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Two-Axial Surfaces of Revolution

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ABSTRACT

Special class of surfaces, two-axial surfaces of revolution created by the Euclidean metric transformation of a simultaneous revolution about two different axes, is presented in the paper. Three specific subclasses of surfaces are classified with respect to the superposition of the two axes ^1o, ^2o of revolution. There are defined several types of two-axial surfaces of revolution specifying the type of the surface basic figure and its position to the axes of revolution.

Key words: composite revolution, two-axial revolution, generalised surfaces of revolution

MSC 2000: 14J26, 15A04, 53A05

Dvoosne rotacijske plohe

SAŽETAK

U radu je prikazana posebna klasa ploha - dvoosne rotacijske plohe - nastala pomoću euklidske metričke transformacije koja se sastoji od dviju istodobnih rotacija oko dvije različite osi. Prema položaju rotacijskih osi ^1o i ^2o razlikuju se tri podklase takvih ploha. Daljnje razvrstavanje na nekoliko tipova ploha provedeno je prema vrsti osnovne figure (čijom rotacijom ploha nastaje) i njenom položaju prema rotacijskim osima.

Ključne riječi: složena rotacija, dvoosne rotacije, poopćene rotacijske plohe

1 Introduction

Composition of two revolutions about two different axes in the space determined as two-axial revolution is a metric transformation. Let $^1o, ^2o$ be the two axes of two revolutions. There can be characterised three distinguished subgroups of the general revolutionary movements in the space composed from the two revolutions, with respect to the superposition of the two axes:

I. cycloidal movement with parallel axes $^1o \parallel {}^2o$,

II. spherical movement with intersect axes $^1o \times {}^2o$,

III. general Euler revolution with skew axes $^1o/^2o$.

Considering the two-axial revolutionary movement, analytic representation of this linear 1-parametric transformation is a regular square matrix of rank 4, with entries in the form of real functions of one real variable defined on an interval of the real numbers

$$\mathrm{T}(v) = (a_{ij}(v)), \quad \text{for} \quad i,j = 1,2,3,4, \quad v \in R.$$

This matrix function can be analytically determined as the product of two square matrix functions representing the separate revolutions about given axes. For the sake of easy formulas, special positions of axes of revolution are chosen, on coordinate axes or parallel to some of the coordinate axes.

Two-axial surfaces of revolution can be created by the two-axial revolutionary movement of a basic curve k defined analytically by a vector function

$$\mathbf{r}(u) = (x_1(u), x_2(u), x_3(u), 1), \quad u \in R.$$

Vector representation of the surface patch defined on the region $\Omega \subset R^2$ is the matrix product

$$\mathbf{p}(u,v) = \mathbf{r}(u).\mathrm{T}(v) = \left(\sum_{i=1}^{4} x_i(u)a_{i1}(v), \sum_{i=1}^{4} x_i(u)a_{i2}(v), \sum_{i=1}^{4} x_i(u)a_{i3}(v), 1\right).$$

There can be distinguished three basic subgroups of two-axial surfaces of revolution according to the type of the generating two-axial revolutionary movement, i.e. according to the superposition of the axes of revolutions $^1o, ^2o$:

I. $^1o \parallel {}^2o$, surfaces of cycloidal type

II. $^1o \times {}^2o$, surfaces of spherical type

III. $^1o/^2o$, surfaces of Euler type.

With respect to the type of the basic curve, we recognise the following types of the two-axial surfaces of revolution:

A. ruled surfaces, for a line (line segment) as basic curve

B. cyclical surfaces, for a circle (circular arc) as basic curve

C. non-specified surfaces, for all other basic curves.

Ruled surfaces can be further classified according to the superposition of the basic line and the two axes of revolution as:

1. cylindrical - basic line is parallel to both axes (for type I exclusively)
2. conical - basic line is intersecting to both axes
3. hyperbolical - basic line is skew to both axes
4. composite - basic line is in different superposition to the two axes.

Cyclical surfaces can be further classified according to the superposition of the plane of the basic circle and the two axes of revolution as:

1. toroidal - basic circle is located in the plane formed by the two axes of revolution (for types I and II only possible)
2. general - basic circle is located in the general plane with respect to the two axes of revolution.

Let us restrict our considerations to the revolutionary movements about two different axes with the same angular velocities. Parametric equations of the defined two-axial surfaces of revolution are derived and several illustrations of the special representatives of all subclasses and types are presented in the following.

Surfaces of Euler type show no symmetry, as there exists no plane formed by the axes of revolution ${}^1o, {}^2o$, unless the basic figure is located in a special "symmetric position" with respect to the two axes. There exists at least one plane of symmetry, $\sigma = {}^1o{}^2o$, for surfaces of spherical and cycloidal types. Special position of the basic figure and axes of revolution may result in existence of at least one more plane of symmetry, perpendicular to the plane σ and passing through one of the axes. Some other planes of symmetry may occur as well, passing through the axis ${}^1o$. Plane σ and one-parametric system of planes perpendicular to σ form symmetry planes of surfaces of cycloidal type -ruled cylindrical.

