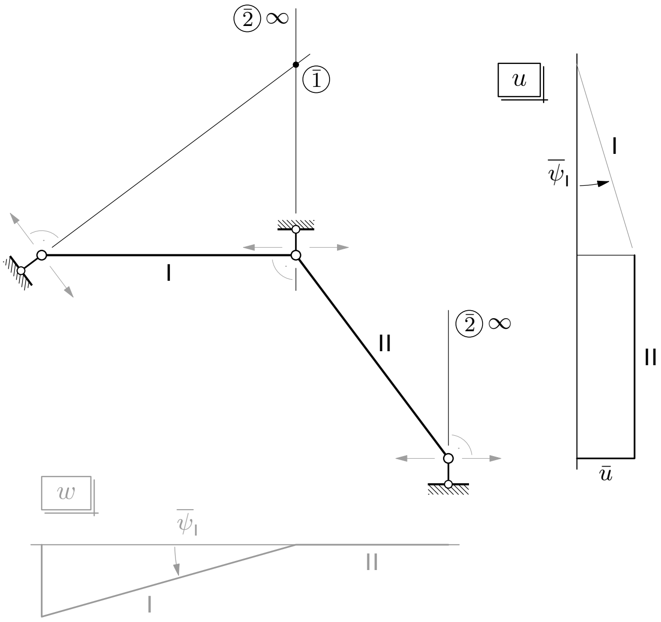
$$\psi_I = -\frac{w_2}{\ell_2} = -\frac{5}{9}\,w_2 = \psi_{\{1,2\}}$$

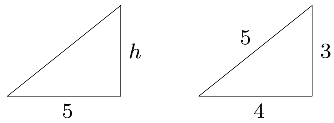
$$\psi_{II} = \frac{w_2}{3} = \psi_{\{2,3\}}$$

(potpunosti radi, skiciran je i dijagram projekcija pomakā na horizontalnu os) zôra radi, plan pomakā za prisilni pomak ležaja:



izračunavanje kutova $\overline{\psi}_{i,j}$ pomoću dijagramā projekcijā pomakā na horizontalnu os:





$$\frac{h}{5} = \frac{3}{4} \quad \Rightarrow \quad h = \frac{15}{4}$$

$$\overline{\psi}_{\mathsf{I}} = \frac{\bar{u}}{h} = \frac{4}{15}\,\bar{u} = \overline{\psi}_{\{1,2\}}$$

$$\overline{\psi}_{\mathsf{II}} = 0 = \overline{\psi}_{\{2,3\}}$$

(potpunosti radi, skiciran je i dijagram projekcija pomakā na vertikalnu os)

vrijednosti momenata upetosti:

$$\overline{M}_{1,2} = \overline{M}_{2,1} = -6\,k_{\{1,2\}}\,\overline{\psi}_{\{1,2\}} = -6\;\frac{EI}{5}\;\frac{4}{15}\,\bar{u}$$

$$= -6 \cdot \frac{162\,000}{5} \cdot \frac{4}{15} \cdot 0{,}002 = -103{,}68\,\mathrm{kNm}$$

(izrazi za) vrijednosti momenata savijanja na krajevima štapova:

$$M_{1,2} = 2\,k_{\{1,2\}}\,\varphi_2 - 6\,k_{\{1,2\}}\,\psi_{\{1,2\}} + \overline{M}_{1,2}$$

$$= 2 \cdot 32\,400\,\varphi_2 - 6 \cdot 32\,400\left(-\frac{5}{9}\,w_2\right) - 103{,}68 = 64\,800\,\varphi_2 + 108\,000\,w_2 - 103{,}68$$

$$
\begin{aligned}
M_{2,1} &= 4\,k_{\{1,2\}}\,\varphi_2 - 6\,k_{\{1,2\}}\,\psi_{\{1,2\}} + \overline{M}_{2,1} \\
&= 4\cdot 32\,400\,\varphi_2 - 6\cdot 32\,400\left(-\frac{5}{9}\,w_2\right) - 103{,}68 = 129\,600\,\varphi_2 + 108\,000\,w_2 - 103{,}68
\end{aligned}
$$

$$
\begin{aligned}
M_{2,3} &= 4\,k_{\{2,3\}}\,\varphi_2 - 6\,k_{\{2,3\}}\,\psi_{\{2,3\}} \\
&= 4\cdot 32\,400\,\varphi_2 - 6\cdot 32\,400\cdot\frac{w_2}{3} = 129\,600\,\varphi_2 - 64\,800\,w_2
\end{aligned}
$$

$$
\begin{aligned}
M_{3,2} &= 2\,k_{\{2,3\}}\,\varphi_2 - 6\,k_{\{2,3\}}\,\psi_{\{2,3\}} \\
&= 2\cdot 32\,400\,\varphi_2 - 6\cdot 32\,400\cdot\frac{w_2}{3} = 64\,800\,\varphi_2 - 64\,800\,w_2
\end{aligned}
$$

jednadžba ravnoteže momenata u čvoru 2:

$$-M_{2,1} + (-M_{2,3}) = 0 \quad \Big|\ \times(-1)$$

$$\big[129\,600\,\varphi_2 + 108\,000\,w_2 - 103{,}68\big] + \big[129\,600\,\varphi_2 - 64\,800\,w_2\big] = 0$$

$$259\,200\,\varphi_2 + 43\,200\,w_2 - 103{,}68 = 0$$

jednadžba rada na virtualnim pomacima:

(„vodeći" je virtualni pomak pomak $\delta\vec{w}_2$ koji odgovara neovisnom translacijskom pomaku $\vec{w}_2$, pa je dijagram projekcija (virtualnih) pomaka na vertikalnu os jednak dijagramu prikazanom na najdonjoj slici na stranici 10, uz zamjene $w_2 \to \delta w_2$ i $\psi_{\mathrm{i}} \to \delta\psi_{\mathrm{i}}$, a veza je kutova $\delta\psi_{\{i,j\}}$ i orijentirane duljine δw_2 ista kao veza $\psi_{\{i,j\}}$ i w_2:

$$\delta\psi_{\{1,2\}} = -5\,\delta w_2/9 \qquad \& \qquad \delta\psi_{\{2,3\}} = \delta w_2/3\ )$$

$$\big(M_{1,2} + M_{2,1}\big)\,\delta\psi_{\{1,2\}} + \big(M_{2,3} + M_{3,2}\big)\,\delta\psi_{\{2,3\}} = 0$$

$$
\begin{aligned}
&\big(64\,800\,\varphi_2 + 108\,000\,w_2 - 103{,}68 + 129\,600\,\varphi_2 + 108\,000\,w_2 - 103{,}68\big)\cdot\left(-\frac{5}{9}\,\delta w_2\right) \\
&\qquad + \big(129\,600\,\varphi_2 - 64\,800\,w_2 + 64\,800\,\varphi_2 - 64\,800\,w_2\big)\cdot\frac{\delta w_2}{3} = 0
\end{aligned}
$$

$$\big(-43\,200\,\varphi_2 - 162\,667\,w_2 + 115{,}2\big)\,\delta w_2 = 0 \quad \forall\,\delta w_2$$

$$-43\,200\,\varphi_2 - 162\,667\,w_2 + 115{,}2 = 0 \quad \Big|\ \times(-1)$$

$$43\,200\,\varphi_2 + 162\,667\,w_2 - 115{,}2 = 0$$

sustav jednadžbi i njegovo rješenje:

$$
\begin{aligned}
259\,200\,\varphi_2 + 43\,200\,w_2 &= 103{,}68 \\
43\,200\,\varphi_2 + 162\,667\,w_2 &= 115{,}2 \qquad \text{(sustav je simetričan)}
\end{aligned}
$$

$$\varphi_2 = 0{,}000\,295\,026 \qquad \& \qquad w_2 = 0{,}000\,629\,844\ \mathrm{m}$$

vrijednosti momenata savijanja na krajevima štapova:

$$M_{1,2} = 64\,800 \cdot 0{,}000\,295\,026 + 108\,000 \cdot 0{,}000\,629\,844 - 103{,}68 = -16{,}775\,2\,\text{kNm}$$

$$M_{2,1} = 129\,600 \cdot 0{,}000\,295\,026 + 108\,000 \cdot 0{,}000\,629\,844 - 103{,}68 = 2{,}578\,52\,\text{kNm}$$

$$M_{2,3} = 129\,600 \cdot 0{,}000\,295\,026 - 64\,800 \cdot 0{,}000\,629\,844 = -2{,}578\,52\,\text{kNm}$$

$$M_{3,2} = 64\,800 \cdot 0{,}000\,295\,026 - 64\,800 \cdot 0{,}000\,629\,844 = -21{,}696\,2\,\text{kNm}$$

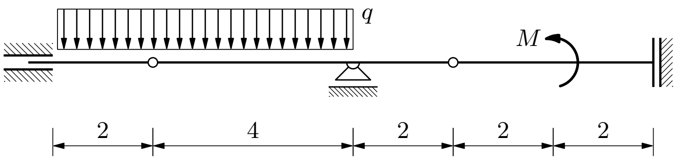
(Nebitne razlike (iza decimalnoga zareza) između ovih vrijednosti i vrijednostī dobivenih rješavanjem pomoću lijevoga osnovnog sistema posljedice su zanemarivanja pogrešaka zaokruživanja (na šest značajnih znamenaka) u pojedinim koracima proračuna i njihova „gomilanja”.)

nastavak je priče (dijagrami *&* duljina pomaka) isti kao za lijevi osnovni sistem
(uz lagana prilagođavanja vrijednostī)

Zadatak 2.

