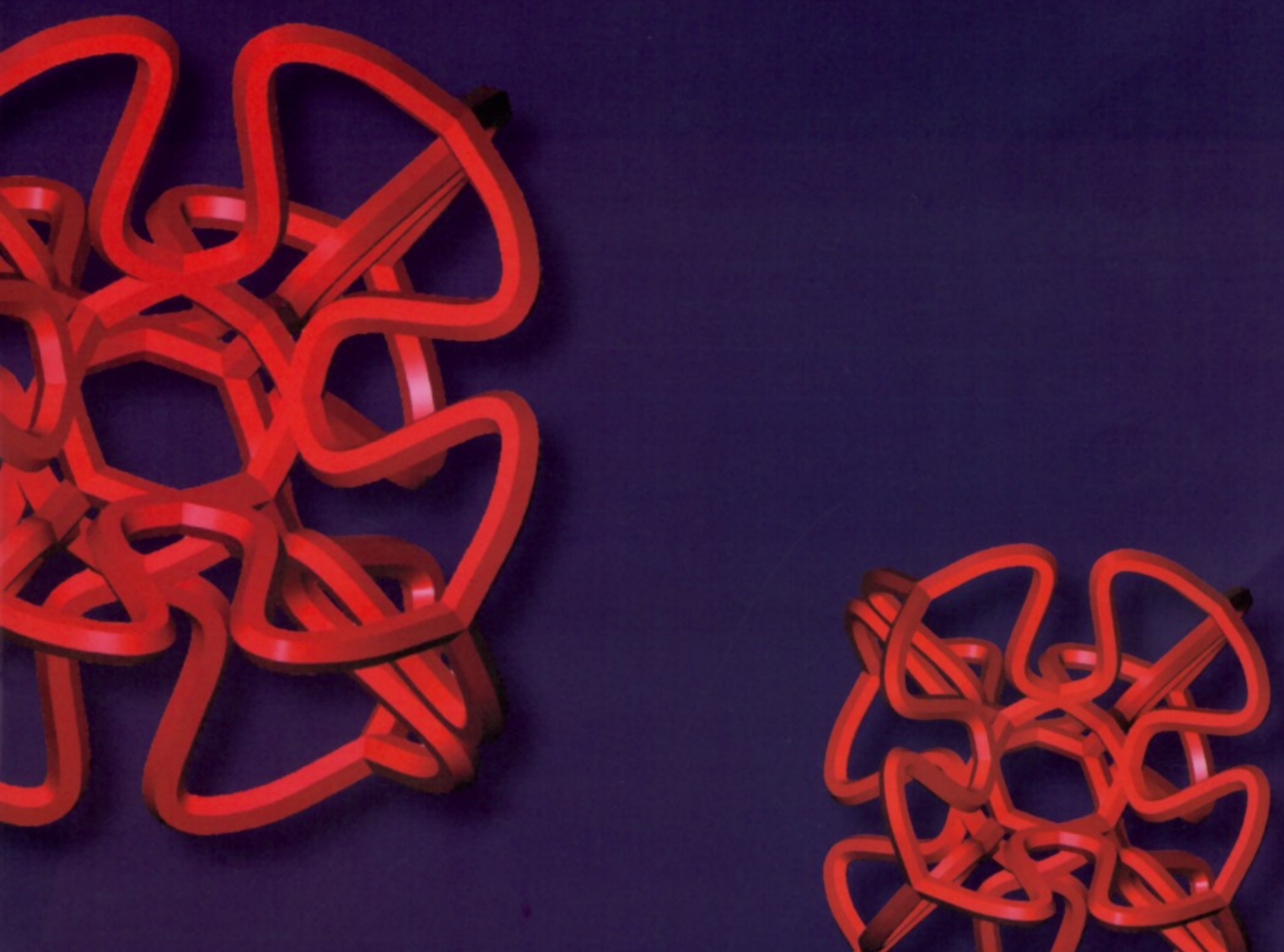
No. 9
ZAGREB, 2005

KoG

SCIENTIFIC-PROFESSIONAL JOURNAL OF
CROATIAN SOCIETY FOR CONSTRUCTIVE GEOMETRY AND COMPUTER GRAPHICS



KoG

Offical publication of the Croatian Society for Constructive Geometry nad Computer Graphics publishes scientific and professional papers from the fields of geometry, appplied geometry and computer graphics.

Founder and Publisher

Croatian Society for Constructive Geometry nad Computer Graphics

Design

Miroslav Ambruš-Kiš

Layout

Sonja Gorjanc, Ema Jurkin

Cover Illustration

S. Mick, O. Röschel, the figure from the paper *Frameworks Generated by the Octahedral Group*

Print

"O-tisak", d.o.o., Zagreb

URL address

http://www.hdgg.hr/kog

Edition

250

Published annually

Guide for authors
Please see the page 76.

KoG is cited in: Mathematical Review, MathSci, Zentralblatt für Mathematik

---

This issue has been financially supported by The Ministry of Science, Education and Sport of the Republic of Croatia.

ISSN 1331–1611

KoG

No. 9

Zagreb, 2005

SCIENTIFIC AND PROFESSIONAL JOURNAL OF
CROATIAN SOCIETY FOR CONSTRUCTIVE GEOMETRY AND COMPUTER GRAPHICS

CONTENTS

REVIEWS

ORIGINAL SCIENTIFIC PAPERS

PROFESSIONAL PAPERS

ISSN 1331–1611

KoG

BROJ 9
Zagreb, 2005

ZNANSTVENO-STRUČNI ČASOPIS
HRVATSKOG DRUŠTVA ZA KONSTRUKTIVNU GEOMETRIJU I KOMPJUTORSKU GRAFIKU

SADRŽAJ

PREGLEDNI ZNANSTVENI RADOVI

ORIGINALNI ZNANSTVENI RADOVI

STRUČNI RADOVI

Review
Accepted 22. 12. 2005.

SYBILLE MICK
OTTO RÖSCHEL

Frameworks Generated by the Octahedral Group

Dedicated to Prof. Hans VOGLER on the occasion of his 70^{th} birthday

Frameworks Generated by the Octahedral Group

ABSTRACT

The paper is devoted to the generation of frameworks of polyhedra as ornaments of the octahedral group O. An hierarchical block structure is used to implement the action of O in a CAD-package. The framework is generated by a starting (prismatic) rod as the motif. We will provide a range of examples and discuss the symmetry of the corresponding ornaments.

Key words: Polyhedra, Octahedral Group, Frameworks, Geometric Modeling in CAD-packages

MSC 2000: 51N05, 68U07, 97D30, 97U70

Strukture nastale pomoću oktaedarske grupe

SAŽETAK

Članak je posvećen nastajanju poliedarskih struktura pomoću ornamenata oktaedarske grupe O. Hijerarhijska blok−struktura upotrebljena je za primjenu djelovanja grupe O u CAD-paketu. Struktura nastaje pomicanjem (prizmatičnog) štapa kao motiva. Prikazati ćemo niz primjera i razmatrati simetriju odgovarajućih ornamenata.

Ključne riječi: poliedar, oktaedarska grupa, struktura, geometrijsko modeliranje u CAD-paketima

1 Introduction

Any polyhedron of the 3-dimensional Euclidean space determines a group of symmetry G. Its elements are direct automorphic displacements of the polyhedron. We start with an object M, called *motif*. The orbit of M with respect to the group G is called *ornament* (G,M) *with motif* M.

There are many interesting publications with fascinating figures dealing with such ornaments (see [1]-[9]). We will present an approach to teach regular polyhedra and their ornaments even for undergraduates. It is an interesting topic to visualize the action of G with CAD-packages (see [10], [11]). We will work with an hierarchic block structure as implementation of the corresponding group of symmetry G. Additionally, the design of the motives trains geometric modeling and needs familiarity with spatial congruence transformations. All considerations can be performed directly in the 3-dimensional space. Former troubles to draw these ornaments by hand are replaced by the use of a CAD-package.[1]

In this paper we restrict our examples to the group $G = O$: This group O is the *set of direct automorphic displacements of a regular octahedron* (or equivalently a cube).

2 Frameworks as Ornaments

In the introduction we have defined ornaments (G,M). We will present examples by using one prismatic rod as motif M. They will be defined by a regular polygon p as profile, which is extruded along its axis g (orthogonal to the plane of the profile).

We will present frameworks where the axes and the edges of the rods form closed rings. They are gained, if g has at least two intersecting neighbors (positions under G). This will happen in the following two cases[2] (which do not exclude each other):

Type A: The axis g of the rod is orthogonal to at least one rotational axis of the group G.

Type B: The axis g of the rod meets at least two rotational axes of the group G.

[1] The figures of this paper are produced in the CAD-packages AutoCAD and MicroStation.
[2] There are more possibilities if G is containing reflections, too.

3 The Octahedral Group

We will use some basic properties of the *octahedral group* O. We will give a list of the elements of O.[3] O contains the following 24 direct displacements (see figure 1):

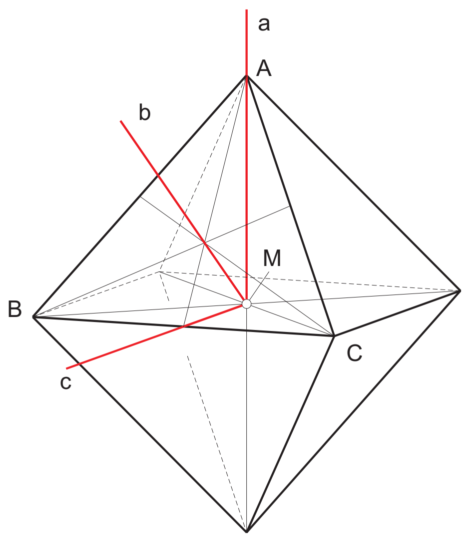


Figure 1: Some axes of rotation of the group O

- The identity id.
- Rotations about 4-fold axes a connecting opposite vertices of the octahedron.
- Rotations about 3-fold axes b connecting the centers of opposite faces.
- Rotations about 2-fold axes c connecting midpoints of opposite edges.

O is a normal subgroup of index 2 in the *full octahedral group* O_h. O_h is the union of O and the coset $O_\rho = O \circ (\rho)$, where (ρ) denotes a reflection about a plane of symmetry ρ of the octahedron.

An investigation of the structure of O leads to various implementations of the action of O in a CAD-package. We suggest to use the following hierarchical procedure, which implements O as a sequence of blocks (or models):

Motif = *Identity* - *Triple* - *Pair* - *Group* = **Ornament**

There the block *Triple* contains 3 rotated copies of the block *Identity* (axis of rotation is the 3-fold axis b of the face A,B,C of the octahedron).

The block *Pair* contains 2 rotated copies of the block *Triple* (axis of rotation is the 2 fold axis c - c contains the midpoint of the edge $[B,C]$).

The block *Group* contains 4 rotated copies of the block *Pair* (axis of rotation is the 4-fold axis a through the vertex A).

Our motif M is used as input into the block *Identity*. The resulting ornament is generated (as output) in the block *Group*. The following figure 2 displays the situation for an arrow as motif M.

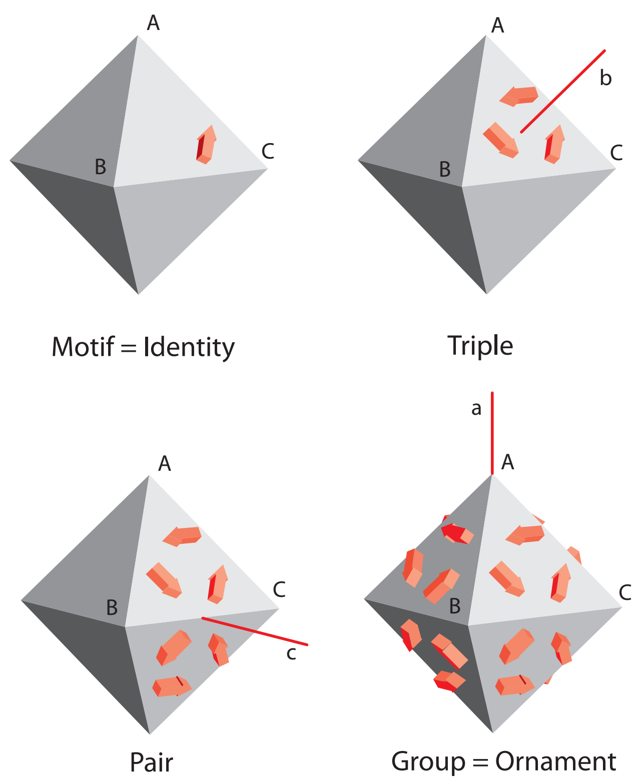


Figure 2: The blocks of our implementation of the ornaments (M,O)

These preparations allow to generate various ornaments by the choice of suitable motives.[4]

4 Interlinked polygons as ornaments of case A

Firstly, we present two nice examples for case A. We will use a cylindric rod as the motif. According to section 2 its axis g is chosen orthogonal and skew to one of the 3- or 4-fold axis of the octahedron.

[3] For more details see the textbooks [1], [3], [5], [6], [7].

[4] The reader is invited to experiment with different motives. Even without deeper geometric considerations there can be gained fascinating ornaments.

Example A1 (Interlinked triangles):

The axis g of the rod is orthogonal to the 3-fold axis b (see figure 3). The ornament is gained by rotating a rod along an edge of the octahedron with respect to the axis b (angle $2\pi/9$).

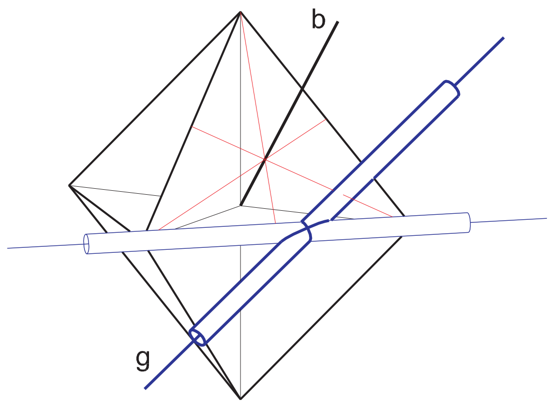


Figure 3: Motif for figures 4 and 5

The block *Triple* contains 3 rotated copies of this motif, which form a triangle. The intersection of two neighbor rods (axis g and g^* respectively) splits into two ellipses. Therefore it is quite natural to use a miter cut in order to get fitting rods. This miter cut follows the plane of symmetry σ of g and g^* through the center M of the polyhedron. The following figure 5 shows the result with an inscribed sphere, which is used to hide some parts in the back and to highlight the structure of the framework. There are 8 triangles, each interlinked with 3 neighbors. It needs some attempts to gain a solution without self-intersections. Figure 4 displays the necessary conditions: In order to get the ornament the triangle (included in the block *Triple*) is rotated about the 2-fold axis c. To guarantee that the original and the rotated triangle are interlinked, the two-fold axis c has to intersect the initial triangle in an inner point. The radius of the rod has to be smaller than the distance of g and of c (see figure 4). Based on these ornaments there exist some art objects (see H.S.M. COXETER [4]).

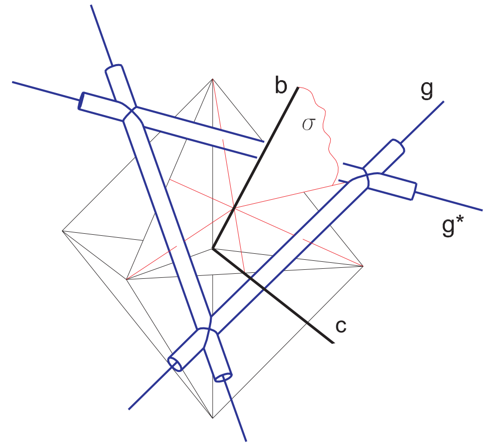


Figure 4: The block *Triple*

Figure 5: Interlinked triangles

Example A2 (Interlinked Quadrangles):

The axis g of the rod is orthogonal to the 4-fold axis a. A rod chosen accordingly will deliver the ornament displayed in figure 6.

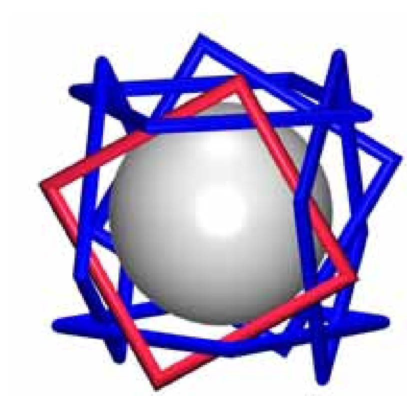


Figure 6: Interlinked squares

5 Frameworks as ornaments of case B

Secondly, we turn to case B and we will use prismatic rods. The axis g of the rod meets two (or more) axes of the octahedron. We generate a prismatic rod R by extrusion of a regular hexagon along the axis g. As in case A the rod R should fit its neighbors. Additionally, we want to get intersecting edges for intersecting prismatic rods.

We consider an $n-$fold axis z intersecting g at a point Z (see figure 7). g^* is the axis of the neighbor rod R^* (rotation ρ of g about the axis z through the angle $2\pi/n$). This rotation can be generated as composition of two reflections $(\sigma)\circ(\varepsilon)$: The first plane is $\varepsilon := [g,z]$, the second plane σ is the plane of symmetry of g and g^* through z. We get intersecting edges on R and R^* if the rotation ρ and the reflection in the plane σ yield the same rod R^*. This is guaranteed if the rod R is plane symmetric with respect to the plane ε (σ is used as the plane of a miter cut). As the $n-$fold axis z contains the center M of the polyhedron we have $\varepsilon = [g,M]$. Hence ε is independent from the axis z. The rod R has a second neighbor R^{**} (rotated copy of R, rotation about z through the angle $-2\pi/n$). As before, we put a second miter cut through R. Figure 8 shows the result. In general the two miter cuts are different - in the case of a 2-fold axis ($n = 2$) the two cuts are coincident.