2 Two-axial surfaces of revolution of cycloidal type

Cycloidal movement is composed from two revolutions about parallel axes ${}^1o \parallel {}^2o$, at the distance $\left|{}^1o{}^2o\right| = d$, $d \neq 0$, with equal angles $\varphi = \psi$. For angles on interval $[0, 2\pi]$ we receive the closed epicycloidal movement, for angles $\varphi = -\psi$ on interval $[0, 2\pi]$ the closed hypocycloidal movement can be obtained. Locating the axis ${}^1o$ on the coordinate axis z and axis ${}^2o$ in the plane xz, we receive the matrix representation of the two-axial revolution - matrix of epicycloidal movement (composition of revolutions in the same directions)

$$\mathrm{T}(v) = \begin{pmatrix} \cos 4\pi v & \sin 4\pi v & 0 & 0 \\ -\sin 4\pi v & \cos 4\pi v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d(\cos 2\pi v - 1) & d\sin 2\pi v & 0 & 1 \end{pmatrix},$$

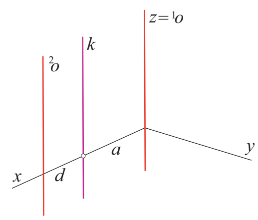
matrix of hypocycloidal movement (composition of revolutions in the opposite directions)

$$\mathrm{T}(v) = \begin{pmatrix} \cos 4\pi v & -\sin 4\pi v & 0 & 0 \\ \sin 4\pi v & \cos 4\pi v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d(\cos 2\pi v - 1) & d\sin 2\pi v & 0 & 1 \end{pmatrix}.$$

2.1 Ruled cylindrical surfaces - group IA1

The basic line k is parallel to both axes of revolution and it is located in the plane they form, i.e. in the coordinate plane xz, while the following metric relations are true

$$\left|{}^1o{}^2o\right| = d, \quad \left|{}^1ok\right| = a, \quad a \neq 0, \quad d \neq 0.$$



Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (a, 0, bu, 1).\mathrm{T}(v) =$$
$$= (a\cos 4\pi v + d(\cos 2\pi v - 1), \pm a\sin 4\pi v + d\sin 2\pi v, bu, 1).$$

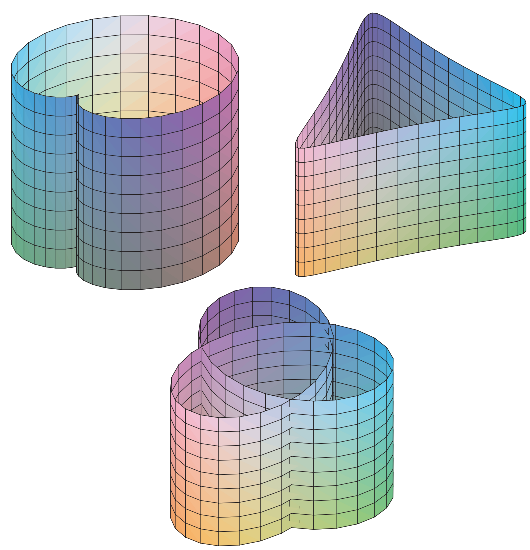


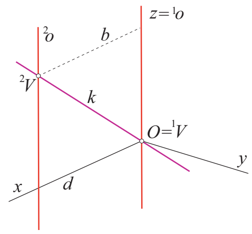
Fig. 1: Two-axial sufaces of revolution of cycloidal type - ruled cylindrical

2.2 Ruled conical surfaces - group IA2

The basic line k intersects both axes of revolution and is located in the plane they form, i.e. in the coordinate plane xz, while the following metric relations are true

$$\left|{}^1o\,{}^2o\right| = d, \quad k \cap {}^1o = {}^1V = (0,0,0,1)$$

$$k \cap {}^2o = {}^2V = (d,0,b,1), \quad b \neq 0, \quad d \neq 0.$$



Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (du,0,bu,1).\mathrm{T}(v) =$$

$$= (d\,u\cos 4\pi v + d(\cos 2\pi v - 1), \pm d\,u\sin 4\pi v + d\sin 2\pi v, bu, 1).$$

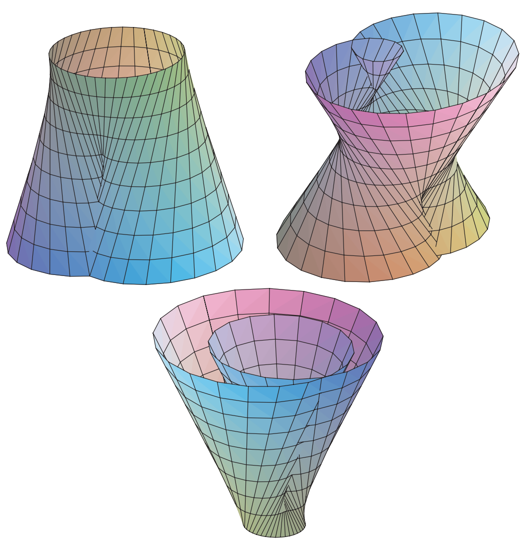


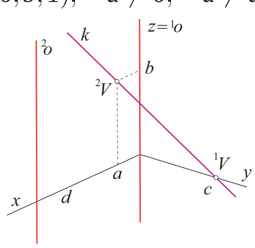
Fig. 2: Two-axial surfaces of revolution of cycloidal type - ruled conical

2.3 Ruled hyperbolical surfaces - group IA3

The basic line k is skew to both axes of revolution, while the following metric relations are true

$$\left|{}^1o\,{}^2o\right| = d, \quad k \cap \pi = {}^1V = (0,c,0,1), \quad c \neq 0, \quad d \neq 0$$

$$k \cap \nu = {}^2V = (a,0,b,1), \quad a \neq 0, \quad a \neq d, \quad b \neq 0.$$