Pomoću utjecajnih linija izračunajte vrijednost vertikalne reakcije u lijevom i vrijednost reaktivnoga momenta u desnom ležaju!

$$q = 62{,}5\,\text{kN/m}$$
$$M = 125\,\text{kNm}$$

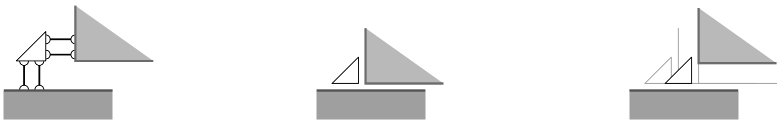


utjecajna linija za vrijednost reakcije u lijevom ležaju (kinematički postupak):

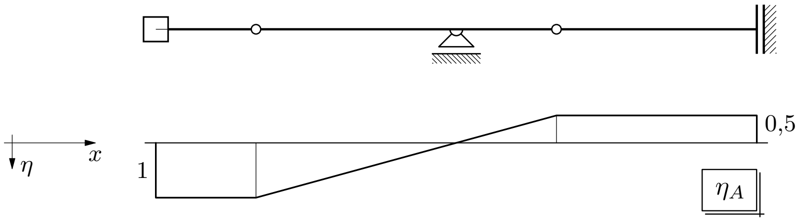
pretpostavljeni smisao djelovanja reakcije: ↑

momentni spoj: spoj koji sprečava zaokret, a dopušta pomak po bilo kojem pravcu

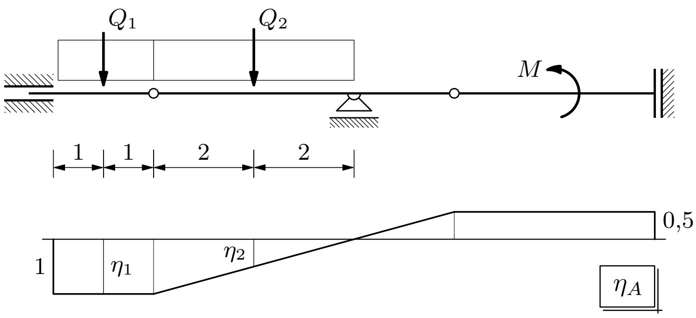
podsjetnik iz *Mehanike 1.*:



mehanizam i utjecajna linija kao plan pomakā za jedinični pomak hvatišta reakcije u smislu suprotnom od smisla djelovanja reakcije:



izračunavanje vrijednosti reakcije:



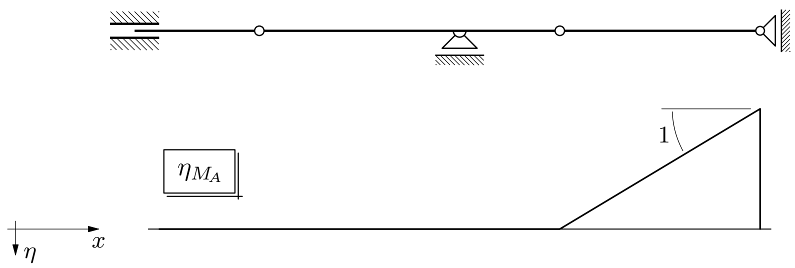
$$A(q, M) = Q_1\,\eta_1 + Q_2\,\eta_2 + M \cdot 0$$
$$= (62{,}5 \cdot 2) \cdot 1 + (62{,}5 \cdot 4) \cdot \frac{1}{2} = 250\,\text{kN}$$

utjecajna linija za vrijednost reaktivnoga momenta u desnom ležaju

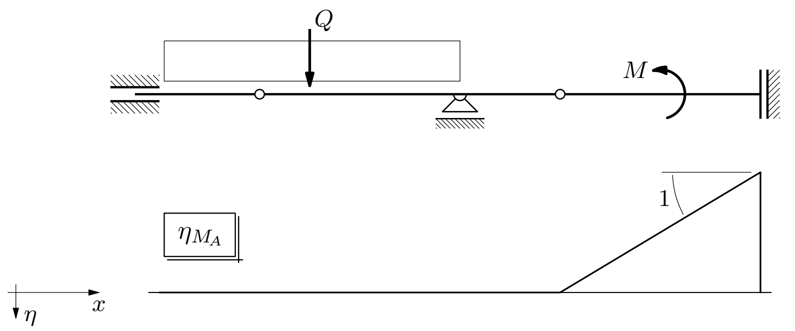
(kinematički postupak):

pretpostavljeni smisao vrtnje momenta: ↻

mehanizam i utjecajna linija kao plan pomakā za jedinični zaokret osi u hvatištu momenta u smislu suprotnom od smisla vrtnje momenta:



izračunavanje vrijednosti reakcije:

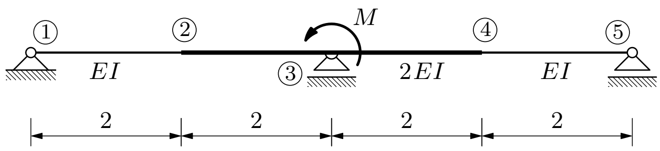


$$M_A(q, M) \;=\; Q \cdot 0 \;+\; M \cdot (-\operatorname{tg}\alpha) \;=\; 125 \cdot \big(-(-1)\big) \;=\; 125$$

Zadatak 3.

Pomoću utjecajne linije izračunajte vrijednost reakcije u srednjem ležaju! (Hvatište vanjskoga momenta je iznad srednjega ležaja.)