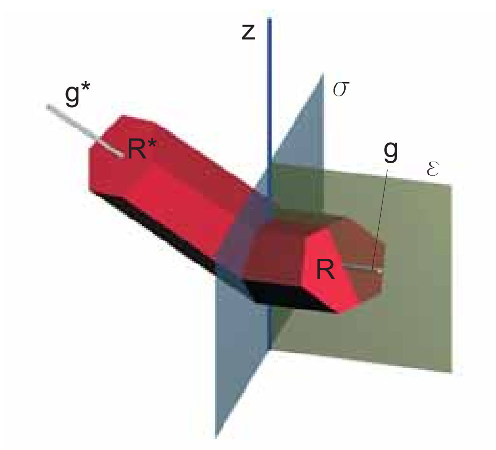


Figure 7: Fitting edges

g meets a second axis $\overline{z}$ of the octahedron. Therefore there exist two further miter cuts (fitting the new neighbors). $\overline{z}$ is a line in ε, too. The symmetry of the rod R with respect to ε guarantees that the edges of R and their new neighbors are intersecting in points of the corresponding miter cut. To get ornaments with closed rings we chose as motif a rod with axis $g = [1,2]$ where 1 and 2 are points of intersection from g and two axes of the octahedron. The planes of the miter cuts are planes through 1 and 2, respectively (see figure 9).

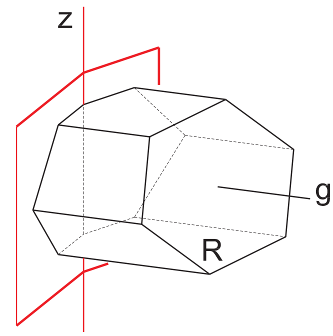


Figure 8: The rod with 2 miter cuts

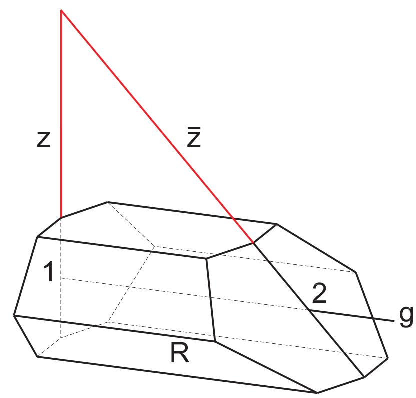


Figure 9: The rod with 4 miter cuts

Remarks: a) As a consequence of these considerations we will use **prismatic rods** R (axis g) **symmetric with respect to the plane** $\varepsilon := [g,M]$ henceforth.

b) If this reflection of the rod R is an automorphic reflection of the octahedron, we gain ornaments from the full octahedral group O_h.[5]

In order to get a structured presentation for examples in case B we will give a complete list of different types of these planes of symmetry ε with respect to the octahedron. As ε has to contain at least two axes of the octahedron, we study the bundle of the rotational axes. All plane sections of this bundle are projectively equivalent. We will use the section with a plane π orthogonal to the 2-fold axis c_1 (see figure 10). The points A_1, A_2 and A_3 denote the intersections of the 4-fold axes, B_1, B_2, B_3, B_4 and $C_1, C_2, C_3, C_4, C_5, C_6$ those with 3-fold and 2-fold axes, respectively. The points A_3, B_3, B_4 and C_6 are points at infinity.

[5] Our next examples show, that there are examples for frameworks as ornaments in the group O which do not belong to the full group O_h even if our motif has a reflection symmetry.

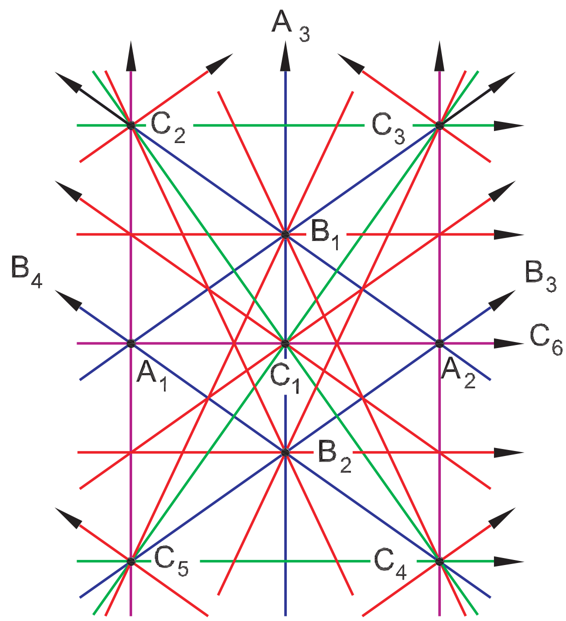


Figure 10: The configuration of the intersections of the axes from O in π

The different types of planes of symmetry ε have lines of intersection in π, which are highlighted by different colors in figure 10:

B1. ε contains exactly two axes of the octahedron (red): ε is spanned exactly by a 3-fold axis (e.g. b_2) and a 2-fold axis (e.g. c_2) which do not belong to one triangular face of the octahedron. ε is no plane of symmetry for the octahedron!

B2. ε contains exactly three axes of the octahedron (green): ε contains three 2-fold axes (e.g. c_1, c_2, c_4). ε is no plane of symmetry for the octahedron!

B3 ε contains exactly four axes of the octahedron: Here we have two distinct cases:

B3a. ε contains two 4-fold and two 2-fold axes of the octahedron (blue, e.g. a_1, c_1, a_2, c_6). ε is a plane of symmetry containing 4 edges of the octahedron.

B3b. ε contains one 4-fold axis, two 3-fold axes and one 2-fold axis of the octahedron (magenta, e.g. a_3, b_1, c_1, b_2). ε is a plane of symmetry of the octahedron.

For B3a and B3b the plane ε is a plane of symmetry of the octahedron. Hence, in these cases our procedure yields ornaments from the full octahedral group O_h. Now we are going to present some *examples*:

[6] If there is only one axis of the same type we omit the indices of the axis.

Case B1:

g intersects a 3-fold axis b and a 2-fold axis c of the octahedron, which do not belong to one face of the octahedron (see figure 11)[6]. The two axes b and c are orthogonal lines. Therefore it is natural to take our line g on one side 12 of a quadrangle (green) with vertices on b and c. The corresponding ornament is displayed in figure 12. It consists of 12 "half - quadrangles", which form closed loops of 6 rods. In order to get its structure they are displayed in different colors.

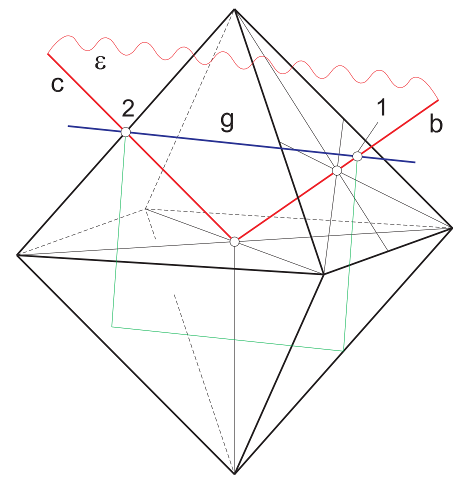


Figure 11: g meeting a 2- and a 3-fold axis

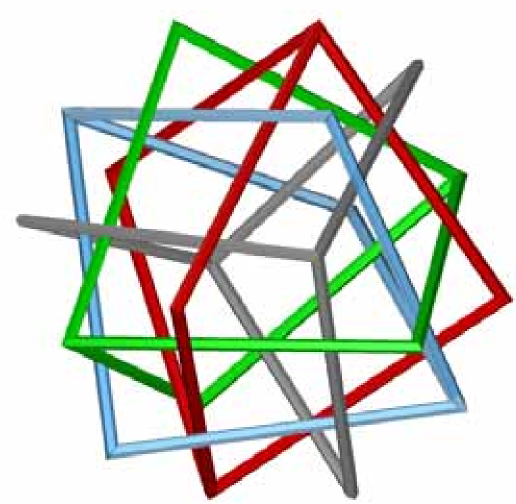


Figure 12: Ornament

Case B2:

g intersects two 2-fold axes c_1 and c_2 of one face of the octahedron (figure 13). The plane $\varepsilon = [g, M] = [c_1, c_2]$ is orthogonal to a 3-fold axis b. As a first choice we take the axis g_1 like in figure 13. The ornament is displayed in figure 14. It consists of four *triangular stars*. A very special version is displayed in figure 15: There the axis g_2 of the rod is orthogonal to the 2-fold axis c_2. Then the two rods meeting at c_2 have the same axis - hence the *star* degenerates into a triangle. The ornament consists of 4 interlinked triangles (each is built up from 6 copies of the motif). Again we gain an ornament known from art (see H.S.M. COXETER [4]).

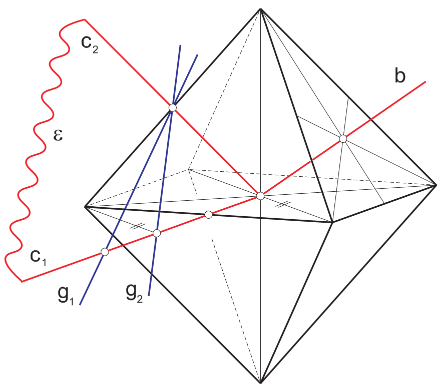


Figure 13: g_1 and g_2 intersecting two 2-fold axes

In this case we have a third 2-fold axis c_3 in ε. The point of intersection on g and c_3 can be an outer or an inner point of 12. The latter case gives ornaments with self-intersections (no example displayed).

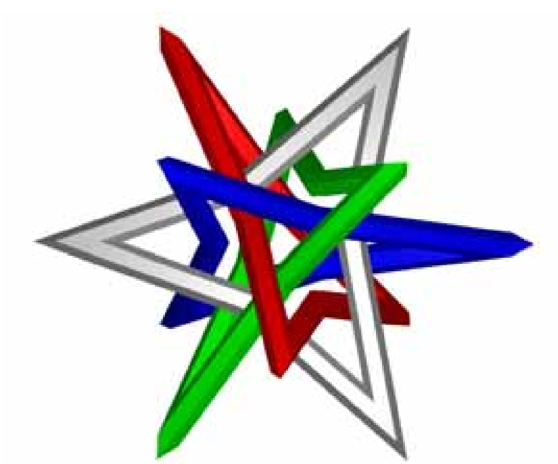


Figure 14: Triangular stars

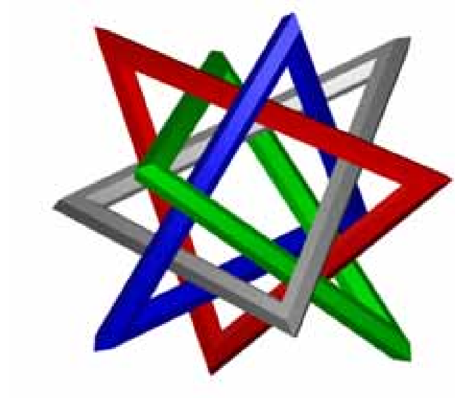


Figure 15: Interlinked triangles

Case B3a:

g is a line in the plane $\varepsilon[a_1, c_1, a_2, c_6]$ (see figure 16). Our rod is restricted to two of these axes (points $1, 2$). If the initial rod follows an edge on the octahedron from the vertex to its midpoint we gain a framework ornament of the octahedron. If one of the other axes intersects $[1, 2]$ in an inner point we get self-intersecting ornaments. The following example (figure 17) is gained by using $1, 2$ as end-points on a 2-fold and a 4-fold axis. The second 2-fold and the second 4-fold axis intersect $[1, 2]$ in inner points. Therefore we get 2-fold and 4-fold self-intersections of the ornament. Figure 18 displays another example of this case with 2-fold self-intersections: The end-points 1 and 2 are chosen on the 4-fold axes a_1 and a_2 (see figure 16).

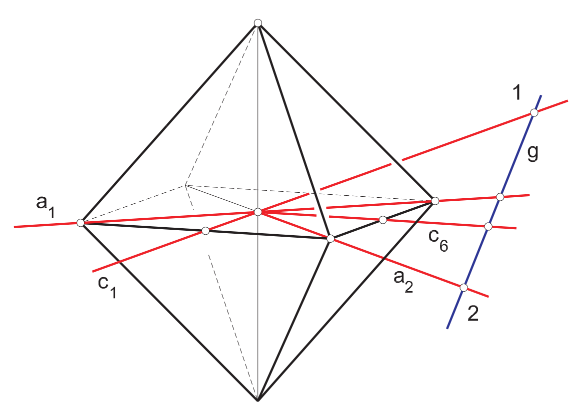


Figure 16: g for case B3a

Figure 17: Example with 2-fold and 4-fold self-intersections

Figure 18: Example with 2-fold self-intersections

Case B3b:

g is a line in the plane $\varepsilon[a_3, b_1, c_1, b_2]$ (see figure 19). We choose the points $1, 2$ on the 4-fold axis a_3 and the 3-fold axis b_1. The rod g_1 is parallel to the second 3-fold axis b_2 in ε, g_2 is parallel to the 2-fold axis c_1 (see figure 19).

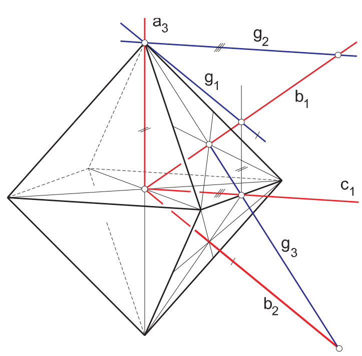


Figure 19: g_1, g_2 and g_3 for case B3b

The corresponding ornaments are the Rhombic Dodecahedron (figure 20) and the famous *Stella Octangula* by J. KEPLER (figure 21). The latter is a compound of two congruent regular tetrahedra which are displayed in different colors.

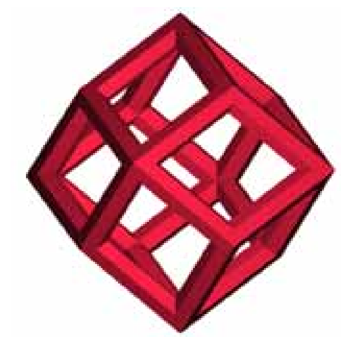


Figure 20: Rhombic Dodecahedron

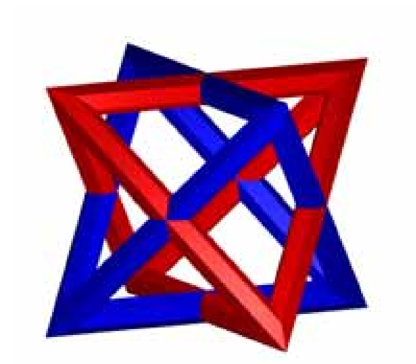


Figure 21: Stella Octangula

There are various cases with further intersections of the rods. Figure 22 displays an example with 2-fold self-intersections. The initial rod has the axis g_3 (see figure 19). It is orthogonal to the 3-fold axis b_1.

Figure 22: Example for case B3b

Of course, the prismatic rod can be replaced by any other motif. If we extrude the hexagon along a curved path we gain further ornaments. Figure 23 displays one example. In order to get fitting edges again we have to guarantee the existence of the corresponding and fitting miter cuts. This implies that the motif has to be symmetric with respect to the corresponding plane of symmetry ε (at least in the neighborhood of the end-points of the path). If the path is in the plane ε this condition holds automatically.

Figure 23: Curved rods

6 Conclusion

We have presented a range of examples generated as ornaments of the octahedral group O. The action of the group O was implemented by a hierarchical block structure. Technically speaking, this was an ideal tool to keep track of the generation of the ornaments. We have started with prismatic rods as motives and have used miter cuts in order to get fitting edges.

In the very last figure 23 we replaced the prisms by other geometric objects. Further generalizations of this step (with a spatial curve path) will reveal ornaments which can be viewed as objects between geometry and art.

Additionally, the presented geometric considerations and methods can as well be applied to symmetric groups of other polyhedra, too. In [10] we have presented some ornaments of the icosahedral group. A paper devoted to frameworks as special ornaments of the icosahedral group is in preparation.

The topic is an ideal training field for spatial transformations. The key to the systematic approach is the use of professional CAD-packages against the backdrop of some geometric knowledge. We do hope that the beauty of the ornaments will inspire some of the readers to make their own experiments in this fascinating field.

References

[1] BÖHM, J., QUAISSER, E., *Schönheit und Harmonie geometrischer Formen*, Akademie Verlag, Berlin 1991.

[2] BILINSKI, S., *Über die Rhombenikosaeder*, Glasnik Math. **15**, 251 - 262 (1960).

[3] COXETER, H. S. M., *Regular Polytopes*, Dover, New York 1973.

[4] COXETER, H. S. M., *Symmetrical Combinations of Three or Four Hollow Triangles*, Math. Intelligencer **16/3**, 25 - 30 (1994).

[5] CROMWELL, P. R., *Polyhedra*, Cambridge Univ. Press, Cambridge 1997.