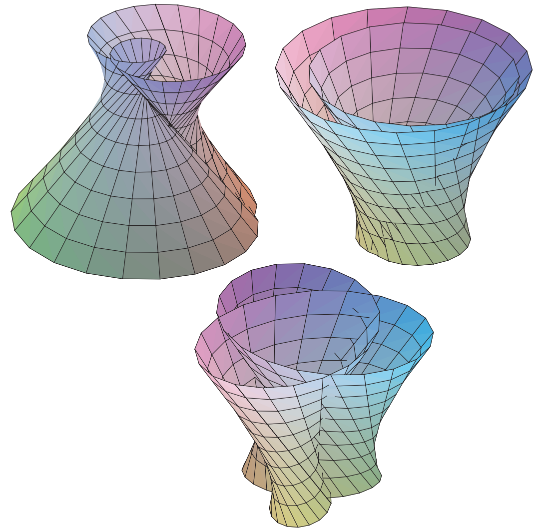
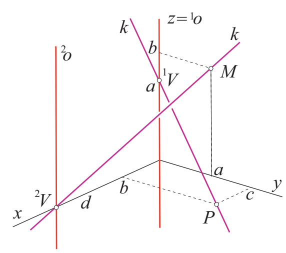


Fig. 3: Two-axial sufaces of revolution of cycloidal type - ruled hyperbolical

2.4 Ruled composite surfaces - group IA4

The basic line k is in different superpostion to both axes of revolution, it is intersecting to one axis and skew to the other one.



a) Let k be intersecting to ^1o, skew to ^2o, and let it intersects coordinate plane $\pi = xz$ in the point P

$$\left|{}^1o\,{}^2o\right| = d, \; k \cap {}^1o = {}^1V = (0,0,a,1), \; a \neq 0, \; d \neq 0$$

$$k \cap \pi = P = (b,c,0,1), \quad b \neq 0, \quad c \neq 0,$$

then the surface contains only one circle, the trajectory of the point ^1V, and it has no planes of symmetry.

Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (bu, cu, a(1-u), 1).\mathrm{T}(v) =$$

$$(bu\cos 4\pi v \mp cu\sin 4\pi v + d(\cos 2\pi v - 1),$$

$$\pm bu\sin 4\pi v + cu\cos 4\pi v + d\sin 2\pi v, a(1-u), 1).$$

b) Let k be intersecting to ${}^2o$ and skew to ${}^1o$, and let it intersects coordinate plane $\mu = zy$ in the point M

$$|{}^1o{}^2o| = d, \quad k \cap {}^2o = {}^2V = (d,0,0,1), \quad d \neq 0$$
$$k \cap \mu = M = (0,a,b,1), \quad a \neq 0, \quad b \neq 0.$$

Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (d(1-u), au, bu, 1).\mathrm{T}(v) =$$
$$(d\,(1-u)\cos 4\pi v \mp a\,u \sin 4\pi v + d\,(\cos 2\pi v - 1),$$
$$\pm d(1-u)\sin 4\pi v + a\,u\cos 4\pi v + d\sin 2\pi v, bu, 1).$$

Some representatives of this group of surfaces are illustrated in figure 4.

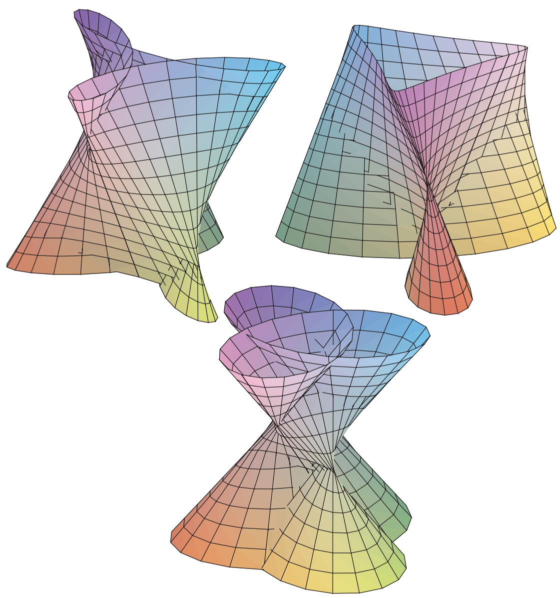
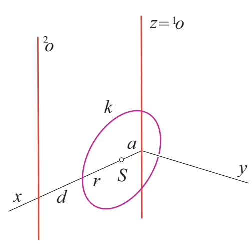


Fig. 4: Two-axial surfaces of revolution of cycloidal type - ruled composite

2.5 Cyclical toroidal surfaces - group IB1

The basic circle $k(S,r)$ is located in the plane xz, while the following metric relations are true

$$|{}^1o{}^2o| = d, \quad S \in x, \quad \left|S{}^1o\right| = a, \quad d \neq 0.$$



Vector function of the surface patch determined on $[0,1]^2$is

$$\mathbf{p}(u,v) = (a + r\cos 2\pi u, 0, r\sin 2\pi u, 1)\,.\mathrm{T}(v),$$

while the separate coordinate functions are in the form

$$x(u,v) = (a + r\cos 2\pi u)\cos 4\pi v + d(\cos 2\pi v - 1)$$
$$y(u,v) = \pm(a + r\cos 2\pi v)\sin 4\pi v + d\sin 2\pi v$$
$$z(u,v) = r\sin 2\pi u.$$