$M = 125\,\text{kNm}$



zdravorazumsko rješenje:

- sistem je osnosimetričan u odnosu na vertikalnu os koja prolazi srednjim ležajem;
- pri pomaku srednjega ležaja (točke na osi simetrije) po vertikalnom pravcu (po osi simetrije) progibna će linija biti osnosimetrična u odnosu na istu os;
- tangenta na progibnu liniju u točki iznad srednjega ležaja, koja je na osi simetrije, bit će horizontalna; drugim riječima, u toj je točki kut zaokreta prognute osi grede u odnosu na njezin početni oblik i položaj (odsječak osi x) jednak nuli;
- prema teoremu Müller–Breslaua utjecajna je linija za reakciju u srednjem ležaju (koja djeluje na vertikalnom pravcu) jednaka progibnoj liniji pri jediničnom pomaku toga ležaja po vertikalnom pravcu (u smislu suprotnom od pozitivnoga smisla djelovanja reakcije);
- utjecaj momenta se pri primjeni utjecajnih linija izračunava prema izrazu

  $$\eta(M) = -M \ \text{tg}\,\alpha_M,$$

  gdje je α_M kut između tangente na utjecajnu liniju u hvatištu momenta i apscisne osi;
- kako je taj kut jednak nuli, bit će

  $$B(M) = -M \ \text{tg}\,0 = -M \cdot 0 = 0.$$

- prema tome, vrijednost je reakcije u srednjem ležaju jednaka nuli.

školsko rješenje (uz crtanje utjecajne linije, kao provjera zdravoga razuma)**:**

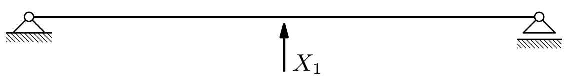
kinematički postupak (primjena teorema Müller–Breslaua):

izbor metode crtanja momentnoga dijagrama:

inženjerska metoda pomakā: nepoznanice: φ_2, φ_3, φ_4, w_2, w_4 $\quad\Rightarrow\quad$ 5 nepoznanica

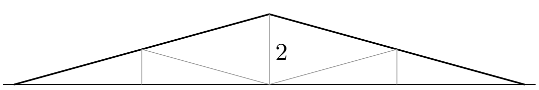
metoda sila: sistem je jedanput statički neodređen $\quad\Rightarrow\quad$ 1 nepoznanica

metoda sila:



o. s.

$$\delta_{1,1}\, X_1 = \bar{\delta}_{1,0} = -1$$



m_1

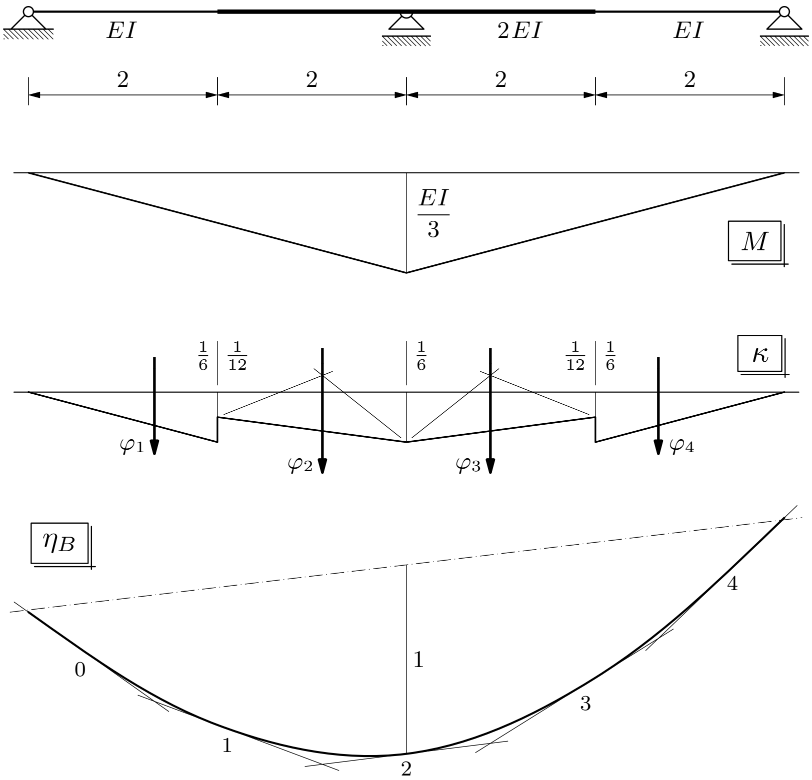
.

$$\delta_{1,1} \;=\; 2\left[\frac{1}{EI}\left(\frac{1}{2}\cdot 1\cdot 2\right)\left(\frac{2}{3}\cdot 1\right) + \frac{1}{2\,EI}\left(\frac{1}{2}\cdot 1\cdot 2\right)\left(\frac{2}{3}\cdot 1+\frac{1}{3}\cdot 2\right) + \frac{1}{2\,EI}\left(\frac{1}{2}\cdot 2\cdot 2\right)\left(\frac{1}{3}\cdot 1+\frac{2}{3}\cdot 2\right)\right] \;=\; \frac{6}{EI}$$

$$\frac{6}{EI}\,X_1 \;=\; -1 \qquad\Rightarrow\qquad X_1 \;=\; -\frac{EI}{6}$$

$$M(x) \;=\; X_1\,m_1(x)$$

momentni dijagram i utjecajna linija:

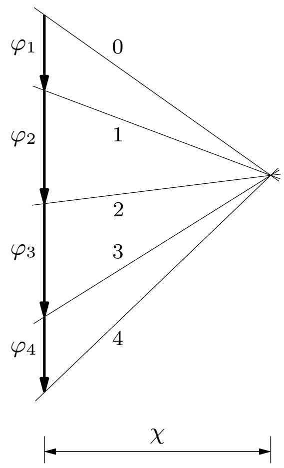


mjerilo duljina:

1 [cm] :: 0,8 [m]

mjerilo kutova:

1 [cm] :: $\frac{1}{6}$



$$\varphi_1 \;=\; \varphi_4 \;=\; \frac{1}{2}\cdot\frac{1}{6}\cdot 2 \;=\; \frac{1}{6}$$

$$\varphi_2 \;=\; \varphi_3 \;=\; \frac{1}{2}\cdot\left(\frac{1}{6}+\frac{1}{12}\right)\cdot 2 \;=\; \frac{1}{4}$$

$$\chi \;=\; \frac{1}{2}$$

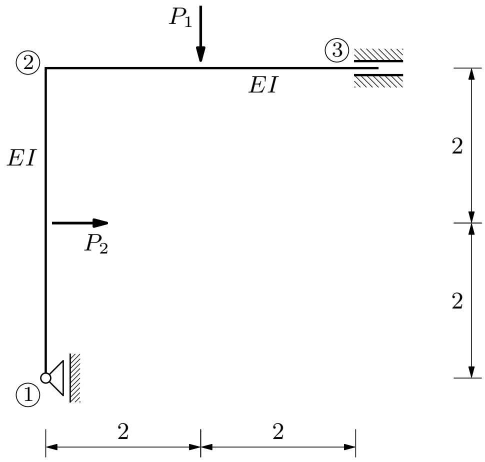
stranica 2 verižnoga poligona (tangenta 2 na verižnu krivulju) je usporedna sa zaključnom linijom $\quad\Rightarrow\quad \alpha_2 = \alpha_M = 0 \quad\Rightarrow\quad \operatorname{tg}\alpha_2 = \operatorname{tg}\alpha_M = 0$

Zadatak 4.

Relaksacijskim postupkom nacrtajte momentni dijagram! Provjerite rezultat!

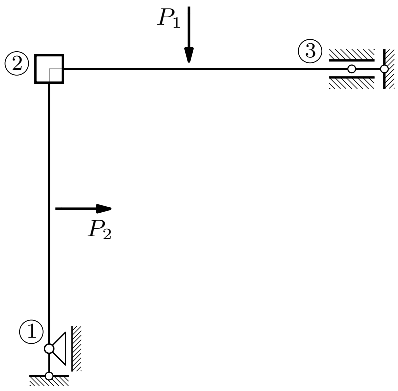
$P_1 = 125$ kN

$P_2 = 50$ kN



prvi korak: Crossov postupak na nepomičnom sistemu:

vrijednosti momenata upetosti:



$$\overline{M}_{2,1}^{(c)} = \overline{M}_{2,1} - \frac{1}{2}\overline{M}_{1,2} = -\frac{P_2\,\ell_{\{1,2\}}}{8} - \frac{1}{2}\,\frac{P_2\,\ell_{\{1,2\}}}{8}$$

$$= -\frac{3\,P_2\,\ell_{\{1,2\}}}{16} = -\frac{3\cdot 50\cdot 4}{16} = -37{,}5\,\text{kNm}$$

$$\overline{M}_{2,3} = \overline{M}_{3,2} = \frac{P_1\,\ell_{\{2,3\}}}{8} = \frac{125\cdot 4}{8} = 62{,}5\,\text{kNm}$$

razdjelni koeficijenti:

$$k_{\{1,2\}} = k_{\{2,3\}} = k$$

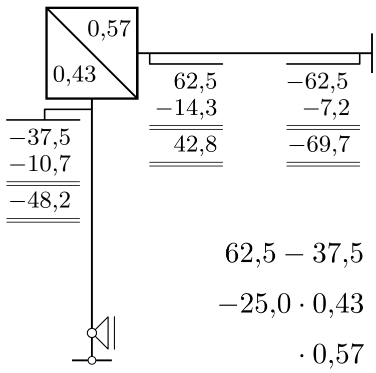
$$k_2 = 3\,k + 4\,k = 7\,k$$

$$\mu_{3,1} = \frac{3\,k}{k_2} = \frac{3}{7} = 0{,}43$$

$$\mu_{3,2} = \frac{4\,k}{k_2} = \frac{4}{7} = 0{,}57$$

$$\mu_{3,1} + \mu_{3,2} = 0{,}43 + 0{,}57 = 1$$

raspodjela i prijenos momenata:



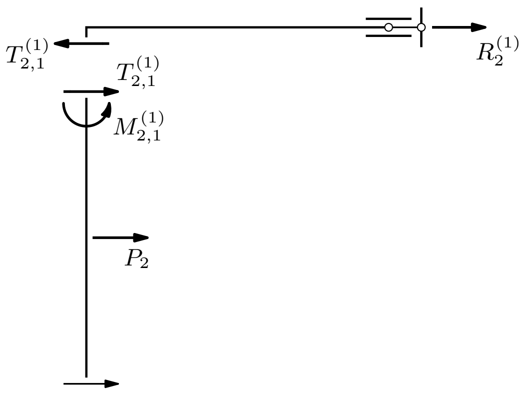
$$62{,}5 - 37{,}5 = 25{,}0$$

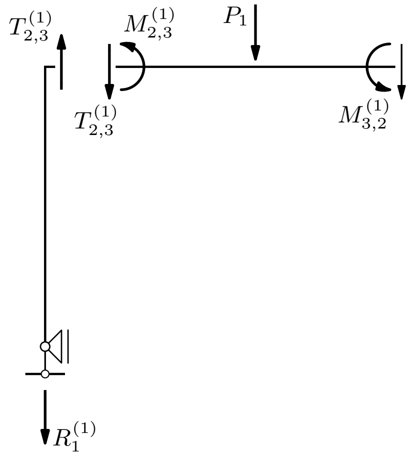
$$-25{,}0\cdot 0{,}43 = -10{,}7$$

$$\cdot\, 0{,}57 = -14{,}3$$

$$-25{,}0$$

vrijednosti reakcija u zamišljenim spojevima:



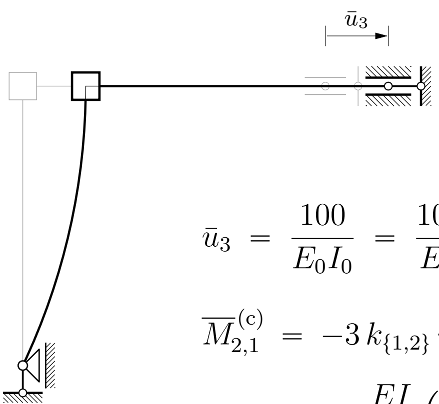


$$-T^{(1)}_{2,1} + R^{(1)}_2 = 0$$
$$R^{(1)}_2 = T^{(1)}_{2,1}$$
$$-4 \cdot T^{(1)}_{2,1} + M^{(1)}_{2,1} - 2 \cdot P_2 = 0$$
$$T^{(1)}_{2,1} = \frac{M^{(1)}_{2,1}}{4} - \frac{P_2}{2} = \frac{-48{,}2}{4} - \frac{50}{2} = -37{,}1\,\text{kN}$$
$$R^{(1)}_2 = -37{,}1\,\text{kN}$$

$$-T^{(1)}_{2,3} + R^{(1)}_1 = 0$$
$$R^{(1)}_1 = T^{(1)}_{2,3}$$
$$4 \cdot T^{(1)}_{2,3} + M^{(1)}_{2,3} + M^{(1)}_{3,2} + 2 \cdot P_1 = 0$$
$$T^{(1)}_{2,3} = -\frac{M^{(1)}_{2,3} + M^{(1)}_{3,2}}{4} - \frac{P_1}{2} = -\frac{48{,}2 - 69{,}7}{4} - \frac{125}{2} = -57{,}1\,\text{kN}$$
$$R^{(1)}_1 = -57{,}1\,\text{kN}$$

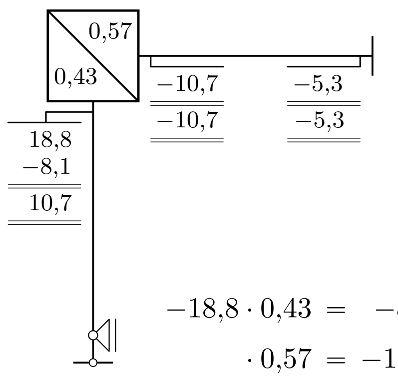
drugi korak: proširenje Crossova postupka:

vrijednost momenta upetosti:



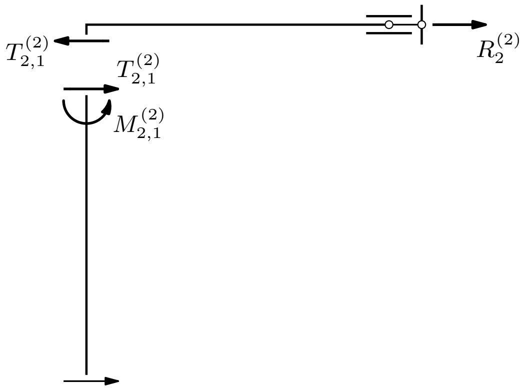
$$\bar{u}_3 = \frac{100}{E_0 I_0} = \frac{100}{EI}$$
$$\overline{M}^{(c)}_{2,1} = -3\,k_{\{1,2\}}\,\bar{\psi}_{\{1,2\}} = -3\,\frac{EI}{4}\left(-\frac{\bar{u}_3}{4}\right) = 3 \cdot \frac{EI}{4} \cdot \frac{\frac{100}{EI}}{4} = 18{,}8\,\text{kNm}$$