[6] FEJES - TOTH, L., *Regular Figures*, Int. Series of Monogr. in Pure and Appl. Math. 48, Pergamon, Oxford 1964.

[7] GRÜNBAUM, B., *Convex Polytopes*, Pure and Applied Mathematics vol. XVI, Wiley, London - New York - Sidney, 1967.

[8] HOHENBERG, F., *Das abgestumpfte Dodekaeder des Archimedes und seine projektiven Eigenschaften*, Sber. Österr. Akad. Wiss II, Bd. 192, 143 - 159 (1983).

[9] JAMNITZER, W., *Perspectiva Corporum Regularium*, Nürnberg 1568.

[10] MICK, S., RÖSCHEL, O., *Ausgewählte Beispiele für Ornamente und Stabmodelle der Ikosaedergruppe*, Informationsblätter Geometrie (IBDG), (in print).

[11] WILTSCHE, A., *Die Kunst ein 3D-CAD-Programm zu erlernen (part 1 and 2)*, Informationsblätter Geometrie (IBDG), **22/2**, 12 - 21 (2003) and **23/1**, 21 - 33 (2004).

Further references are listed in the textbooks [1], [3], [5], [6] and [7].

Sybille Mick

email: mick@tugraz.at

Otto Röschel

email: roeschel@tugraz.at

Institute of Geometry TU Graz
Kopernikusgasse 24
A-8010 Graz, Austria

Original scientific paper
Accepted 20. 12. 2005.

DANIELA VELICHOVÁ

Two-Axial Surfaces of Revolution

Two-Axial Surfaces of Revolution

ABSTRACT

Special class of surfaces, two-axial surfaces of revolution created by the Euclidean metric transformation of a simultaneous revolution about two different axes, is presented in the paper. Three specific subclasses of surfaces are classified with respect to the superposition of the two axes ^1o, ^2o of revolution. There are defined several types of two-axial surfaces of revolution specifying the type of the surface basic figure and its position to the axes of revolution.

Key words: composite revolution, two-axial revolution, generalised surfaces of revolution

MSC 2000: 14J26, 15A04, 53A05

Dvoosne rotacijske plohe

SAŽETAK

U radu je prikazana posebna klasa ploha - dvoosne rotacijske plohe - nastala pomoću euklidske metričke transformacije koja se sastoji od dviju istodobnih rotacija oko dvije različite osi. Prema položaju rotacijskih osi ^1o i ^2o razlikuju se tri podklase takvih ploha. Daljnje razvrstavanje na nekoliko tipova ploha provedeno je prema vrsti osnovne figure (čijom rotacijom ploha nastaje) i njenom položaju prema rotacijskim osima.

Ključne riječi: složena rotacija, dvoosne rotacije, poopćene rotacijske plohe

1 Introduction

Composition of two revolutions about two different axes in the space determined as two-axial revolution is a metric transformation. Let $^1o, ^2o$ be the two axes of two revolutions. There can be characterised three distinguished subgroups of the general revolutionary movements in the space composed from the two revolutions, with respect to the superposition of the two axes:

I. cycloidal movement with parallel axes $^1o \parallel {}^2o$,

II. spherical movement with intersect axes $^1o \times {}^2o$,

III. general Euler revolution with skew axes $^1o / {}^2o$.

Considering the two-axial revolutionary movement, analytic representation of this linear 1-parametric transformation is a regular square matrix of rank 4, with entries in the form of real functions of one real variable defined on an interval of the real numbers

$$\mathrm{T}(v) = (a_{ij}(v)), \quad \text{for} \quad i, j = 1, 2, 3, 4, \quad v \in R.$$

This matrix function can be analytically determined as the product of two square matrix functions representing the separate revolutions about given axes. For the sake of easy formulas, special positions of axes of revolution are chosen, on coordinate axes or parallel to some of the coordinate axes.

Two-axial surfaces of revolution can be created by the two-axial revolutionary movement of a basic curve k defined analytically by a vector function

$$\mathbf{r}(u) = (x_1(u), x_2(u), x_3(u), 1), \quad u \in R.$$

Vector representation of the surface patch defined on the region $\Omega \subset R^2$ is the matrix product

$$\mathbf{p}(u,v) = \mathbf{r}(u).\mathrm{T}(v) = \left(\sum_{i=1}^{4} x_i(u) a_{i1}(v), \sum_{i=1}^{4} x_i(u) a_{i2}(v), \sum_{i=1}^{4} x_i(u) a_{i3}(v), 1 \right).$$

There can be distinguished three basic subgroups of two-axial surfaces of revolution according to the type of the generating two-axial revolutionary movement, i.e. according to the superposition of the axes of revolutions $^1o, ^2o$:

I. $^1o \parallel {}^2o$, surfaces of cycloidal type

II. $^1o \times {}^2o$, surfaces of spherical type

III. $^1o / {}^2o$, surfaces of Euler type.

With respect to the type of the basic curve, we recognise the following types of the two-axial surfaces of revolution:

A. ruled surfaces, for a line (line segment) as basic curve

B. cyclical surfaces, for a circle (circular arc) as basic curve

C. non-specified surfaces, for all other basic curves.

Ruled surfaces can be further classified according to the superposition of the basic line and the two axes of revolution as:

1. cylindrical - basic line is parallel to both axes (for type I exclusively)
2. conical - basic line is intersecting to both axes
3. hyperbolical - basic line is skew to both axes
4. composite - basic line is in different superposition to the two axes.

Cyclical surfaces can be further classified according to the superposition of the plane of the basic circle and the two axes of revolution as:

1. toroidal - basic circle is located in the plane formed by the two axes of revolution (for types I and II only possible)
2. general - basic circle is located in the general plane with respect to the two axes of revolution.

Let us restrict our considerations to the revolutionary movements about two different axes with the same angular velocities. Parametric equations of the defined two-axial surfaces of revolution are derived and several illustrations of the special representatives of all subclasses and types are presented in the following.

Surfaces of Euler type show no symmetry, as there exists no plane formed by the axes of revolution ${}^1o, {}^2o$, unless the basic figure is located in a special "symmetric position" with respect to the two axes. There exists at least one plane of symmetry, $\sigma = {}^1o{}^2o$, for surfaces of spherical and cycloidal types. Special position of the basic figure and axes of revolution may result in existence of at least one more plane of symmetry, perpendicular to the plane σ and passing through one of the axes. Some other planes of symmetry may occur as well, passing through the axis ${}^1o$. Plane σ and one-parametric system of planes perpendicular to σ form symmetry planes of surfaces of cycloidal type -ruled cylindrical.

2 Two-axial surfaces of revolution of cycloidal type

Cycloidal movement is composed from two revolutions about parallel axes ${}^1o \parallel {}^2o$, at the distance $\left|{}^1o{}^2o\right| = d$, $d \neq 0$, with equal angles $\varphi = \psi$. For angles on interval $[0,2\pi]$ we receive the closed epicycloidal movement, for angles $\varphi = -\psi$ on interval $[0,2\pi]$ the closed hypocycloidal movement can be obtained. Locating the axis ${}^1o$ on the coordinate axis z and axis ${}^2o$ in the plane xz, we receive the matrix representation of the two-axial revolution - matrix of epicycloidal movement (composition of revolutions in the same directions)

$$\mathrm{T}(v) = \begin{pmatrix} \cos 4\pi v & \sin 4\pi v & 0 & 0 \\ -\sin 4\pi v & \cos 4\pi v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d(\cos 2\pi v - 1) & d\sin 2\pi v & 0 & 1 \end{pmatrix},$$

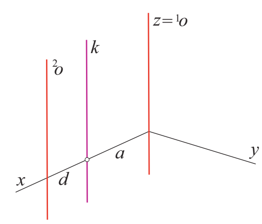
matrix of hypocycloidal movement (composition of revolutions in the opposite directions)

$$\mathrm{T}(v) = \begin{pmatrix} \cos 4\pi v & -\sin 4\pi v & 0 & 0 \\ \sin 4\pi v & \cos 4\pi v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d(\cos 2\pi v - 1) & d\sin 2\pi v & 0 & 1 \end{pmatrix}.$$

2.1 Ruled cylindrical surfaces - group IA1

The basic line k is parallel to both axes of revolution and it is located in the plane they form, i.e. in the coordinate plane xz, while the following metric relations are true

$$\left|{}^1o{}^2o\right| = d, \quad \left|{}^1ok\right| = a, \quad a \neq 0, \quad d \neq 0.$$



Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (a, 0, bu, 1).\mathrm{T}(v) =$$
$$= (a\cos 4\pi v + d(\cos 2\pi v - 1), \pm a\sin 4\pi v + d\sin 2\pi v, bu, 1).$$

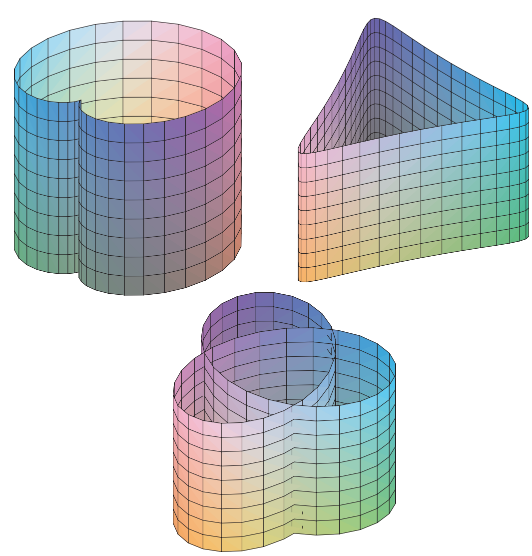
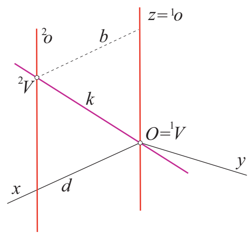


Fig. 1: Two-axial sufaces of revolution of cycloidal type - ruled cylindrical

2.2 Ruled conical surfaces - group IA2

The basic line k intersects both axes of revolution and is located in the plane they form, i.e. in the coordinate plane xz, while the following metric relations are true

$$|{}^1o{}^2o| = d, \quad k \cap {}^1o = {}^1V = (0,0,0,1)$$
$$k \cap {}^2o = {}^2V = (d,0,b,1), \quad b \neq 0, \quad d \neq 0.$$



Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (du,0,bu,1).\mathrm{T}(v) =$$
$$= (d\,u\cos 4\pi v + d(\cos 2\pi v - 1), \pm d\,u\sin 4\pi v + d\sin 2\pi v, bu, 1).$$

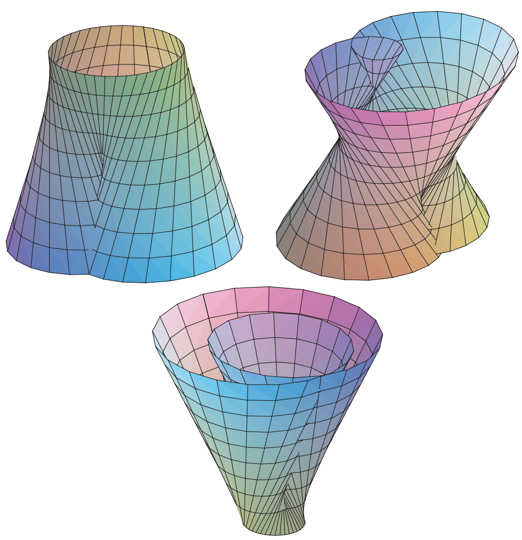
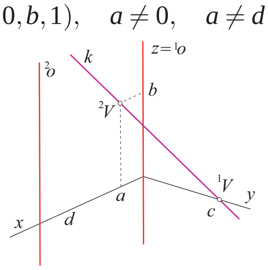


Fig. 2: Two-axial surfaces of revolution of cycloidal type - ruled conical

2.3 Ruled hyperbolical surfaces - group IA3

The basic line k is skew to both axes of revolution, while the following metric relations are true

$$|{}^1o{}^2o| = d, \quad k \cap \pi = {}^1V = (0,c,0,1), \quad c \neq 0, \quad d \neq 0$$
$$k \cap \nu = {}^2V = (a,0,b,1), \quad a \neq 0, \quad a \neq d, \quad b \neq 0.$$



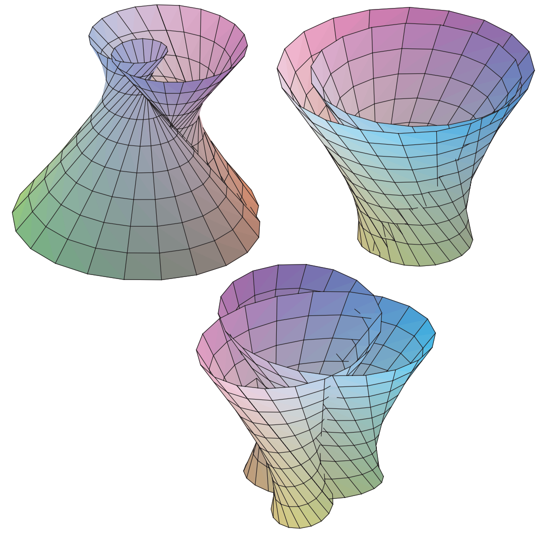
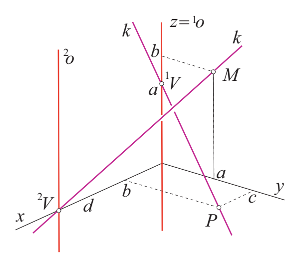


Fig. 3: Two-axial sufaces of revolution of cycloidal type - ruled hyperbolical

2.4 Ruled composite surfaces - group IA4

The basic line k is in different superpostion to both axes of revolution, it is intersecting to one axis and skew to the other one.



a) Let k be intersecting to ${}^1o$, skew to ${}^2o$, and let it intersects coordinate plane $\pi = xz$ in the point P

$$|{}^1o{}^2o| = d, \; k \cap {}^1o = {}^1V = (0,0,a,1), \; a \neq 0, \; d \neq 0$$
$$k \cap \pi = P = (b,c,0,1), \quad b \neq 0, \quad c \neq 0,$$

then the surface contains only one circle, the trajectory of the point ${}^1V$, and it has no planes of symmetry.

Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (bu, cu, a(1-u), 1).\mathrm{T}(v) =$$
$$(bu\cos 4\pi v \mp cu\sin 4\pi v + d(\cos 2\pi v - 1),$$
$$\pm bu\sin 4\pi v + cu\cos 4\pi v + d\sin 2\pi v, a(1-u), 1).$$

b) Let k be intersecting to ^2o and skew to ^1o, and let it intersects coordinate plane $\mu = zy$ in the point M

$$|^1o^2o| = d, \quad k \cap {}^2o = {}^2V = (d,0,0,1), \quad d \neq 0$$
$$k \cap \mu = M = (0,a,b,1), \quad a \neq 0, \quad b \neq 0.$$

Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (d(1-u), au, bu, 1).\mathrm{T}(v) =$$
$$(d(1-u)\cos 4\pi v \mp au \sin 4\pi v + d(\cos 2\pi v - 1),$$
$$\pm d(1-u)\sin 4\pi v + au\cos 4\pi v + d\sin 2\pi v, bu, 1).$$

Some representatives of this group of surfaces are illustrated in figure 4.

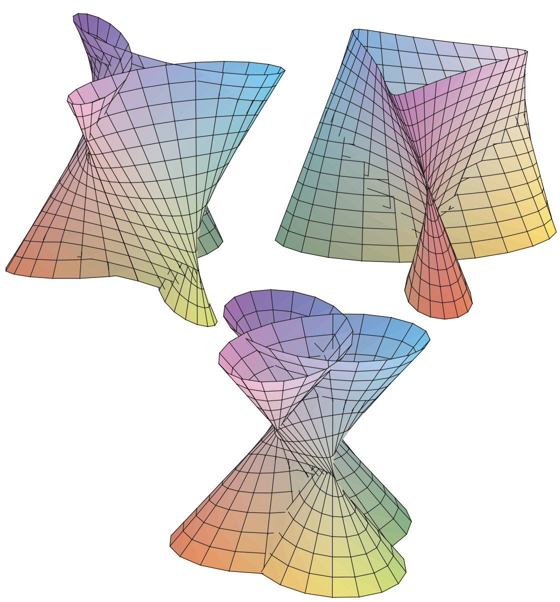
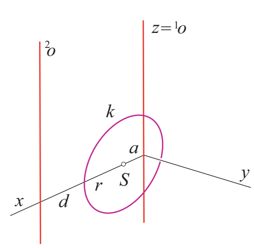


Fig. 4: Two-axial surfaces of revolution of cycloidal type - ruled composite

2.5 Cyclical toroidal surfaces - group IB1

The basic circle $k(S,r)$ is located in the plane xz, while the following metric relations are true

$$|^1o^2o| = d, \quad S \in x, \quad |S^1o| = a, \quad d \neq 0.$$



Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (a + r\cos 2\pi u, 0, r\sin 2\pi u, 1).\mathrm{T}(v),$$

while the separate coordinate functions are in the form

$$x(u,v) = (a + r\cos 2\pi u)\cos 4\pi v + d(\cos 2\pi v - 1)$$
$$y(u,v) = \pm(a + r\cos 2\pi v)\sin 4\pi v + d\sin 2\pi v$$
$$z(u,v) = r\sin 2\pi u.$$

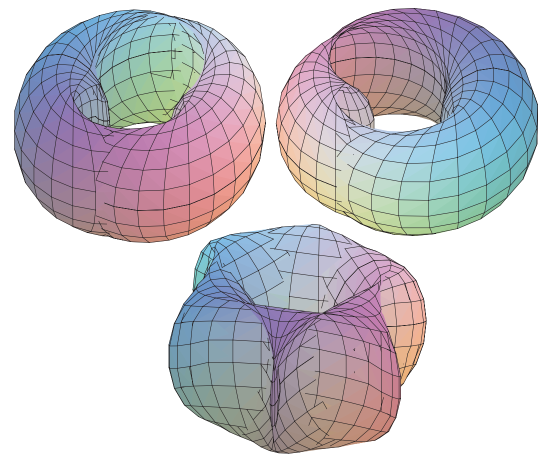
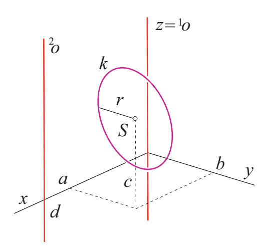


Fig. 5: Two-axial surfaces of revolution of cycloidal type - cyclical toroidal

2.6 Cyclical general surfaces - group IB2

The basic circle $k(S,r)$ can be located in the arbitrary plane, let it be the plane parallel to the plane yz, while the following metric relations are true

$$|^1o^2o| = d, \quad S = (a,b,c,1), \quad a \neq 0, \quad b \neq 0, \quad d \neq 0.$$



Vector function of the surface patch determined on $[0,1]^2$ is

$$\mathbf{p}(u,v) = (a, b + r\cos 2\pi u, c + r\sin 2\pi u, 1).\mathrm{T}(v),$$

while the separate coordinate functions are in the form

$$x(u,v) = a\cos 4\pi v \mp (b + r\cos 2\pi u)\sin 4\pi v + d(\cos 2\pi v - 1)$$
$$y(u,v) = \pm a\sin 4\pi v + (b + r\cos 2\pi v)\cos 4\pi v + d\sin 2\pi v$$
$$z(u,v) = c + r\sin 2\pi u.$$

Some representatives of the surfaces in this group can be seen in figure 6.