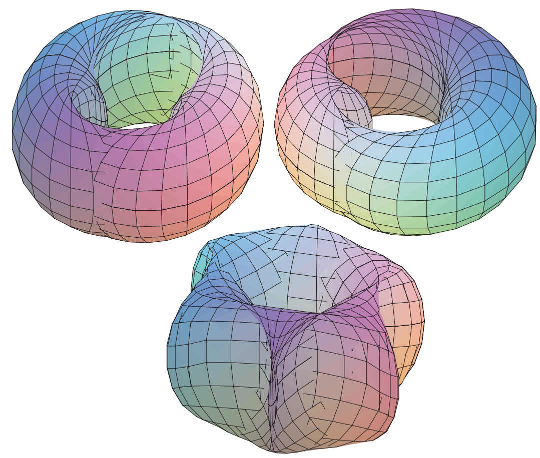
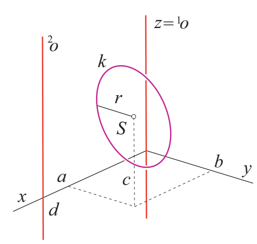


Fig. 5: Two-axial surfaces of revolution of cycloidal type - cyclical toroidal

2.6 Cyclical general surfaces - group IB2

The basic circle $k(S,r)$ can be located in the arbitrary plane, let it be the plane parallel to the plane yz, while the following metric relations are true

$$|{}^1o{}^2o| = d, \quad S = (a,b,c,1), \quad a \neq 0, \quad b \neq 0, \quad d \neq 0.$$



Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (a, b + r\cos 2\pi u, c + r\sin 2\pi u, 1)\,.\mathrm{T}(v),$$

while the separate coordinate functions are in the form

$$x(u,v) = a\cos 4\pi v \mp (b + r\cos 2\pi u)\sin 4\pi v + d(\cos 2\pi v - 1)$$
$$y(u,v) = \pm a\sin 4\pi v + (b + r\cos 2\pi v)\cos 4\pi v + d\sin 2\pi v$$
$$z(u,v) = c + r\sin 2\pi u.$$

Some representatives of the surfaces in this group can be seen in figure 6.

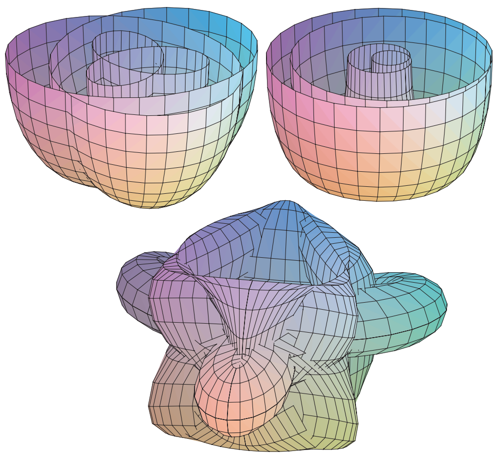


Fig. 6: Two-axial surfaces of revolution of cycloidal type - cyclical general

3 Two axial surfaces of revolution of spherical type

Two-axial revolution determined by intersecting axes ${}^1o \times {}^2o$ is a spherical movement. Let us locate the axis ${}^1o$ on the coordinate axis z, and axis ${}^2o$ on the coordinate axis y. Analytic representation of the two-axial surface of revolution of spherical type determined on the region $\Omega \subset R^2$, with the basic curve represented by the vector equation

$$\mathbf{r}(u) = (x(u), y(u), z(u), 1), \quad u \in R$$

is in the form

$$\mathbf{p}(u,v) = (x(u,v), y(u,v), z(u,v), 1), \quad (u,v) \in \Omega$$

where

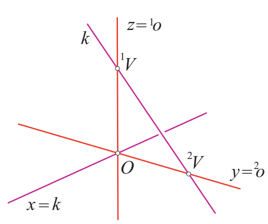
$$x(u,v) = x(u)\cos^2 2\pi v - y(u)\sin 2\pi v \cos 2\pi v + z(u)\sin 2\pi v$$

$$y(u,v) = x(u)\sin 2\pi v + y(u)\cos 2\pi v$$

$$z(u,v) = -x(u)\sin 2\pi v \cos 2\pi v + y(u)\sin^2 2\pi v + z(u)\cos 2\pi v.$$

3.1 Ruled conical surfaces - group IIA2

The basic line k is intersecting to both axes of revolution.



a) Let k be on the coordinate axes x, therefore intersecting both axes of revolution in their common point O, then surface does not contain any circle, and it has 3 planes of symmetry.

b) Let k be in the plane of axes of revolution, i.e. in the coordinate plane yz

$$k \cap {}^1o = {}^1V = (0,0,b,1), \quad k \cap {}^2o = {}^2V = (0,a,0,1)$$
$$a \neq 0, \quad b \neq 0$$

then the surface contains only one circle, the trajectory of the point ${}^1V$, and it has a unique plane of symmetry.