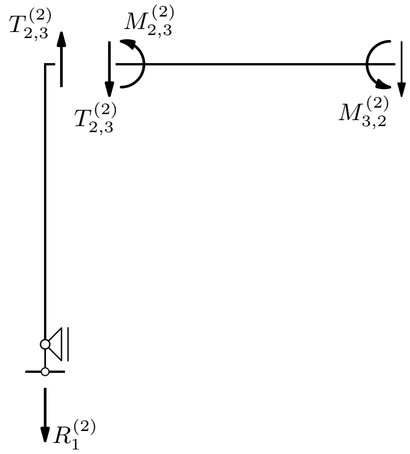
raspodjela i prijenos momenata:



$$-18{,}8 \cdot 0{,}43 = -8{,}1$$
$$\cdot\, 0{,}57 = -10{,}7$$
$$-18{,}8$$

vrijednosti reakcija u zamišljenim spojevima:





$$-T_{2,1}^{(2)} + R_2^{(2)} = 0$$

$$-4 \cdot T_{2,1}^{(2)} + M_{2,1}^{(2)} = 0$$

$$T_{2,1}^{(2)} = \frac{M_{2,1}^{(2)}}{4} = \frac{10{,}7}{4} = 2{,}7\,\mathrm{kN}$$

$$R_2^{(2)} = 2{,}7\,\mathrm{kN}$$

$$-T_{2,3}^{(2)} + R_1^{(2)} = 0$$

$$4 \cdot T_{2,3}^{(2)} + M_{2,3}^{(2)} + M_{3,2}^{(2)} = 0$$

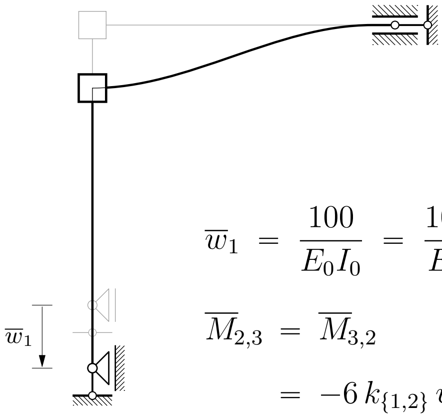
$$T_{2,3}^{(2)} = -\frac{M_{2,3}^{(2)} + M_{3,2}^{(2)}}{4}$$

$$= -\frac{-10{,}7 - 5{,}3}{4} = 4{,}0\,\mathrm{kN}$$

$$R_1^{(2)} = 4{,}0\,\mathrm{kN}$$

treći korak: proširenje Crossova postupka:

vrijednosti momenta upetosti:



$$\overline{w}_1 = \frac{100}{E_0 I_0} = \frac{100}{EI}$$

$$\overline{M}_{2,3} = \overline{M}_{3,2}$$

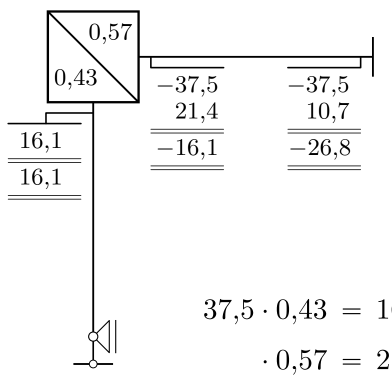
$$= -6\,k_{\{1,2\}}\,\bar{\psi}_{\{2,3\}}$$

$$= -6\,\frac{EI}{4}\,\frac{\overline{w}_1}{4}$$

$$= -6 \cdot \frac{EI}{4} \cdot \frac{\frac{100}{EI}}{4}$$

$$= -37{,}5\,\mathrm{kNm}$$

raspodjela i prijenos momenata:

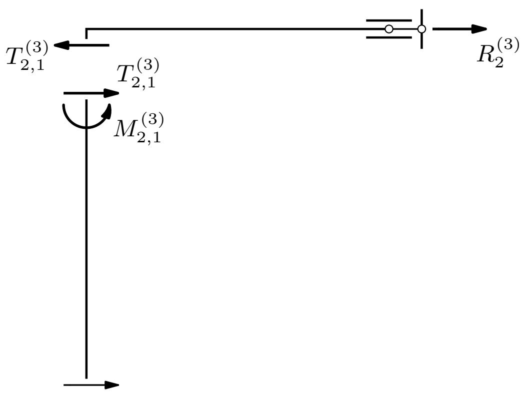


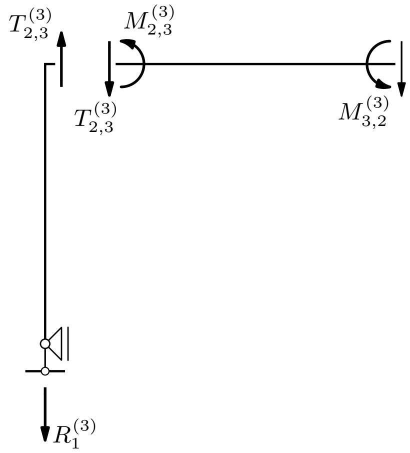
$$37{,}5 \cdot 0{,}43 = 16{,}1$$

$$\cdot\, 0{,}57 = 21{,}4$$

$$37{,}5$$

vrijednosti reakcija u zamišljenim spojevima:





$$-T_{2,1}^{(3)} + R_2^{(3)} = 0$$

$$-4 \cdot T_{2,1}^{(3)} + M_{2,1}^{(3)} = 0$$

$$T_{2,1}^{(3)} = \frac{M_{2,1}^{(3)}}{4} = \frac{16{,}1}{4} = 4{,}0\,\text{kN}$$

$$R_2^{(3)} = 4{,}0\,\text{kN}$$

$$-T_{2,3}^{(3)} + R_1^{(3)} = 0$$

$$4 \cdot T_{2,3}^{(3)} + M_{2,3}^{(3)} + M_{3,2}^{(3)} = 0$$

$$T_{2,3}^{(3)} = -\frac{M_{2,3}^{(3)} + M_{3,2}^{(3)}}{4} = -\frac{-16{,}1 - 26{,}8}{4} = 10{,}7\,\text{kN}$$

$$R_1^{(3)} = 10{,}7\,\text{kN}$$

. . . i, na kraju:

$$R_1^{(1)} + R_1^{(2)}\,\varrho_1 + R_1^{(3)}\,\varrho_2 = 0$$

$$R_2^{(1)} + R_2^{(2)}\,\varrho_1 + R_2^{(3)}\,\varrho_2 = 0$$

$$-57{,}1 + 4{,}0\,\varrho_1 + 10{,}7\,\varrho_2 = 0$$

$$-37{,}1 + 2{,}7\,\varrho_1 + 4{,}0\,\varrho_2 = 0$$

$$\varrho_1 = 13{,}08 \qquad \& \qquad \varrho_2 = 0{,}45$$

konačne vrijednosti momenata:

$$M_{i,j} = M_{i,j}^{(1)} + \varrho_1\,M_{i,j}^{(2)} + \varrho_2\,M_{i,j}^{(3)}$$

$$M_{2,1} = -48{,}2 + 13{,}08 \cdot 10{,}7 + 0{,}45 \cdot 16{,}1 = 99{,}0\,\text{kNm}$$

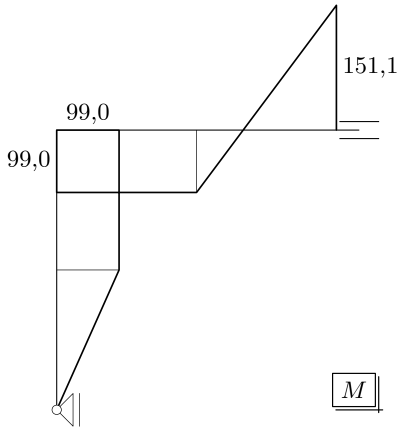
$$M_{2,3} = 48{,}2 + 13{,}08 \cdot (-10{,}7) + 0{,}45 \cdot (-16{,}1) = -99{,}0\,\text{kNm}$$

$$M_{3,2} = -69{,}7 + 13{,}08 \cdot (-5{,}3) + 0{,}45 \cdot (-26{,}8) = -151{,}1\,\text{kNm}$$

u horizontalnom štapu nema uzdužne sile, pa u vertikalnom štapu između čvora 2 i hvatišta sile P_2 nema poprečne sile, a moment se ne mijenja

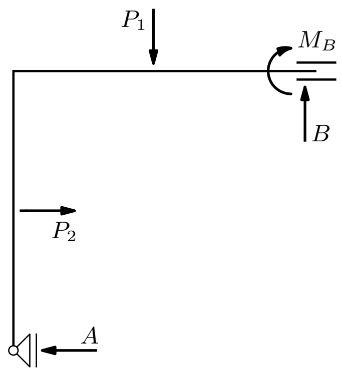
uzdužne sile nema ni u vertikalnom štapu, pa u horizontalnom štapu između čvora 2 i hvatišta sile P_1 nema poprečne sile, a moment se ne mijenja

dijagram momenata:



provjera:

sistem je statički određen



$$\sum F_x = 0 : \quad P_2 - A = 0 \quad \Rightarrow \quad A = P_2 = 50\,\text{kN}$$

$$\sum F_z = 0 : \quad P_1 - B = 0 \quad \Rightarrow \quad B = P_1 = 125\,\text{kN}$$

$$\sum M_{/B} = 0 : \quad -A \cdot 4 + P_2 \cdot 2 + P_1 \cdot 2 - M_B = 0$$

$$\Rightarrow \quad M_B = -A \cdot 4 + P_2 \cdot 2 + P_1 \cdot 2 = 150\,\text{kNm}$$

$M_{h/2} = A \cdot 2 = 100\,\text{kNm}$ (u hvatištu sile P_2)

A i P_2 tvore spreg, pa se iznad hvatišta do čvora 2 moment ne mijenja: $M_{2,\text{dolje}} = M_{h,2}$

ravnoteža čvora 2: $M_{2,\text{dolje}} = M_{2,\text{desno}}$

moment se u horizontalnom štapu do hvatišta sile P_1 ne mijenja

iz svega slijedi da je dijagram momenata isti kao dijagram na vrhu stranice, s neznatno drukčijim (i, u stvari, točnijim) vrijednostima (u Crossovom smo postupku zaokruživanjem na jednu decimalu izgubili točnost)