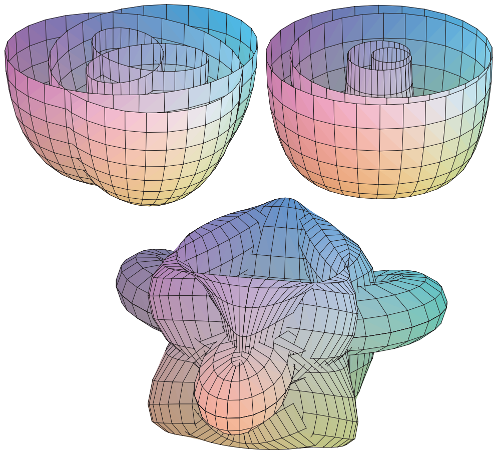


Fig. 6: Two-axial surfaces of revolution of cycloidal type - cyclical general

3 Two axial surfaces of revolution of spherical type

Two-axial revolution determined by intersecting axes ${}^1o \times {}^2o$ is a spherical movement. Let us locate the axis ${}^1o$ on the coordinate axis z, and axis ${}^2o$ on the coordinate axis y. Analytic representation of the two-axial surface of revolution of spherical type determined on the region $\Omega \subset R^2$, with the basic curve represented by the vector equation

$$\mathbf{r}(u) = (x(u), y(u), z(u), 1), \quad u \in R$$

is in the form

$$\mathbf{p}(u,v) = (x(u,v), y(u,v), z(u,v), 1), \quad (u,v) \in \Omega$$

where

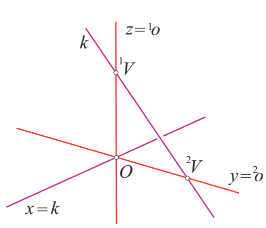
$$x(u,v) = x(u)\cos^2 2\pi v - y(u)\sin 2\pi v \cos 2\pi v + z(u)\sin 2\pi v$$

$$y(u,v) = x(u)\sin 2\pi v + y(u)\cos 2\pi v$$

$$z(u,v) = -x(u)\sin 2\pi v \cos 2\pi v + y(u)\sin^2 2\pi v + z(u)\cos 2\pi v.$$

3.1 Ruled conical surfaces - group IIA2

The basic line k is intersecting to both axes of revolution.



a) Let k be on the coordinate axes x, therefore intersecting both axes of revolution in their common point O, then surface does not contain any circle, and it has 3 planes of symmetry.

b) Let k be in the plane of axes of revolution, i.e. in the coordinate plane yz

$$k \cap {}^1o = {}^1V = (0,0,b,1), \quad k \cap {}^2o = {}^2V = (0,a,0,1)$$

$$a \neq 0, \quad b \neq 0$$

then the surface contains only one circle, the trajectory of the point ${}^1V$, and it has a unique plane of symmetry.

Parametric equations of the two surfaces (Fig. 7) defined on the region $[0,1]^2 \subset R^2$ are in the forms

$$x(u,v) = a u \cos^2 2\pi v$$

$$y(u,v) = a u \sin 2\pi v$$

$$z(u,v) = -a u \sin 2\pi v \cos 2\pi v$$

and

$$x(u,v) = -a u \sin 2\pi v \cos 2\pi v$$

$$y(u,v) = a \cos 2\pi v$$

$$z(u,v) = -a u \sin^2 2\pi v + b(1-u)\cos 2\pi v.$$

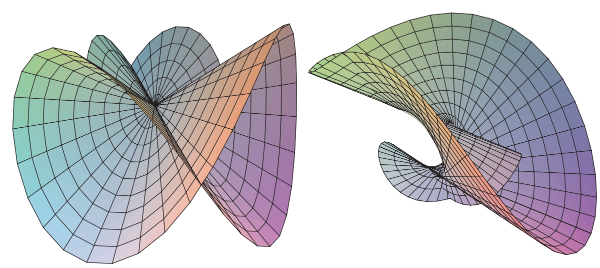
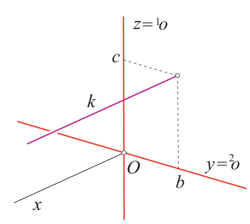


Fig. 7: Two-axial surfaces of revolution of spherical type - ruled conical

3.2 Ruled hyperbolical surfaces - group IIA3

The basic line k is skew to both axes of revolution, and let it be parallel to the coordinate axis x.



Parametric equations of the ruled hyperbolic surfaces (Fig. 8.) defined on the region $[0,1]^2 \subset R^2$ are in the form

$$x(u,v) = a u \cos^2 2\pi v - b \sin 2\pi v \cos 2\pi v + c \sin 2\pi v$$

$$y(u,v) = a u \sin 2\pi v + b \cos 2\pi v$$

$$z(u,v) = -a u \sin 2\pi v \cos 2\pi v + b \sin^2 2\pi v + c \cos 2\pi v$$

and there are no circles on the surface.

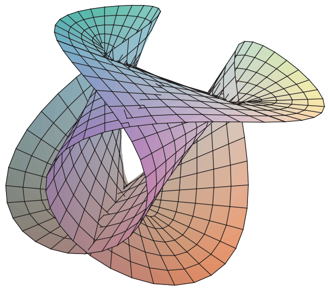


Fig. 8: Two-axial surface of revolution of spherical type - ruled hyperbolical

3.3 Ruled composite surfaces 1 - group IIA4

The basic line k is parallel to one axis of revolution and it is intersecting to the other axis. Let k be located in the plane of the two axes of revolution, coordinate plane yz, and

a) let k be parallel to ${}^1o$

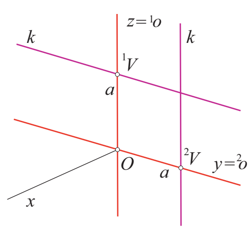
$\left|{}^1ok\right| = a,\ k \cap {}^2o = {}^2V = (0,a,0,1),\ a \neq 0$

then surface contains no circles, and it has 2 planes of symmetry;

b) let k be parallel to ${}^2o$

$\left|{}^2ok\right| = a, \quad k \cap {}^1o = {}^1V = (0,0,a,1), \quad a \neq 0$

then surface contains the only one circle, trajectory of the point ${}^1V$, and it has a unique plane of symmetry.



Parametric equations of this specific ruled composite surfaces (Fig. 9) defined on the region $[0,1]^2 \subset R^2$ are in the form

a) k is parallel to ${}^1o$, intersecting to ${}^2o$

$x(u,v) = -a\,u \sin 2\pi v \cos 2\pi v + b\,u \sin 2\pi v$

$y(u,v) = a \cos 2\pi v$

$z(u,v) = a \sin^2 2\pi v + b\,u \cos 2\pi v;$

b) k is intersecting to ${}^1o$, parallel to ${}^2o$

$x(u,v) = -b\,u \sin 2\pi v \cos 2\pi + a \sin 2\pi v$

$y(u,v) = b\,u \cos 2\pi v$

$z(u,v) = b\,u \sin^2 2\pi v + a \cos 2\pi v.$

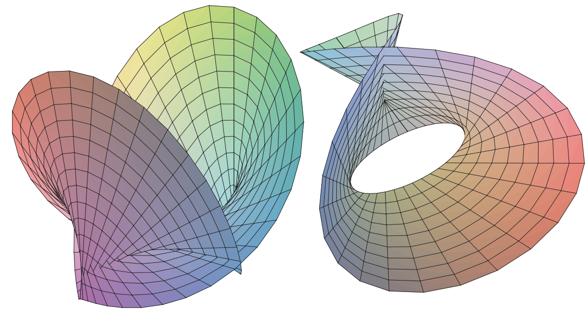
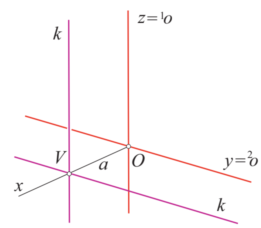


Fig. 9: Two-axial surfaces of revolution of spherical type - ruled composite - 1

3.4 Ruled composite surfaces 2 - group IIA4

The basic line k is parallel to one axis of revolution and it is skew to the other axis.



Let line k intersects the coordinate axis x,

$k \cap x = V = (a,0,0,1)$ and

a) let k be parallel to ${}^1o$, $\quad \left|{}^1ok\right| = a, \quad a \neq 0$, then the surface has only one plane of symmetry;

b) let k be parallel to o, $\quad \left|{}^2ok\right| = a, \quad a \neq 0$, then the surface has 2 planes of symmetry.

There are no circles on the surfaces.

Parametric equations of these specific ruled composite surfaces (Fig. 10) defined on the region $[0,1]^2 \subset R^2$ are in the form

a) k is parallel to ${}^1o$, skew to ${}^2o$

$x(u,v) = a \cos^2 2\pi v + b\,u \sin 2\pi v$

$y(u,v) = a \sin 2\pi v$

$z(u,v) = -a \sin 2\pi v \cos 2\pi v + b\,u \cos 2\pi v;$

b) k is skew to ${}^1o$, parallel to ${}^2o$

$x(u,v) = a \cos^2 2\pi v - b\,u \sin 2\pi v \cos 2\pi v$

$y(u,v) = a \sin 2\pi v + b\,u \cos 2\pi v$

$z(u,v) = -a \sin 2\pi v \cos 2\pi v + b\,u \sin^2 2\pi v.$

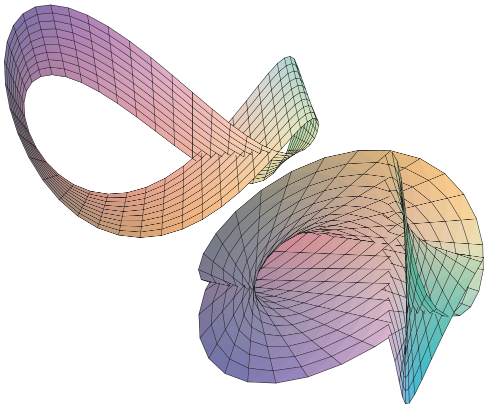
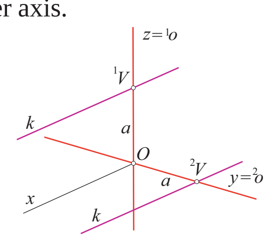


Fig. 10: Two-axial surfaces of revolution of spherical type - ruled composite - 2

3.5 Ruled composite surfaces 3 - group IIA4

The basic line k intersects one axis of revolution and is skew to the other axis.



Let line k be parallel to the coordinate axis x, and

a) let k be intersecting to ${}^1o$, skew to ${}^2o$,
$k \cap {}^1o = {}^1V = (0,0,a,1)$,
then the surface contains only one circle, the trajectory of the point ${}^1V$, and it has 2 planes of symmetry;

b) let k be intersecting to ${}^2o$ and skew to ${}^1o$,
$k \cap {}^2o = {}^2V = (0,a,0,a)$,
then the surface contains no circles, and it has a unique plane of symmetry.

Parametric equations of these specific ruled composite surfaces (Fig. 11) defined on the region $[0,1]^2 \subset R^2$are in the form

a) k is intersecting to ${}^1o$, skew to ${}^2o$

$$x(u,v) = b\,u\cos^2 2\pi v + a\sin 2\pi v$$
$$y(u,v) = b\,u\sin 2\pi v$$
$$z(u,v) = -b\,u\sin 2\pi v\cos 2\pi v + a\cos 2\pi v;$$

b) k is skew to ${}^1o$, intersecting to ${}^2o$

$$x(u,v) = b\,u\cos^2 2\pi v - a\sin 2\pi v\cos 2\pi v$$
$$y(u,v) = b\,u\sin 2\pi v + a\cos 2\pi v$$
$$z(u,v) = -b\,u\sin 2\pi v\cos 2\pi v + a\sin^2 2\pi v.$$

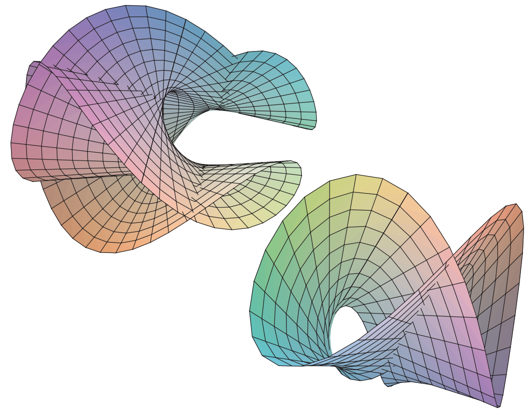
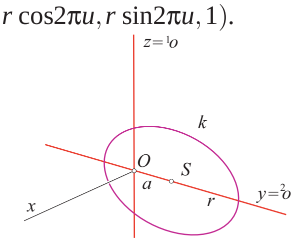


Fig. 11: Two-axial surfaces of revolution of spherical type - ruled composite - 3

3.6 Cyclical toroidal surfaces - Group IIB1

The basic circle $k(S,r)$ is located in the plane determined by the two axes of revolution, in the coordinate plane yz, while the vector equation of the circle for $u \in [0,1]$ is

$$\mathbf{r}(u) = (0, a + r\cos 2\pi u, r\sin 2\pi u, 1).$$



Number of circles as trajectories of the points on the basic circle depends on the superposition of the basic circle and the axis of revolution ${}^1o$, i.e. on the common relation of parameters a and r. There exist no circular trajectories for $a > r$, exactly one circular trajectory for $a = r$, and two circular trajectories for $a < r$.

Parametric equations of the surface defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = -(a + r\cos 2\pi u)\sin 2\pi v\cos 2\pi v + r\sin 2\pi u\sin 2\pi v$$
$$y(u,v) = (a + r\cos 2\pi u)\cos 2\pi v$$
$$z(u,v) = (a + r\cos 2\pi u)\sin^2 2\pi v + r\sin 2\pi u\cos 2\pi v.$$

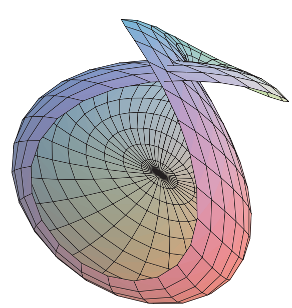
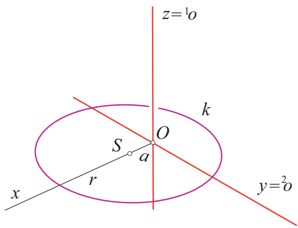


Fig. 12: Two-axial surface of revolution of spherical type - cyclical toroidal

3.7 Group IIB2 - Cyclical general surfaces

The basic circle $k(S,r)$ is not located in the plane determined by the two axes of revolution; let it be located in the coordinate plane xy, then the vector equation of the circle for $u \in [0,1]$ is in the form

$$r(u) = (a + r\cos 2\pi u, r\sin 2\pi u, 0, 1).$$



Number of circles as trajectories of the points on the basic circle depends on the superposition of the basic circle and the axis of revolution ^1o, i.e. on the relation of parameters a and r, in the same way as it was in the previous type of surfaces.