Parametric equations of the two surfaces (Fig. 7) defined on the region $[0,1]^2 \subset R^2$ are in the forms

$$x(u,v) = a\,u \cos^2 2\pi v$$
$$y(u,v) = a\,u \sin 2\pi v$$
$$z(u,v) = -a\,u \sin 2\pi v \cos 2\pi v$$

and

$$x(u,v) = -a\,u \sin 2\pi v \cos 2\pi v$$
$$y(u,v) = a \cos 2\pi v$$
$$z(u,v) = -a\,u \sin^2 2\pi v + b(1-u)\cos 2\pi v.$$

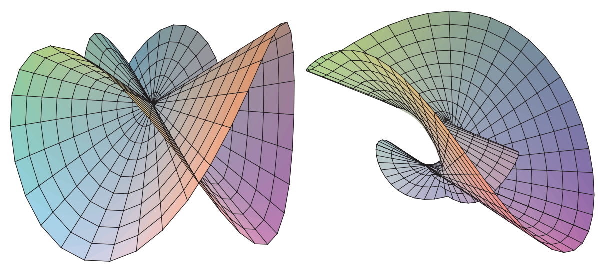
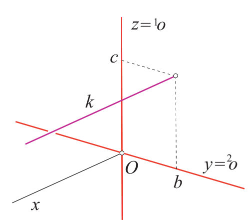


Fig. 7: Two-axial surfaces of revolution of spherical type - ruled conical

3.2 Ruled hyperbolical surfaces - group IIA3

The basic line k is skew to both axes of revolution, and let it be parallel to the coordinate axis x.



Parametric equations of the ruled hyperbolic surfaces (Fig. 8.) defined on the region $[0,1]^2 \subset R^2$ are in the form

$$x(u,v) = a\,u \cos^2 2\pi v - b \sin 2\pi v \cos 2\pi v + c \sin 2\pi v$$
$$y(u,v) = a\,u \sin 2\pi v + b \cos 2\pi v$$
$$z(u,v) = -a\,u \sin 2\pi v \cos 2\pi v + b \sin^2 2\pi v + c \cos 2\pi v$$

and there are no circles on the surface.

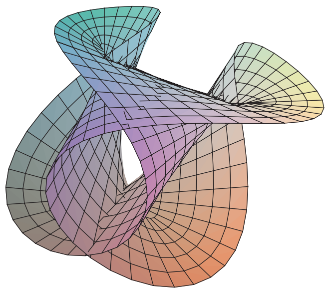


Fig. 8: Two-axial surface of revolution of spherical type - ruled hyperbolical

3.3 Ruled composite surfaces 1 - group IIA4

The basic line k is parallel to one axis of revolution and it is intersecting to the other axis. Let k be located in the plane of the two axes of revolution, coordinate plane yz, and

a) let k be parallel to ${}^1o$

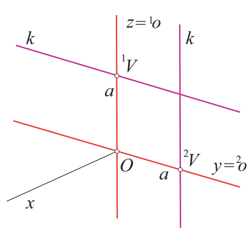
$\left|{}^1ok\right| = a,\ k \cap {}^2o = {}^2V = (0,a,0,1),\ a \neq 0$

then surface contains no circles, and it has 2 planes of symmetry;

b) let k be parallel to ${}^2o$

$\left|{}^2ok\right| = a, \quad k \cap {}^1o = {}^1V = (0,0,a,1), \quad a \neq 0$

then surface contains the only one circle, trajectory of the point ${}^1V$, and it has a unique plane of symmetry.



Parametric equations of this specific ruled composite surfaces (Fig. 9) defined on the region $[0,1]^2 \subset R^2$ are in the form

a) k is parallel to ${}^1o$, intersecting to ${}^2o$

$x(u,v) = -a\,u\,\sin 2\pi v\,\cos 2\pi v + b\,u\,\sin 2\pi v$

$y(u,v) = a\,\cos 2\pi v$

$z(u,v) = a\sin^2 2\pi v + b\,u\,\cos 2\pi v;$

b) k is intersecting to ${}^1o$, parallel to ${}^2o$

$x(u,v) = -b\,u\,\sin 2\pi v\,\cos 2\pi + a\,\sin 2\pi v$

$y(u,v) = b\,u\,\cos 2\pi v$

$z(u,v) = b\,u\sin^2 2\pi v + a\,\cos 2\pi v.$

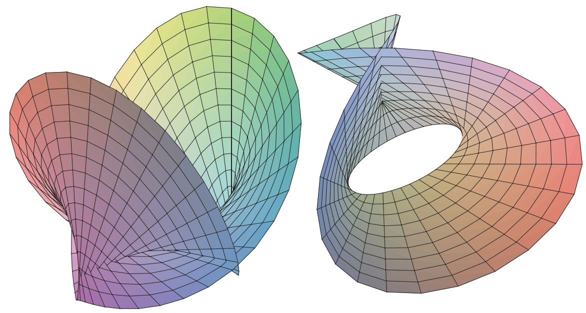
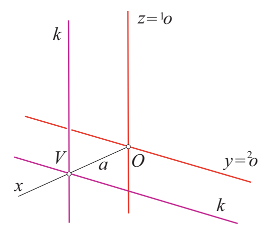


Fig. 9: Two-axial surfaces of revolution of spherical type - ruled composite - 1

3.4 Ruled composite surfaces 2 - group IIA4

The basic line k is parallel to one axis of revolution and it is skew to the other axis.



Let line k intersects the coordinate axis x,

$k \cap x = V = (a,0,0,1)$ and

a) let k be parallel to ${}^1o$, $\quad \left|{}^1ok\right| = a, \quad a \neq 0$, then the surface has only one plane of symmetry;

b) let k be parallel to o, $\quad \left|{}^2ok\right| = a, \quad a \neq 0$, then the surface has 2 planes of symmetry.

There are no circles on the surfaces.