Parametric equations of the surface defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = (a + r\cos 2\pi u)\cos^2 2\pi v - r\sin 2\pi u \sin 2\pi v \cos 2\pi v$$

$$y(u,v) = (a + r\cos 2\pi u)\sin 2\pi v + r\sin 2\pi u \cos 2\pi v$$

$$z(u,v) = -(a + r\cos 2\pi u)\sin 2\pi v \cos 2\pi v + r\sin 2\pi u \sin^2 2\pi v.$$

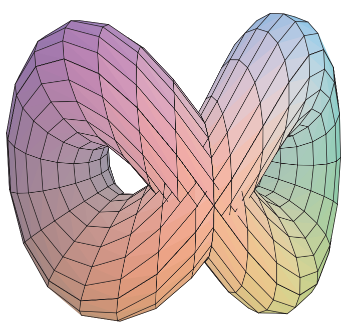


Fig. 13: Two-axial surface of revolution of spherical type - cyclical general

4 Two axial surfaces of revolution of Euler type

Two-axial revolution determined by skew axes $^1o/^2o$ is a general Euler revolution. Let us locate the axis ^1o on the coordinate axis z, and axis ^2o parallel to the coordinate axis x. Analytic representation of the two-axial surface of revolution of the spherical type determined on the region $\Omega \subset R^2$, with the basic curve represented by the vector equation

$$\mathbf{r}(u) = (x(u), y(u), z(u), 1)\,,\ u \in [o,1]$$

is in the form

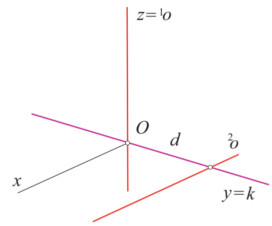
$$x(u,v) = x(u)\cos 2\pi v - y(u)\sin 2\pi v$$

$$y(u,v) = x(u)\sin 2\pi v \cos 2\pi v + y(u)\cos^2 2\pi v - z(u)\sin 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = x(u)\sin^2 2\pi v + y(u)\sin 2\pi v \cos 2\pi v + z(u,v)\cos 2\pi v - d\sin 2\pi v.$$

4.1 Ruled conical surfaces - group IIIA2

The basic line k is intersecting to both axes of revolution, let it be located on the coordinate axis y with the vector equation $\mathbf{r}(u) = (0, a\,u, 0, 1)$, $u \in [0,1]$.



Surface is of Möbius type (Fig. 14) and parametric equations defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = -a\,u\sin 2\pi v$$

$$y(u,v) = a\,u\cos^2 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = a\,u\sin 2\pi v\cos 2\pi v - d\sin 2\pi v.$$

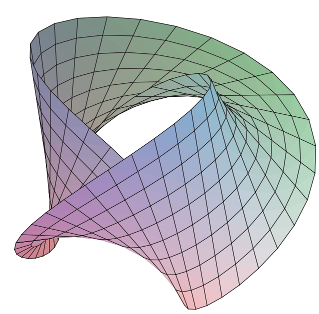
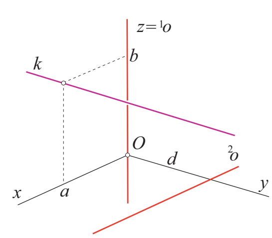


Fig. 14: Two-axial surface of revolution of Euler type - ruled conical

4.2 Ruled hyperbolical surfaces - group IIIA3

The basic line k is skew to both axes of revoluton and let it be parallel to the coordinate axis y, then its vector equation has the form $r(u) = (a, c\,u, b, 1)$, $u \in [0,1]$.



Parametric equations of the surfaces defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = a\cos 2\pi v - c\,u\sin 2\pi v$$

$$y(u,v) = a\sin 2\pi v\cos 2\pi v + c\,u\cos^2 2\pi v - b\sin 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = a\sin^2 2\pi v + c\,u\sin 2\pi v\cos 2\pi v + b\cos 2\pi v - d\sin 2\pi v.$$

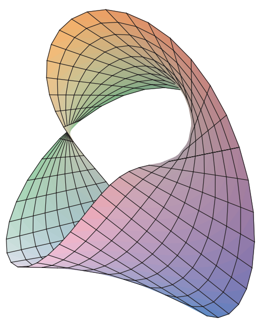


Fig. 15: Two-axial surface of revolution of Euler type - ruled hyperbolical

4.3 Ruled composite surfaces - group IIIA4

The basic line k is in different superposition to the two axes of revolution. From the similar three subgroups of ruled composite surfaces as there were presented for the two-axial surfaces of revolution of spherical type, some examples of specific representatives are chosen in illustration figure 16.

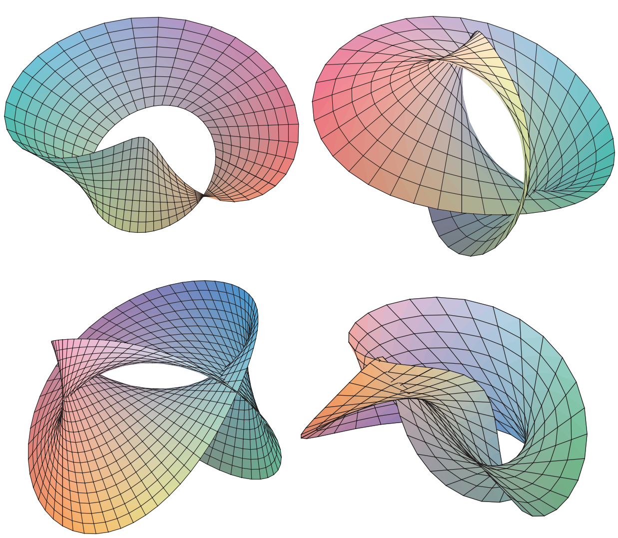
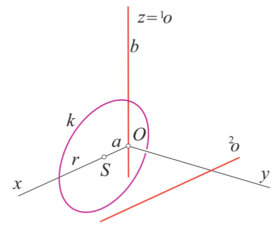


Fig. 16: Two-axial surfaces of revolution of Euler type - ruled composite

4.4 Cyclical general surfaces - group IIIB2

a) The basic circle $k(S,r)$ is located in the coordinate plane xz, and its equation is given in the form

$$\mathbf{r}(u) = (a + r\cos 2\pi u, 0, r\sin 2\pi u, 1),\ u \in [0,1].$$



Parametric equations of the surface defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = (a + r\cos 2\pi u)\cos 2\pi v$$

$$y(u,v) = (a + r\cos 2\pi u)\sin 2\pi v\cos 2\pi v - r\sin 2\pi u\sin 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = (a + r\cos 2\pi u)\sin^2 2\pi v + r\sin 2\pi u\cos 2\pi v - d\sin 2\pi v.$$

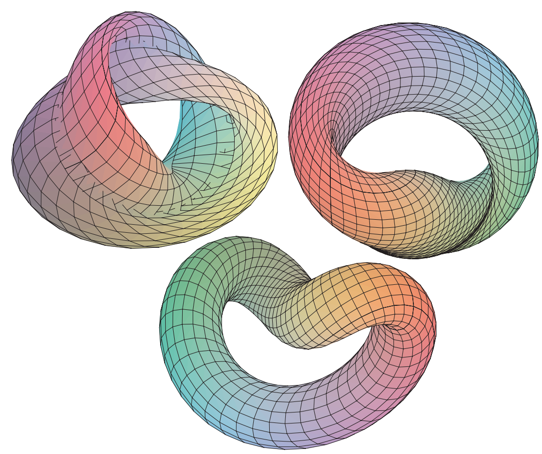
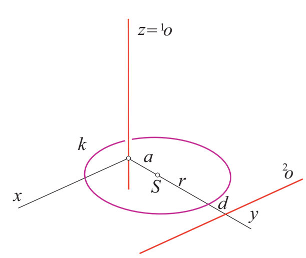


Fig. 17: Two-axial surfaces of revolution of Euler type - cyclical general a)

b) The basic circle $k(S,r)$ is located in the coordinate plane xy, and its equation is

$$\mathbf{r}(u) = (r\cos 2\pi u,\ a + r\sin 2\pi u, 0, 1), u \in [0,1].$$



Parametric equations of the surface (Fig. 18) defined on the region $[0,1]^2 \subset R^2$ have the form

$$x(u,v) = r\cos 2\pi u \cos 2\pi v - (a + r\sin 2\pi u)\sin 2\pi v$$

$$y(u,v) = r\cos 2\pi u \sin 2\pi v \cos 2\pi v + (a + r\sin 2\pi u)\cos^2 2\pi v + d(1-\cos 2\pi v)$$

$$z(u,v) = r\cos 2\pi u \sin^2 2\pi v + (a + r\sin 2\pi u)\sin 2\pi v \cos 2\pi v - d\sin 2\pi v.$$

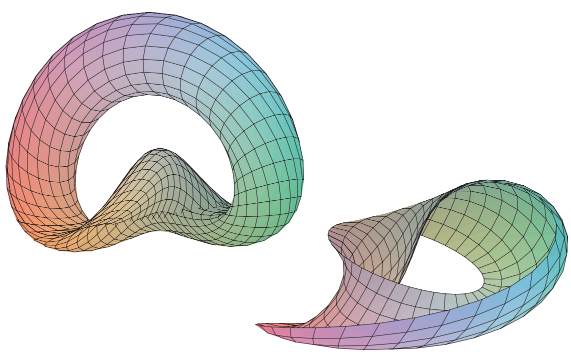


Fig. 18: Two-axial surfaces of revolution of Euler type - cyclical general b)

The number of circular trajectories of the points on the basic circle depends on the superposition of the basic circle and the axis of revolution ${}^1o$, i.e. on the relation of parameters a and r. There can be no circular trajectories for $a > r$, exactly one circular trajectory for $a = r$, and two circular trajectories for $a < r$. Special surfaces can be modelled by setting $a = 0$, with one double circular trajectory.

References

[1] GOLDSTEIN, H., The *Euler Angles* and *Euler Angles in Alternate Conventions*, 2nd ed. Reading, MA: Addison-Wesley, 1980.

[2] VELICHOVÁ, D., *Euler angles*, Proceedings 3rd International Conference on Applied Mathematics APLIMAT 2004, SjF STU Bratislava, SR, ISBN 80-227-1995-1, 2004, p. 191 - 198.

[3] VELICHOVÁ, D., *Two-Axial Surfaces of Revolutio*, Proceedings of the 8th International Scientific Conference Mechanical Engineering 2004, SjF STU, Bratislava 2004, SR, CD, ISBN 80-27-2105-0, p. S2-60 - S2-65.

[4] Euler angles in Spider WEB, animations on web. http://www.wadsworth.org/spider_doc/spider/docs/euler.html

[5] Wikipedia, the free encyclopedia on the web, Wikimedia Foundation, Inc., 2001. http://en.wikipedia.org/wiki/Euler_angles

Daniela Velichová

e-mail: daniela.velichova@stuba.sk

Department of Mathematics
Faculty of Mechanical Engineering
Slovak University of Technology in Bratislava

Originäre wissenschaftliche Arbeit
Angenommen am 22.12.2005.

LÁSZLÓ VÖRÖS

Reguläre Körper und mehrdimensionale Würfel

Regular Bodies and Hypercubes

ABSTRACT

The paper gives useful connections between regular bodies, bodies originated from them and the 2D and 3D projections of multidimensional cubes. The problem of graphic representation has been solved with AutoCAD and Autolisp programs which we developed .

Key words: multidimensional axonometry, zonotopes, Minkowski sum

MSC 2000: 51M20, 68U07

Pravilna tijela i hiperkocke

SAŽETAK

U članku su dane korisne veze između pravilnih tijela i tijela nastalih pomoću njih sa 2D i 3D projekcijama višedimenzionalnih kocki. Problemi grafičke prezentacije riješeni su pomoću AutoCADa i Autolisp programa koje smo razvili.

Ključne riječi: višedimenzionalna aksonometrija, zonotopi, Minkowskijev zbroj

Die Fragen:

– Was ist der Durchschnitt der fünf Würfel, die in ein Dodekaeder einschreibbar sind (Abb. 1.)?

– Welche dreidimensonale Gitterkonstruktion ergibt einen Grundriss, der gleich ist der Titelblattfigur des *Journal for Geometry and Graphics* (Abb. 2.)?

Abbildung 1.

Abbildung 2.

Antworten und eine weitere Frage:

Falls die Kanten der zueinander dualen Hexa- und Oktaeder einander halbieren, ergeben die 12-12 Kanten die Diagonalen von 12 kongruenten Rauten, die ein Rhombendodekaeder begrenzen. Das ist die Hülle des dreidimensionalen Gitters der Projektion eines vierdimensionalen Würfels. Die inneren Ecken fallen in der Mitte zusammen (Abb. 3.).

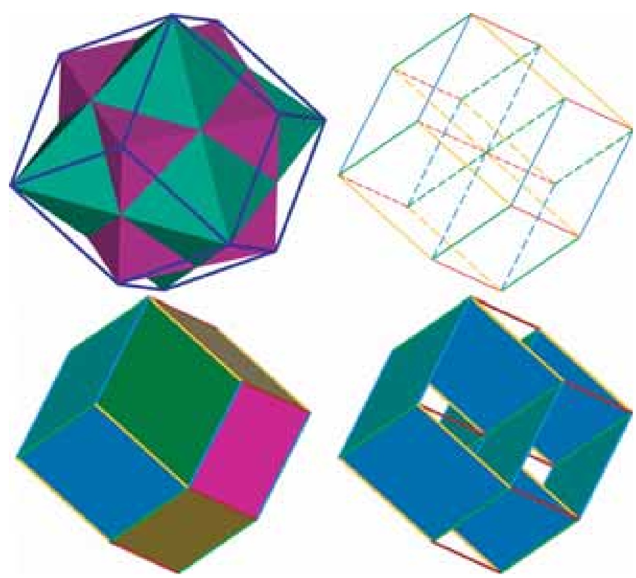


Abbildung 3.

Wenn die Kanten der zueinander dualen Dodeka- und Ikosaeder einander halbieren, ergeben die 30-30 Kanten die Diagonalen von 30 kongruenten Rauten, und diese begrenzen einen halbregulären Körper (Abb. 4.). Die Diagonalen der Rauten verhalten sich nach dem goldenen Schnitt. Wenn wir dem oben erwähnten Dodekaeder in bekannter Weise fünf Würfel einschreiben, ergibt deren Durchschnitt einen Körper ähnlich dem obigen Triakontaeder (Abb. 5.). Sie verhalten sich nach dem goldenen Schnitt (Abb. 6.). Verschieben wir die Kanten, die verschiedene räumliche Stellungen haben, in die Eckpunkte, so erhalten wir das 3D Modell eines 6D Würfels (Abb. 7.).

Die Kanten dieser 3D Gitter haben alle dieselbe Länge, und benachbarte Kanten treffen sich unter gleichen Winkeln - ähnlich wie bei der isogonal-isometrischen Axonometrie. Die Titelblattfigur des *Journal for Geometry and Graphics* (Abb. 2.) kann offenbar als zweidimensionales Bild eines vierdimensionalen Würfels verstanden werden, besser gesagt, als dessen dreidimensionales Gitter, wenn wir die dargestellten Überdeckungen der Kanten in Betracht ziehen. Aus den Richtungen der Bilder von vier nichtparallelen Kanten können zwei-zwei kongruente Monge-Bilder des 3D Raumgitters gebaut werden, so dass wir voraussetzen, dass die Kanten mit gleichen Längen zur ersten Bildebene unter demselben frei gewählten Winkel geneigt sind (Abb. 8.).

Abbildung 4.

Abbildung 5.

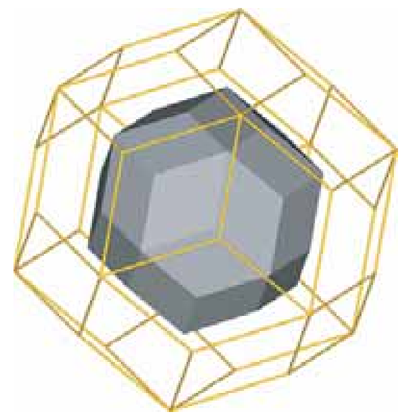


Abbildung 6.

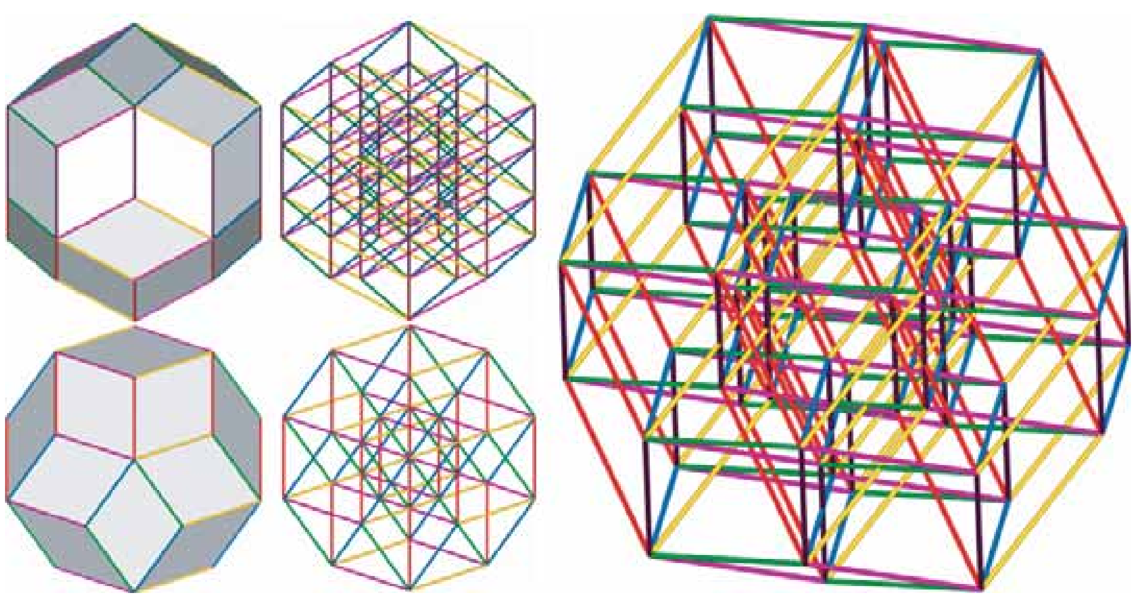


Abbildung 7.