Parametric equations of these specific ruled composite surfaces (Fig. 10) defined on the region $[0,1]^2 \subset R^2$ are in the form

a) k is parallel to ${}^1o$, skew to ${}^2o$

$x(u,v) = a\cos^2 2\pi v + b\,u\,\sin 2\pi v$

$y(u,v) = a\,\sin 2\pi v$

$z(u,v) = -a\,\sin 2\pi v\,\cos 2\pi v + b\,u\,\cos 2\pi v;$

b) k is skew to ${}^1o$, parallel to ${}^2o$

$x(u,v) = a\cos^2 2\pi v - b\,u\,\sin 2\pi v\,\cos 2\pi v$

$y(u,v) = a\,\sin 2\pi v + b\,u\,\cos 2\pi v$

$z(u,v) = -a\,\sin 2\pi v\,\cos 2\pi v + b\,u\sin^2 2\pi v.$

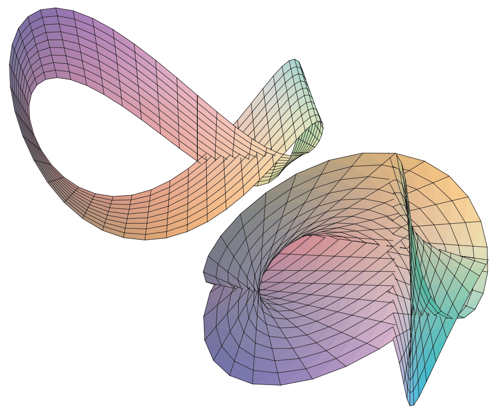


Fig. 10: Two-axial surfaces of revolution of spherical type - ruled composite - 2

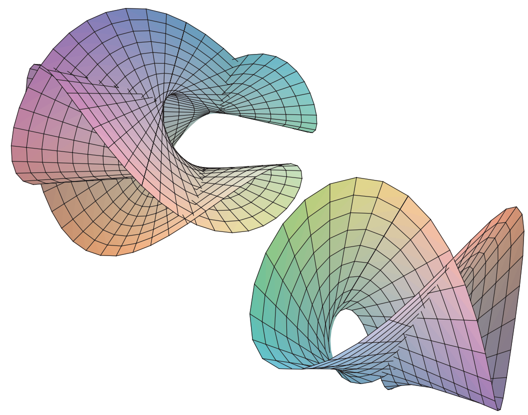
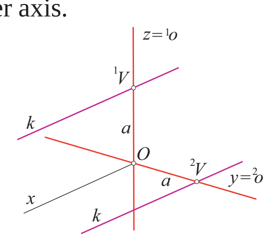


Fig. 11: Two-axial surfaces of revolution of spherical type - ruled composite - 3

3.5 Ruled composite surfaces 3 - group IIA4

The basic line k intersects one axis of revolution and is skew to the other axis.



Let line k be parallel to the coordinate axis x, and

a) let k be intersecting to ${}^1o$, skew to ${}^2o$,
$k \cap {}^1o = {}^1V = (0,0,a,1)$,
then the surface contains only one circle, the trajectory of the point ${}^1V$, and it has 2 planes of symmetry;

b) let k be intersecting to ${}^2o$ and skew to ${}^1o$,
$k \cap {}^2o = {}^2V = (0,a,0,a)$,
then the surface contains no circles, and it has a unique plane of symmetry.

Parametric equations of these specific ruled composite surfaces (Fig. 11) defined on the region $[0,1]^2 \subset R^2$ are in the form

a) k is intersecting to ${}^1o$, skew to ${}^2o$

$$x(u,v) = b\,u\cos^2 2\pi v + a\sin 2\pi v$$
$$y(u,v) = b\,u\sin 2\pi v$$
$$z(u,v) = -b\,u\sin 2\pi v\cos 2\pi v + a\cos 2\pi v;$$

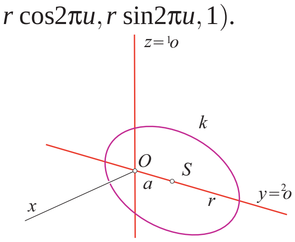
b) k is skew to ${}^1o$, intersecting to ${}^2o$

$$x(u,v) = b\,u\cos^2 2\pi v - a\sin 2\pi v\cos 2\pi v$$
$$y(u,v) = b\,u\sin 2\pi v + a\cos 2\pi v$$
$$z(u,v) = -b\,u\sin 2\pi v\cos 2\pi v + a\sin^2 2\pi v.$$

3.6 Cyclical toroidal surfaces - Group IIB1

The basic circle $k(S,r)$ is located in the plane determined by the two axes of revolution, in the coordinate plane yz, while the vector equation of the circle for $u \in [0,1]$ is

$\mathbf{r}(u) = (0, a + r\cos 2\pi u, r\sin 2\pi u, 1)$.



Number of circles as trajectories of the points on the basic circle depends on the superposition of the basic circle and the axis of revolution ${}^1o$, i.e. on the common relation of parameters a and r. There exist no circular trajectories for $a > r$, exactly one circular trajectory for $a = r$, and two circular trajectories for $a < r$.

Parametric equations of the surface defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = -(a + r\cos 2\pi u)\sin 2\pi v\cos 2\pi v + r\sin 2\pi u\sin 2\pi v$$
$$y(u,v) = (a + r\cos 2\pi u)\cos 2\pi v$$
$$z(u,v) = (a + r\cos 2\pi u)\sin^2 2\pi v + r\sin 2\pi u\cos 2\pi v.$$

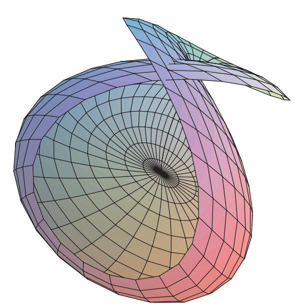
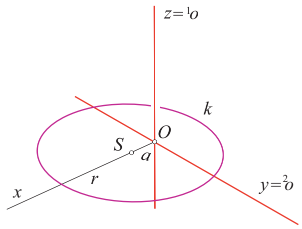


Fig. 12: Two-axial surface of revolution of spherical type - cyclical toroidal

3.7 Group IIB2 - Cyclical general surfaces

The basic circle $\mathrm{k}(S,r)$ is not located in the plane determined by the two axes of revolution; let it be located in the coordinate plane xy, then the vector equation of the circle for $u \in [0,1]$ is in the form

$$r(u) = (a + r\cos 2\pi u, r\sin 2\pi u, 0, 1).$$



Number of circles as trajectories of the points on the basic circle depends on the superposition of the basic circle and the axis of revolution ${}^1o$, i.e. on the relation of parameters a and r, in the same way as it was in the previous type of surfaces.

Parametric equations of the surface defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = (a + r\cos 2\pi u)\cos^2 2\pi v - r\sin 2\pi u \sin 2\pi v \cos 2\pi v$$

$$y(u,v) = (a + r\cos 2\pi u)\sin 2\pi v + r\sin 2\pi u \cos 2\pi v$$

$$z(u,v) = -(a + r\cos 2\pi u)\sin 2\pi v \cos 2\pi v + r\sin 2\pi u \sin^2 2\pi v.$$

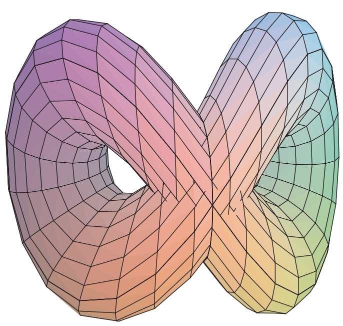


Fig. 13: Two-axial surface of revolution of spherical type - cyclical general

4 Two axial surfaces of revolution of Euler type

Two-axial revolution determined by skew axes ${}^1o/{}^2o$ is a general Euler revolution. Let us locate the axis ${}^1o$ on the coordinate axis z, and axis ${}^2o$ parallel to the coordinate axis x. Analytic representation of the two-axial surface of revolution of the spherical type determined on the region $\Omega \subset R^2$, with the basic curve represented by the vector equation

$$\mathbf{r}(u) = (x(u), y(u), z(u), 1)\,,\; u \in [o,1]$$

is in the form

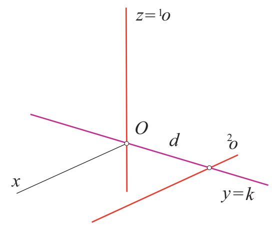
$$x(u,v) = x(u)\cos 2\pi v - y(u)\sin 2\pi v$$

$$y(u,v) = x(u)\sin 2\pi v \cos 2\pi v + y(u)\cos^2 2\pi v - z(u)\sin 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = x(u)\sin^2 2\pi v + y(u)\sin 2\pi v \cos 2\pi v + z(u,v)\cos 2\pi v - d\sin 2\pi v.$$

4.1 Ruled conical surfaces - group IIIA2

The basic line k is intersecting to both axes of revolution, let it be located on the coordinate axis y with the vector equation $\mathbf{r}(u) = (0, a\,u, 0, 1)$, $u \in [0,1]$.



Surface is of Möbius type (Fig. 14) and parametric equations defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = -a\,u\sin 2\pi v$$

$$y(u,v) = a\,u\cos^2 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = a\,u\sin 2\pi v\cos 2\pi v - d\sin 2\pi v.$$

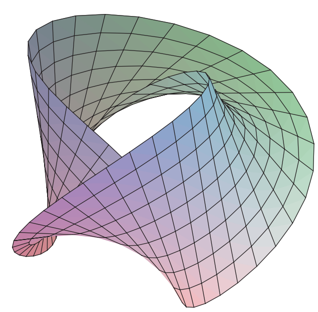
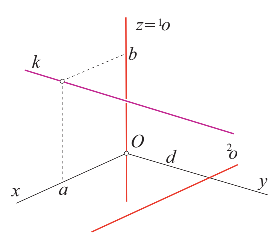


Fig. 14: Two-axial surface of revolution of Euler type - ruled conical

4.2 Ruled hyperbolical surfaces - group IIIA3

The basic line k is skew to both axes of revoluton and let it be parallel to the coordinate axis y, then its vector equation has the form $r(u) = (a, c\,u, b, 1)$, $u \in [0,1]$.



Parametric equations of the surfaces defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = a\cos 2\pi v - c\,u\sin 2\pi v$$

$$y(u,v) = a\sin 2\pi v\cos 2\pi v + c\,u\cos^2 2\pi v - b\sin 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = a\sin^2 2\pi v + c\,u\sin 2\pi v\cos 2\pi v + b\cos 2\pi v - d\sin 2\pi v.$$

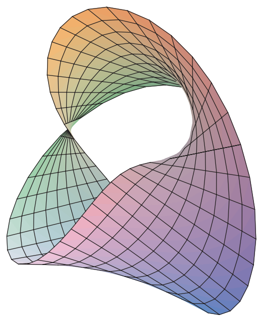


Fig. 15: Two-axial surface of revolution of Euler type - ruled hyperbolical

4.3 Ruled composite surfaces - group IIIA4

The basic line k is in different superposition to the two axes of revolution. From the similar three subgroups of ruled composite surfaces as there were presented for the two-axial surfaces of revolution of spherical type, some examples of specific representatives are chosen in illustration figure 16.