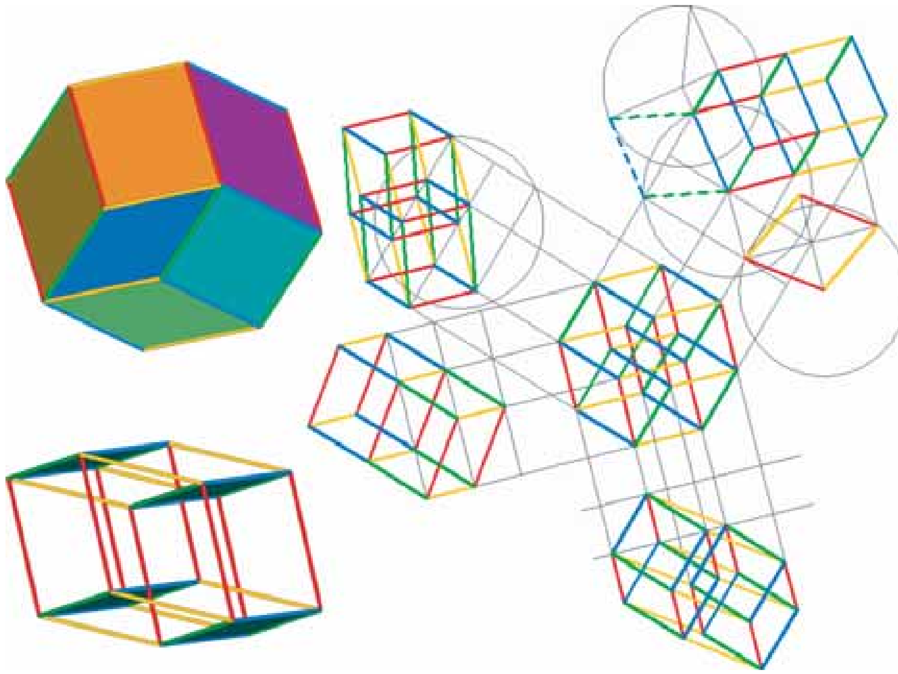


Abbildung 8.

Auf Grund dieser Voraussetzungen kann ein k-dimensionaler Würfel in einem 2D Bild dargestellt werden, das ähnlich zur isogonal-isometrischen Axonometrie die folgenden Eigenschaften hat: Die Bilder benachbarter Kanten schließen gleiche Winkel ein, alle Bildkanten haben dieselbe Länge, die Bildkontur zeigt ein reguläres Polygon, dessen Seitenanzahl mindestens k ist oder $2k$, sofern die Anzahl der deckungsgleichen Bildpunkte möglichst reduziert wird. Die Achsenbilder sind also parallel zu den Seiten eines k-seitigen Polygons oder zu den Diagonalen eines $2k$-seitigen Polygons. Um das 3D Bild zu konstruieren, müssen die Kanten unter gleichem Winkel gegenüber der Bildebene geneigt werden (Abb. 9.).

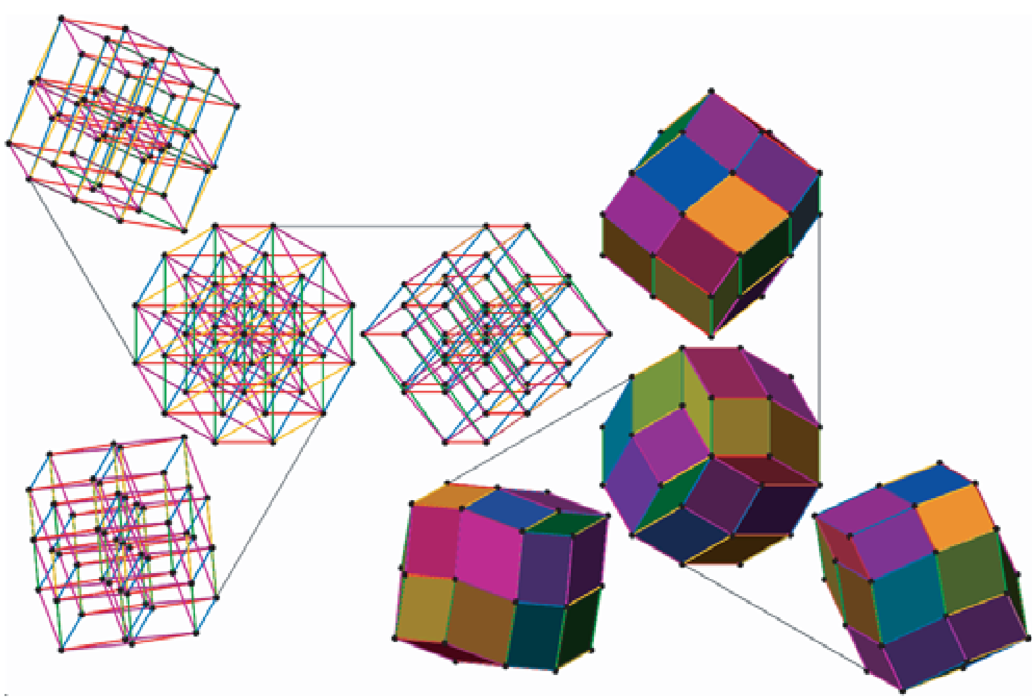


Abbildung 9.

Im dreidimensionalen Fall ergibt das Verfahren mit einem k-seitigen Polygon ein besonders schönes und statisch gut behandelbares Raumgitter. Seine Ecken sind in verschiedenen Höhenstufe platziert, wobei das einheitliche Höhenintervall regulierbar ist durch den Kantenwinkel zur horizontalen Ebene. In dem dargestellten 8D Beispiel sind die horizontalen und vertikalen Ausdehnungen des Modells ausgeglichen; die Gesetzmäßigkeiten der Modellkonstruktion sind leicht ablesbar (Abb. 10.).
Die Frage, inwieweit (nicht nur auf Grund der Analogien) diese zwei- und dreidimensionalen Bilder als durch mehrdimensionale Parallelprojektionen entstandene isogonal-isometrische Axonometrien angesehen werden können, ist noch offen, da die Gültigkeit des Satzes von K. Pohlke in mehrdimensionalen euklidischen Räumen beschränkt ist [2], [5], siehe auch den Satz am Ende dieser Arbeit.

Die Kanten können natürlich auch mit beliebigen Längen und Winkeln gewählt werden, und mittels entsprechender Verschiebungen sind die dreidimensionalen "axonometrischen Gitter" der mehrdimensionalen Würfel zu konstruieren. Eine Möglichkeit zur Darstellung von mehr oder weniger regulären 3D Modellen besteht darin, die Kanten der platonischen und archimedischen oder der von diesen auf verschiedenen Weisen hergeleiteten Körper zu wählen. Ich habe deswegen zum Beispiel die Verbindung des Ikosaeders und des Würfels studiert und die folgenden Körper bekommen (Abb. 11-14.).

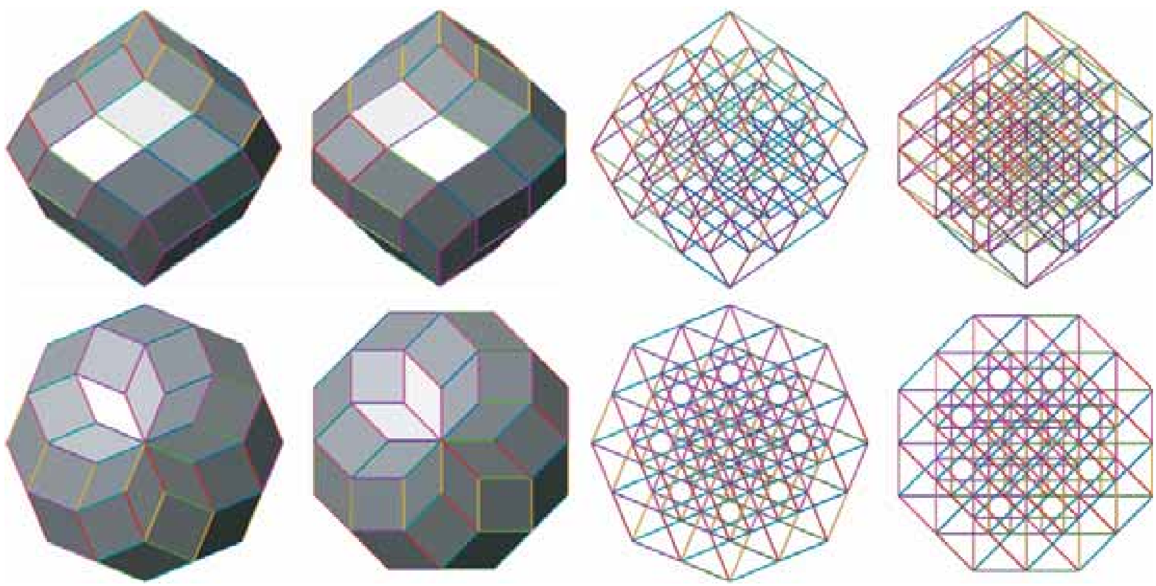


Abbildung 10.

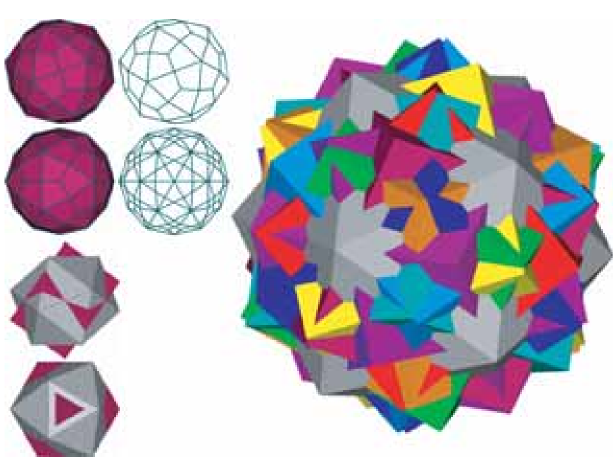


Abbildung 11.

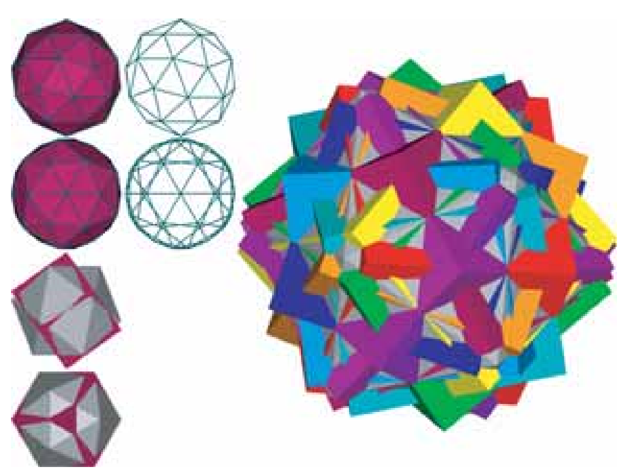


Abbildung 12.

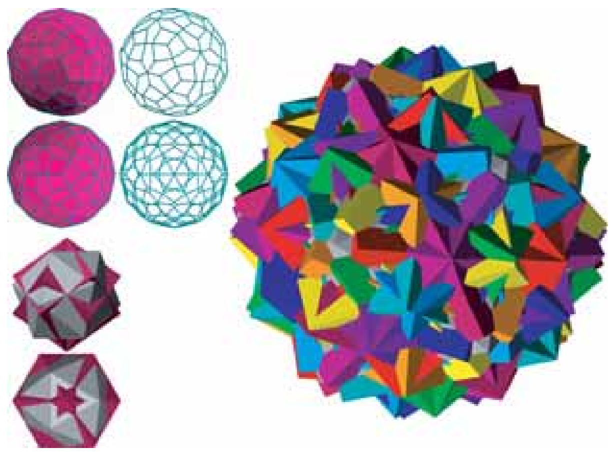


Abbildung 13.

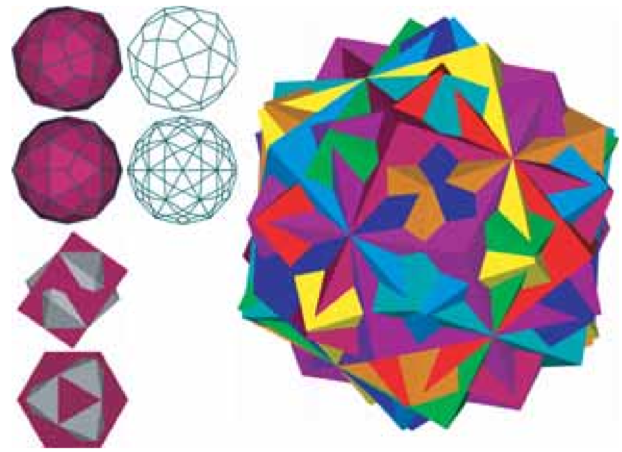


Abbildung 14.

Zwei von diesen sind schon vorgekommen als Körper, die gleichmäßig die Symmetrieeigenschaften des Dodekaeders und des Ikosaeders bewahren [3]. Im Interesse einer schnellen Konstruktion der 3D Modelle höherdimensionaler Würfel und um die diesbezüglichen darstellerischen Möglichkeiten von AutoCAD auszunutzen, habe ich ein Programm in Autolisp geschrieben. Die verschiedene Stellungen besitzenden Kanten dieser Körpern bestimmen bereits Würfel von derart hoher Dimension, dass ein herkömmlicher PC deren 3D Modell mit allen Kanten und Seiten gar nicht mehr darstellen kann.

Ein 3D Modell der k-dimensionalen Würfel kann aber auch auf folgende Art erzeugt werden: Wenn die Kanten je einen Vektor bedeuten, zeigen die zu derselben Ecke gehörenden j Vektoren auf Punkte derselben Ebene und sie bestimmen je eine Ecke, die mit Kanten verbunden und der ein Polygon umschrieben ist. Diese werden mit den weiteren $(k-j)$ Vektoren verschoben. Die Gesamtheit der so herstellbaren ebenen Netzen bildet das Modell. Nehmen wir also die in einer gewählten Ecke zusammentreffenden Kanten, von diesen die möglichen Kantengruppen, die zur gleichen Ebenen gehören, und bilden wir nach allen Ebenen die Summe der zwei Gruppen der Kantenvektoren, die von der gegebenen Ebene nach verschiedenen Seiten ausragen. Verschieben wir die um eine Ecke herstellbaren Polygonen mit den zu diesen gehörenden zwei-zwei Summenvektoren, bekommen wir die Hülle des 3D Modells eines höherdimensionalen Würfels. So können wir zum Beispiel aus den Kanten des gemeinsamen Teils eines Oktaeders und eines 3D Würfels die Hülle des 3D Modells eines 6D Würfels herstellen (Abb. 15.).

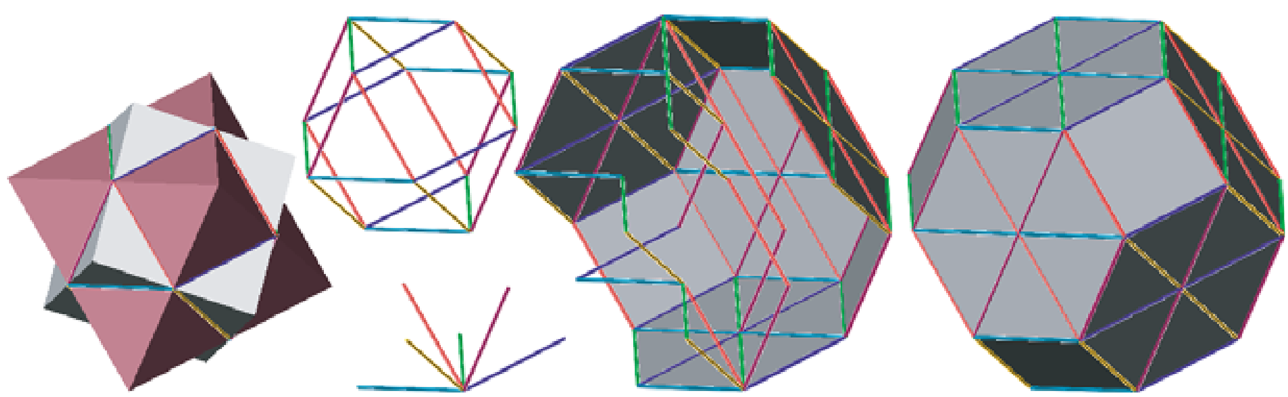


Abbildung 15.

Abbildung 16: Grundriß, Aufriß, Kreuzriß

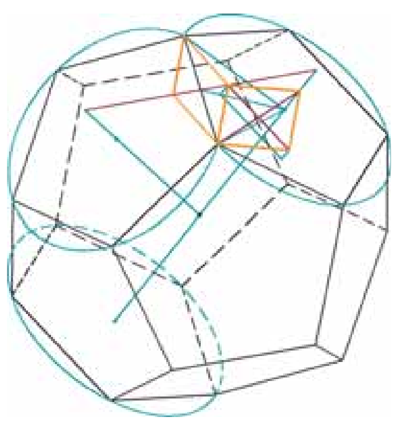


Abbildung 17.