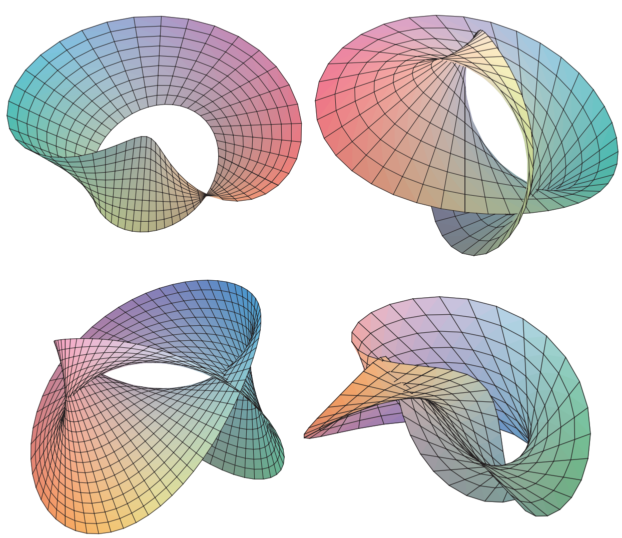
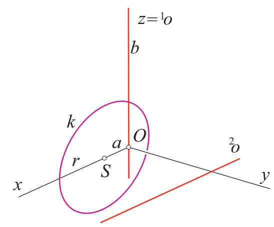


Fig. 16: Two-axial surfaces of revolution of Euler type - ruled composite

4.4 Cyclical general surfaces - group IIIB2

a) The basic circle $k(S,r)$ is located in the coordinate plane xz, and its equation is given in the form

$$\mathbf{r}(u) = (a + r\cos 2\pi u, 0, r\sin 2\pi u, 1),\; u \in [0,1].$$



Parametric equations of the surface defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = (a + r\cos 2\pi u)\cos 2\pi v$$

$$y(u,v) = (a + r\cos 2\pi u)\sin 2\pi v\cos 2\pi v - r\sin 2\pi u\sin 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = (a + r\cos 2\pi u)\sin^2 2\pi v + r\sin 2\pi u\cos 2\pi v - d\sin 2\pi v.$$

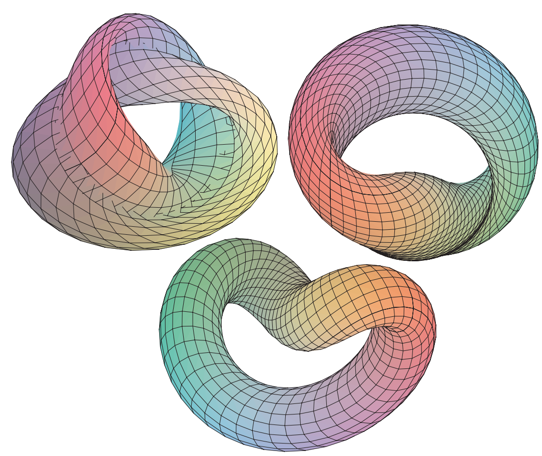
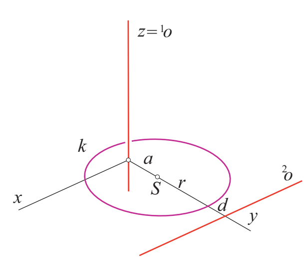


Fig. 17: Two-axial surfaces of revolution of Euler type - cyclical general a)

b) The basic circle $k(S,r)$ is located in the coordinate plane xy, and its equation is

$$\mathbf{r}(u) = (r\cos 2\pi u,\; a + r\sin 2\pi u, 0, 1), u \in [0,1].$$



Parametric equations of the surface (Fig. 18) defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = r\cos 2\pi u \cos 2\pi v - (a + r\sin 2\pi u)\sin 2\pi v$$

$$y(u,v) = r\cos 2\pi u \sin 2\pi v \cos 2\pi v + (a + r\sin 2\pi u)\cos^2 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = r\cos 2\pi u \sin^2 2\pi v + (a + r\sin 2\pi u)\sin 2\pi v \cos 2\pi v - d\sin 2\pi v.$$

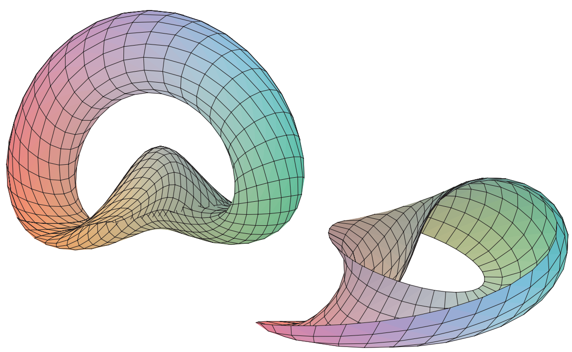


Fig. 18: Two-axial surfaces of revolution of Euler type - cyclical general b)

The number of circular trajectories of the points on the basic circle depends on the superposition of the basic circle and the axis of revolution ^1o, i.e. on the relation of parameters a and r. There can be no circular trajectories for $a > r$, exactly one circular trajectory for $a = r$, and two circular trajectories for $a < r$. Special surfaces can be modelled by setting $a = 0$, with one double circular trajectory.

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