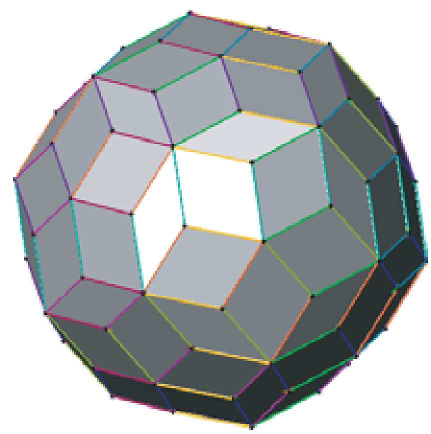


Abbildung 18.

Die Kanten des Pentakisdodekaeders (Abb. 11.) zum Beispiel bestimmen das gleichkantige 3D Modell des 45D Würfels. Dessen Hülle ist bereits durch mein Programm herstellbar und besteht aus 270 Vierecken, 190 Sechsecken, 15 Achtecken und 6 Zehnecken (Abb. 16.).

Aus den oben erwähnten Eigenschaften kann vermutet werden, dass zentralsymmetrische dreidimensionale Körper, die von Polygonen mit geraden Seitenanzahl begrenzt sind, selbst 3D Projektionen mehrdimensionaler Würfel sein können, wie auch die am Anfang erwähnten Rombendodekaeder und Triakontaeder. Dies gilt auch für den Körper in der Abbildung 15., der ein raumfüllendes Polyeder ist. Die Kombination aus einem Triakontaeder und ein Deltoid-Hexekontaeder [3] kann auch nach der Abbildung 17. konstruiert werden. Dieses Polyeder ist die Hülle eines gleichkantigen 3D Modells des 10D Würfels (Abb. 18.).

Zusammenfassend können wir den folgenden Satz formulieren, der mit dem Begriff des Zonotopes (des n-dimensionalen Zonoeders) verbunden ist, also mit der Minkowski-Summe von n Strecken, und eben mit den linearen Abbildungen des n-Würfels auf den zwei- oder dreidimensionalen Raum (siehe [6] und [8]).

Satz:
Wenn n Strecken, die von einem Punkt O ausgehen, axonometrische 2D oder 3D Projektion (oder allgemein k-Projektion, $2 \leq k < n$) der Eckenfigur eines n-dimensionalen Würfels sein können, dann ist die Minkowski-Summe der Strecken ein zwei- oder dreidimensionales (k-dimensionales) axonometrisches Bild des n-Würfels.

Wie kurz aufgeführt, sollen wir hier die n Strecken als Vektoren auffassen und die möglichen Verschiebungen auf die n Strecken anwenden.

Formal ist die *Minkowski-Summe von zwei Punktmengen A und B* in einem euklidischen k-Raum wie folgt *definiert*: Wir nehmen einen beliebiger Aufpunkt O an und bilden mit $X \in A$ und $Y \in B$ die Vektorsumme $\overrightarrow{OX} + \overrightarrow{OY} = \overrightarrow{OZ}$. Die so gewonnenen Punkte Z beschreiben die Punktmenge C, d.h. die Minkowski-Summe $C := A + B$, bis auf eine Translation.
Nun haben wir in dieser Arbeit vorausgesetzt und benutzt, dass eine isogonal-isometrische Axonometrie eines n-Würfels, bei welcher die n gleichen Strecken und ihre "entgegengesetzten" eine entsprechende "gleichwinklige Sternfigur" bilden (Abb. 2.), in einem k-Raum (jetzt $k = 2,3$) realisierbar ist. Der genauere Satz und sein Beweis wären interessant (ist mir aber bisher nicht bekannt)!

Literatur

[1] BOLTYANSKI, V., MARTINI, H., SOLTAN, P.S., *Excursions into Combinatorial Geometry*, Springer, 1977.

[2] BRAUNER, H., *Zum Satz von K. Pohlke in n-dimensionalen euklidischen Räumen*, Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl., Band 195, Heft 8-10 (1986)

[3] GÉVAY, G., *Icosahedral Morphology*, in: *Fivefold Symmetry*, edited by István Hargittai, World Scientific Publishing Co. Pte. Ltd., 1992 & reprinted in: Sándor Kabai, *Mathematical Graphics - The Icosahedron*, Uniconstant, 2005.

[4] MIYAZAKI, K., *Adventure in Multidimensional Space: The Art and Geometry of Polygons, Polyhedra and Polytopes*, Wiley, New York, 1986.

[5] STACHEL, H., *Mehrdimensionale Axonometrie*, Proceedings of the Congress of Geometry, Thessaloniki, 159-168 (1987)

[6] http://home.inreach.com/rtowle/Zonohedra.html

[7] http://icai.voros.pmmf.hu

[8] http://www.decatur.de/personal/zono/index.html

László Vörös
e-mail: vorosl@witch.pmmf.hu

Universität Pécs
M. Pollack Technische Fakultät

Original scientific paper
Accepted 23. 12. 2005.

Márta Szilvási-Nagy[1]
Ildikó Szabó

C^1-Continuous Coons-type Blending of Triangular Patches

C^1-continuous Coons-type blending of triangular patches

SAŽETAK

A Gordon–Coons-type surface construction starts from three differentiable triangular surface patches, which are defined on the same triangular parameter domain. If one boundary curve of each fits a curvilinear triangle, then the defined surface interpolates to these curves. The connection between the resulting surface and the constituents is C^1 continuous along the common boundary curves with the exception of the corner points. This surface definition is an extension of the Gordon–Coons definiton of a triangular surface patch constructed from three boundary curves.

Key words: blending surface, surface modelling, CAGD

MSC 2000: 68U05

Coonsovo povezivanje klase C^1 trokutnih dijelova

SAŽETAK

Gordon-Coonsova konstrukcija plohe kreće od tri diferencijabilna trokutna plošna dijela koji su definirani na iston trokutnom parametarskom području. Ako po jedna rubna krivulja svakog od njih odgovara krivuljnom trokutu, tada definirana ploha interpolira te krivulje. Veza između dobivene plohe i sastavnih dijelova je klase C^1 duž zajedničkih rubnih krivulja, s izuzetkom vrhova. Ova definicija plohe je proširenje Gordon–Coonsove definicije trokutnog plošnog dijela konstruiranog iz tri granične točke.

Ključne riječi: povezivanje ploha, modeliranje ploha, CAGD

1 Introduction

The presented surface definition is based on a classical interpolation method, where the constructed function of two variables has given values on the boundary of a given triangle. The original formulation of the solution of this interpolation problem is the following [1].

If the real-valued function $F(x,y)$ is continuous on the triangle T with vertices $(0,0)$, $(1,0)$ and $(0,1)$ in the xy plane, then the function given by

$$\begin{aligned} W(x,y) = \frac{1}{2}\Big\{ & \left[\frac{1-x-y}{1-y}F(0,y) + \frac{x}{1-y}F(1-y,y)\right] \\ & + \left[\frac{1-x-y}{1-x}F(x,0) + \frac{y}{1-x}F(x,1-x)\right] \\ & + \left[\frac{x}{x+y}F(x+y,0) + \frac{y}{x+y}F(0,x+y)\right] \\ & - [xF(1,0) + yF(0,1) + (1-x-y)F(0,0)] \Big\} \end{aligned}$$

is continuous over T and interpolates to the values of F on its boundary, i.e. along the curves $x=0$, $y=0$ and $1-x-y=0$ [3, §8.2].

A geometric interpretation of this interpolation problem is the construction of Gordon–Coons triangular surface patches, which is the triangular version of the well-known construction of rectangular Coons patches [2] extended in [6]. A Gordon–Coons surface patch is generated by the above formula from three continuous curve segments forming a spatial curvilinear triangle, which are the boundary curves of the generated patch. The boolean sum of convex combinations of three pairs of the given curves is corrected with a convex combination of the vertex points (Fig 1).

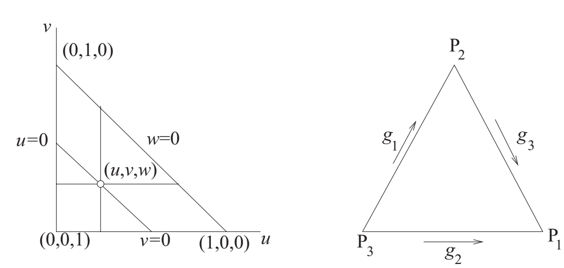


Figure 1: The parameter triangle and boundary curves.

As the convex combination is invariant with respect to affine transformations, the standard parameter triangle can

[1] Supported by the Hungarian National Foundation OTKA No. T047276 and the Foundation TeT HR-29/2004.

be transformed affinely, and barycentric coordinates can be used with respect to a base triangle with the vertices $(0,0,1)$, $(1,0,0)$ and $(0,1,0)$ [4, §18]. The parameter triangle is determined by $0 \le u,v,w \le 1$ and $u+v+w=1$. The three continuous input curves are defined on the boundaries of the parameter triangle. Their representing vector functions are expressed with barycentric coordinates written in a symmetric form.

$\mathbf{g}_1(0,v,1-v)$ is defined over the edge $u=0$, $\mathbf{g}_2(u,0,1-u)$ over the edge $v=0$ and $\mathbf{g}_3(u,1-u,0)$ over the edge $w=0$, which can be written also as $\mathbf{g}_3(1-v,v,0)$ substituting $u=1-v$.

If the three curves satisfy the boundary conditions

$$\begin{aligned}
&\mathbf{g}_2(1,0,0)=\mathbf{g}_3(1,0,0)=\mathbf{P}_1,\\
&\mathbf{g}_1(0,1,0)=\mathbf{g}_3(0,1,0)=\mathbf{P}_2 \text{ and}\\
&\mathbf{g}_1(0,0,1)=\mathbf{g}_2(0,0,1)=\mathbf{P}_3,
\end{aligned}$$

then the surface patch given by the vector function

$$\begin{aligned}
\mathbf{r}(u,v,w) = \frac{1}{2}\Big\{ &\left[\frac{w}{u+w}\mathbf{g}_1(0,v,1-v)+\frac{u}{u+w}\mathbf{g}_3(1-v,v,0)\right]\\
&+\left[\frac{w}{v+w}\mathbf{g}_2(u,0,1-u)+\frac{v}{v+w}\mathbf{g}_3(u,1-u,0)\right]\\
&+\Big[\frac{u}{u+v}\mathbf{g}_2(u+v,0,1-u-v)\\
&\qquad+\frac{v}{u+v}\mathbf{g}_1(0,u+v,1-u-v)\Big]\\
&-\left[u\mathbf{g}_3(1,0,0)+v\mathbf{g}_1(0,1,0)+w\mathbf{g}_2(0,0,1)\right]\Big\},\\
&0 \le u,v,w \le 1, \quad u+v+w=1
\end{aligned} \tag{1}$$

interpolates the input curves along the edges of the parameter triangle.

The other surface definition, which we use in our surface construction, was given for the construction of a C^1 continuous triangular interpolant in [5] as follows.

If three functions F_i, $i=1,2,3$ are C^2 differentiable on the triangle T described with the barycentric coordinates u, v and w, $0 \le u,v,w \le 1$, $u+v+w=1$, and each of them interpolates one vertex of T and a vector field along its opposite side, then the function given by

$$DF = \frac{u^2w^2F_1+v^2w^2F_2+u^2v^2F_3}{u^2w^2+v^2w^2+u^2v^2} \tag{2}$$

is differentiable, and interpolates the values and the first partial derivatives of the given "underlying" surfaces F_1, F_2 and F_3 (consequently, also the given vector fields) along the edge $u=0$, $v=0$ and $w=0$ of the triangle, respectively.

This convex combination scheme was applied and investigated for three differentiable triangular surface patches defined on the same parameter domain in [7]. However, the problem ensuring the compatibility conditions for the input surfaces at the corner points is not solved in general. Therefore, the continuity of the defined surface at the vertices is not ensured.

A generalization of the Gordon–Coons surface construction in (1) was given with three triangular surface constituents in [8] as follows.

Let $\mathbf{r}_1(u,v,w)$, $\mathbf{r}_2(u,v,w)$ and $\mathbf{r}_3(u,v,w)$ be continuous vector functions defined on the parameter triangle $0 \le u,v,w \le 1$, $u+v+w=1$, representing three triangular surface patches with common corner points (Fig 2)

$$\begin{aligned}
&\mathbf{r}_1(0,0,1)=\mathbf{r}_2(0,0,1)=\mathbf{P}_3,\\
&\mathbf{r}_1(0,1,0)=\mathbf{r}_3(0,1,0)=\mathbf{P}_2,\\
&\mathbf{r}_2(1,0,0)=\mathbf{r}_3(1,0,0)=\mathbf{P}_1.
\end{aligned}$$

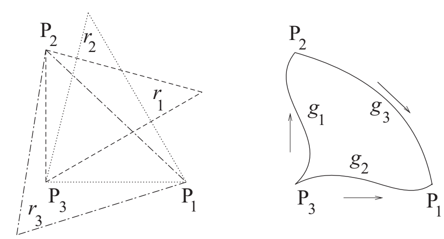


Figure 2: Three input surface patches and auxiliary curves.

The weighting ("blending") functions are

$$\begin{aligned}
\mu_1 &= \frac{(1-\lambda_1)w^2}{\lambda_1u^2+(1-\lambda_1)w^2},\\
\mu_2 &= \frac{(1-\lambda_2)u^2}{\lambda_2v^2+(1-\lambda_2)u^2},\\
\mu_3 &= \frac{(1-\lambda_3)v^2}{\lambda_3w^2+(1-\lambda_3)v^2},
\end{aligned}$$

where $0 \le \lambda_1,\lambda_2,\lambda_3 \le 1$ are shape parameters of values between 0 and 1, and at the corner points

$$\mu_1(0,1,0) := 1, \quad \mu_2(0,0,1) := 1, \quad \mu_3(1,0,0) := 1$$

are required.

Definition 1. The surface patch is defined by the vector function

$$\begin{aligned}
\mathbf{f}(u,v,w) = \frac{1}{2}\big[&\mu_1\mathbf{r}_1+(1-\mu_1)\mathbf{r}_3+\mu_2\mathbf{r}_2+(1-\mu_2)\mathbf{r}_1\\
&+\mu_3\mathbf{r}_3+(1-\mu_3)\mathbf{r}_2-\mathbf{q}(u,v,w)\big],
\end{aligned} \tag{3}$$

where $\quad \mathbf{q}(u,v,w) =$

$$\begin{aligned}
&\frac{v^2w^2\mathbf{g}_1(0,v,1-v)+u^2w^2\mathbf{g}_2(u,0,1-u)+u^2v^2\mathbf{g}_3(u,1-u,0)}{u^2w^2+v^2w^2+u^2v^2}\\
&0 \le u,v,w \le 1, \quad u+v+w=1
\end{aligned} \tag{4}$$

is a correction term generated from the auxiliary curves $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ over the boundaries of the parameter triangle.

$$\mathbf{g}_1(0,v,1-v) = [\mu_3\mathbf{r}_3 + (1-\mu_3)\mathbf{r}_2]_{(0,v,1-v)} ,$$
$$\mathbf{g}_2(u,0,1-u) = [\mu_1\mathbf{r}_1 + (1-\mu_1)\mathbf{r}_3]_{(u,0,1-u)} ,$$
$$\mathbf{g}_3(u,1-u,0) = [\mu_2\mathbf{r}_2 + (1-\mu_2\mathbf{r}_1]_{(u,1-u,0)}$$

are blended curves over the sides $u=0$, $v=0$ and $w=0$, respectively of the triangular parameter domain. $\diamond$

The surface $\mathbf{f}(u,v,w)$ matches the boundary curves $\mathbf{r}_1(0,v,1-v)$, $\mathbf{r}_2(u,0,1-u)$ and $\mathbf{r}_3(u,1-u,0)$, $0 \le u \le 1$, $0 \le v \le 1$ [8].

The structure of this scheme is similar to that of Gordon–Coons' construction, where the boolean sum of three convex combinations of the given constituents is corrected according to the interpolation condition. Here the correction function has the structure of the scheme in (2) and fits the auxiliary curves:

$$\begin{aligned}
&\mathbf{q}(0,v,1-v) = \mathbf{g}_1(0,v,1-v),\\
&\mathbf{q}(u,0,1-u) = \mathbf{g}_2(u,0,1-u),\\
&\mathbf{q}(1-v,v,0) = \mathbf{q}(u,1-u,0) =\\
&\qquad \mathbf{g}_3(1-v,v,0) = \mathbf{g}_3(u,1-u,0).
\end{aligned}$$

The connection between the resulting surface and the input surface constituents is C^0 along the common boundary curves.

The shape parameters $(\lambda_1,\lambda_2,\lambda_3) = \underline{\lambda}$ are either specified by the user, or can be determined from a fairing condition. We have used the linearized thin plate energy function with $\overline{\mathbf{f}}(u,v) = \mathbf{f}(u,v,w)\big|_{w=1-u-v}$,

$$E(\underline{\lambda}) = \int_A (\overline{\mathbf{f}}^2_{uu} + 2\overline{\mathbf{f}}^2_{uv} + \overline{\mathbf{f}}^2_{vv})\, dA, \quad A = [0,1]\times[0,1]. \tag{5}$$

The optimal values of λ_1, λ_2 and λ_3 are computed by minimizing $E(\underline{\lambda})$. (In the equations the indices u and v denote the partial derivatives with respect to u and v, respectively.) The integral has been approximated by an integral sum computed at 9 inner points, and the numerical minimization has been carried out by the symbolic algebraic program package Mathematica.

For drawing triangular patches with Mathematica the parameter triangle had to be transformed into a rectangle by substituting $u = t - st$, $v = st$, $s,t \in [0,1]$. Therefore, the patches appear in the figures with s and t parameter lines.

2 Examples

In Fig 3 three triangular surface patches are shown, which are defined as quadratic Bézier surfaces. One auxiliary curve and the correction term $\mathbf{q}(u,v,w)$ is shown in Fig 4 and in Fig 5, respectively. The resulting surface defined in (3) is shown in Fig 6. It joins to the input surfaces with C^0 continuity along their connection curves.

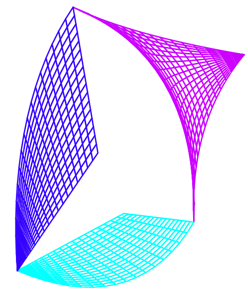


Figure 3: Three Bézier patches.

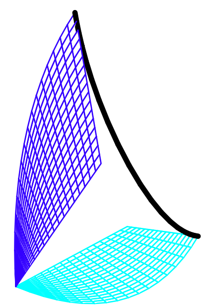


Figure 4: One auxiliary curve.

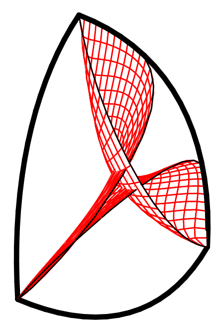


Figure 5: The correction function defined from the auxiliary curves.

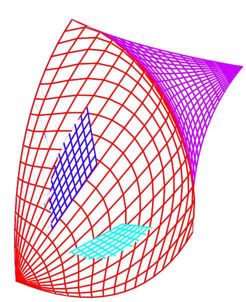


Figure 6: The resulting surface.

3 C^1 continuous blending surface constructed from differentiable patches

In this chapter a new definition of a triangular Gordon–Coons-type surface patch will be given. It is determined by three differentiable triangular patches, where three boundary curves, one of each patch, form a curvilinear triangle. The resulting patch fits these boundary curves and has a C^1-continuous connection to the given constituents along them.

Now we investigate the partial derivatives of the vector function defined in (3) along the edges of the parameter triangle T. Computing the partial derivatives with barycentric coordinates we get the following.

Along the edge $u = 0$ $\mathbf{f}_v = \mathbf{r}_{1v}$, $\mathbf{f}_w = \mathbf{r}_{1w}$ and

$$\mathbf{f}_u = \left[\mathbf{r}_{1u} + \frac{1}{2}\left(\mu_3\mathbf{r}_{3u} + (1-\mu_3)\mathbf{r}_{2u}\right)\right]\Big|_{u=0}. \tag{6}$$

Along the edge $v = 0$ $\mathbf{f}_u = \mathbf{r}_{2u}$, $\mathbf{f}_w = \mathbf{r}_{2w}$ and

$$\mathbf{f}_v = \left[\mathbf{r}_{2v} + \frac{1}{2}\left(\mu_1\mathbf{r}_{1v} + (1-\mu_1)\mathbf{r}_{3v}\right)\right]\Big|_{v=0}. \tag{7}$$

Along the edge $w = 0$ $\mathbf{f}_u = \mathbf{r}_{3u}$, $\mathbf{f}_v = \mathbf{r}_{3v}$ and

$$\mathbf{f}_w = \left[\mathbf{r}_{3w} + \frac{1}{2}\left(\mu_2\mathbf{r}_{2w} + (1-\mu_2)\mathbf{r}_{1w}\right)\right]\Big|_{w=0}. \tag{8}$$

In order to get C^1 continuous connection between the resulting surface represented by $\mathbf{f}(u,v,w)$ and the constituents

$$\mathbf{f}_u\big|_{u=0} = \mathbf{r}_{1u}\big|_{u=0}, \quad \mathbf{f}_v\big|_{v=0} = \mathbf{r}_{2v}\big|_{v=0}, \quad \mathbf{f}_w\big|_{w=0} = \mathbf{r}_{3w}\big|_{w=0},$$

must be ensured. For this an additional correction term is needed in Definition 1. Its value has to be zero along the boundary curves, and its partial derivatives have to annulate the second terms of the partial derivatives in the expressions (6), (7) and (8). The following vector function satisfies these requirements

$$\begin{aligned}\mathbf{s}(u,v,w) = \frac{1}{2}\big[&\kappa_1\left(\mu_3\mathbf{r}_{3u} + (1-\mu_3)\mathbf{r}_{2u}\right)\big|_{u=0} \\ &+ \kappa_2\left(\mu_1\mathbf{r}_{1v} + (1-\mu_1)\mathbf{r}_{3v}\right)\big|_{v=0} \\ &+ \kappa_3\left(\mu_2\mathbf{r}_{2w} + (1-\mu_2)\mathbf{r}_{1w}\right)\big|_{w=0}\big]\end{aligned} \tag{9}$$

with the blending functions

$$\begin{aligned}\kappa_1 = \frac{uv^2w^2}{\Sigma}, \quad \kappa_2 = \frac{vu^2w^2}{\Sigma}, \quad \kappa_3 = \frac{wu^2v^2}{\Sigma}, \\ \Sigma = u^2w^2 + v^2w^2 + u^2v^2.\end{aligned} \tag{10}$$

Obviously,

$$\begin{aligned}&\kappa_i\big|_{u=0} = 0, \quad \kappa_i\big|_{v=0} = 0, \quad \kappa_i\big|_{w=0} = 0, \quad i = 1,2,3, \\ &\kappa_{1u}\big|_{u=0} = 1, \quad \kappa_{1v}\big|_{v=0} = 0, \quad \kappa_{1w}\big|_{w=0} = 0, \\ &\kappa_{2u}\big|_{u=0} = 0, \quad \kappa_{2v}\big|_{v=0} = 1, \quad \kappa_{2w}\big|_{w=0} = 0, \\ &\kappa_{3u}\big|_{u=0} = 0, \quad \kappa_{3v}\big|_{v=0} = 0, \quad \kappa_{3w}\big|_{w=0} = 1.\end{aligned}$$

The required surface is defined by extending Definition 1 in the following way.

Definition 2.

$$\begin{aligned}\mathbf{f}(u,v,w) = \frac{1}{2}\big[&\mu_1\mathbf{r}_1 + (1-\mu_1)\mathbf{r}_3 + \mu_2\mathbf{r}_2 + (1-\mu_2)\mathbf{r}_1 \\ &+ \mu_3\mathbf{r}_3 + (1-\mu_3)\mathbf{r}_2\big] - \mathbf{q}(u,v,w) - \mathbf{s}(u,v,w), \\ &0 \le u,v,w \le 1, \quad u+v+w = 1,\end{aligned} \tag{11}$$

where $\mathbf{q}(u,v,w)$ is defined in (4), $\mathbf{s}(u,v,w)$ in (9) with the weighting functions in (10) $\diamond$

Considering the computed derivatives, we have obtained the following theorem.

Theorem 1. *Assume that three surface paches are given by the differentiable vector functions $\mathbf{r}_1(u,v,w)$, $\mathbf{r}_2(u,v,w)$ and $\mathbf{r}_3(u,v,w)$ on the parameter triangle $0 \le u,v,w \le 1$, $u+v+w = 1$ with common corner points, i.e.*

$$\begin{aligned}\mathbf{r}_1(0,0,1) &= \mathbf{r}_2(0,0,1), \\ \mathbf{r}_1(0,1,0) &= \mathbf{r}_3(0,1,0), \\ \mathbf{r}_2(1,0,0) &= \mathbf{r}_3(1,0,0).\end{aligned}$$

Then the surface represented by the vector function in Definition 2 interpolates the boundary curves $\mathbf{r}_1\big|_{u=0}$, $\mathbf{r}_2\big|_{v=0}$ and $\mathbf{r}_3\big|_{w=0}$, and joins to the corresponding surface patch C^1 continuously along the common boundary with the exception of the corner points.

Proof. The proof follows from the computations above. However, the compatibility conditions of the differentiability of the resulting surface at the vertices require further investigations. $\square$

4 Examples

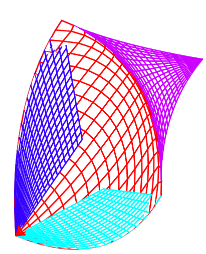


Figure 7: C^1 continuous surface defined from the constituents in Fig 3.

In Fig 7 the surface constructed by Definition 2 is shown. It is generated from the same quadratic Bézier patches as the surface in Fig 6. There is a visible difference between the C^0 and C^1 results. While the C^0 surface is rather round, and intersects the constituents, the C^1 result has common tangent planes with them along the common boundary curves. The next two figures illustrate the effect of the shape parameters λ_i included in the blending coefficients μ_i, $i = 1, 2, 3$. In the equation of the resulting surface in Fig 7 the shape parameters have been determined from the fairing condition by minimizing the energy function in (5). The same surface is shown from a side view in Fig 8.

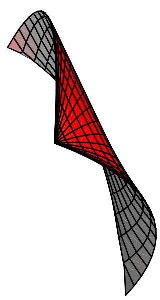


Figure 8: The surface in Fig 7 from the side.

In Fig 9 the surface is generated from the same constituents, but the shape parameters have been given as user inputs. The value of λ_3 influencing the weight of the given patch on the right hand side has been raised. Consequently, the result is less concave in the middle.

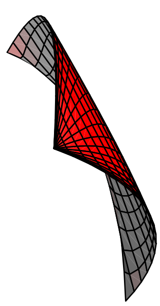


Figure 9: The surface generated with different shape parameters.

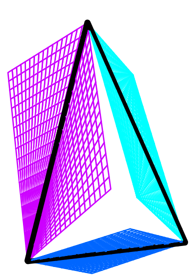


Figure 10: An open corner on a prism.

In Fig 10 an open corner on a prism is shown modelled with triangular Bézier patches. The boundary of the triangular hole is drawn with heavy lines. The constituents in the surface definition are in the inside of this triangle. The neighbouring triangles are coplanar extensions of them.

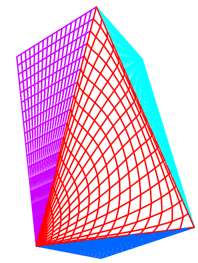


Figure 11: The C^1 resulting surface with the extensions of the constituents.

The constructed surface is shown in Fig 11. It fits the boundary and has common tangent planes with the neighbouring surfaces.

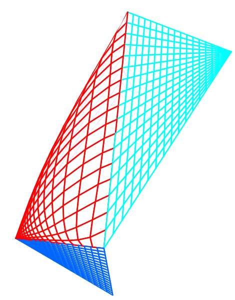


Figure 12: The same solution from a different view.

In Fig 12 the same surface is shown from a side view in order to make the comparison with the next examples easier.

The next figures show the shaping effect of the constituents. In Fig 13 different constituents with the same boundary curves and tangent planes are shown, the resulting C^1 surface is shown in Fig 14.

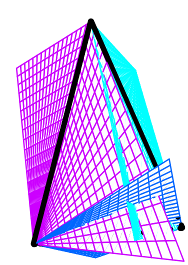


Figure 13: Constituents with the same boundaries.

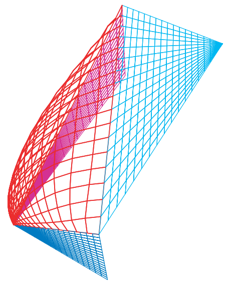


Figure 14: The result has a different shape.

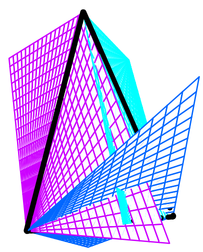


Figure 15: Changing one input patch.

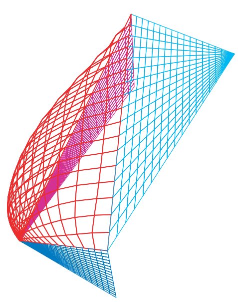


Figure 16: The effect on the inner shape of the resulting surface.

In Fig 15 the input patch on the lower side has been changed while keeping its boundary fixed. The result with these constituents is shown in Fig 16.

5 Conclusions

We have presented a new surface definition, which generates a triangular patch from three triangular surface patches. Novel in this definition is that the inner shape of the resulting surface can be modified by changing the input surface patches while keeping the boundary conditions fixed. Moreover, new is the introduction of shape parameters in the blending functions. This surface construction can be applied for filling triangular holes which occur in modelling of composite surfaces.

References

[1] BARNHILL, R.E., BIRKHOFF, G. AND GORDON, W.S.: *Smooth Interpolation in Triangles*, Journal of Approximation Theory, 8, 1973. pp. 114–128.

[2] COONS, S.A.: *Surfaces for computer-aided design of space forms*, Project MAC report, 1964.

[3] HOSCHEK, J., LASSER, D.: Grundlagen der geometrischen Datenverarbeitung, B. G. Teubner Stuttgart, 1992.

[4] FARIN, G.: Curves and Surfaces for Computed Aided Geometric Design, Academic Press, London, 1990.

[5] NIELSON, G.M.: *The Side-vertex Method for Interpolation in Triangles*, Journal of Approximation Theory, 25, 1979. pp. 318–336.

[6] SZILVÁSI-NAGY, M., VENDEL, T.P., STACHEL, H.: *C^2 filling of gaps by convex combination of surfaces under boundary constrains*, KoG, 6, 2002. pp. 41–48.

[7] SZILVÁSI-NAGY, M.: *Filling triangular holes by convex combination of surfaces*, Periodica Polytechnica Mech. Engrg., 47, 2003. pp. 81–89.

[8] SZILVÁSI-NAGY, M., SZABÓ, I.: *Generalization of Coons' Construction*, manuscript, submitted in computers & graphics

Márta Szilvási-Nagy
Dept. of Geometry
Budapest University of Technology and Economics
H-1521 Budapest, Hungary
e-mail: szilvasi@math.bme.hu

Ildikó Szabó
Dept. of Geometry
Budapest University of Technology and Economics
H-1521 Budapest, Hungary
e-mail: szabo@math.bme.hu

Stručni rad
Prihvaćeno 15. 12. 2004.

ANA SLIEPČEVIĆ
JASNA KOS -MODOR

Neke planimetrijske konstrukcije u H-ravnini

Some Planimetric Constructions in the H-plane

ABSTRACT

On the Klein's model of the hyperbolic plane we can define the central collineation between absolute and H-circle. Using that central collineation it's possible to make the elementary geometric constructions in the H-plane by the instruments of the euclidean geometry. Two problems have been solved in this way.

Key words: hyperbolic geometry, hyperbolic circle, central collineation

MSC 2000: 51 M 10, 51 M 15

Neke planimetrijske konstrukcije u H-ravnini

SAŽETAK

Na Kleinovom modelu hiperboličke ravnine uspostavja se centralna kolineacija između apsolute i H-kružnice. Pokazuje se kako je moguće uz pomoć ove centralne kolineacije, sredstvima euklidske geometrije izvoditi elementarne geometrijske konstrukcije u H-ravnini. U tom su smislu riješena dva zadatka.

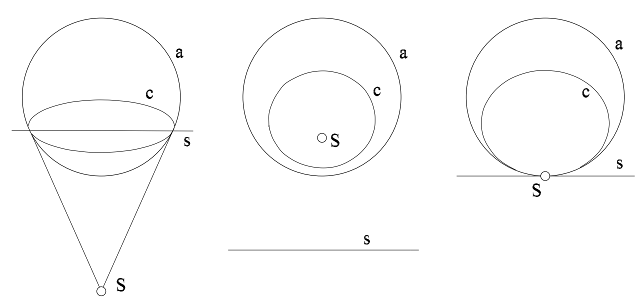
Ključne riječi: hiperbolička geometrija, hiperbolička kružnica, centralna kolineacija

U hiperboličkoj su ravnini planimetrijske konstrukcije neizvedive zbog nepostojanja "hiperboličkog ravnala" i "hiperboličkog šestara", pa su često pri objašnjavanju prisutne jedino skice. Mnogi zadaci postaju konstruktibilni tek na nekom od euklidskih modela H-ravnine. U tom je smislu najpogodniji Kleinov model i to onaj s euklidskom kružnicom kao apsolutom.

Neka je kružnicom *a* zadana apsolutna konika Kleinovog modela hiperboličke ravnine. Točke unutar apsolute zovemo pravim, one izvan apsolute nepravim ili idealnim, a točke na apsoluti graničnim točkama H-ravnine. Za dva pravca koji se sijeku u pravoj točki kaže se da su ukršteni. Ako im je sjecište na apsoluti, pravci su paralelni, a ako se sijeku izvan apsolute, kaže se da su hiperparalelni.

U euklidskoj se ravnini smatra da je točka geometrijski točno konstruirana ukoliko je određena kao sjecište dvaju pravaca, pravca s kružnicom ili kao sjecište dviju kružnica. Kako se ovdje radi o euklidskom modelu hiperboličke geometrije, za očekivati je da se i u njemu mogu elementarne geometrijske konstrukcije izvoditi ravnalom i šestarom. Pokazuje se da je to često doista moguće, uz napomenu da su konstrukcije znatno složenije od analognih konstrukcija u euklidskoj ravnini.

Hiperboličkom kružnicom u ovom modelu zovemo svaku onu koniku koja dodiruje apsolutu u dvije realne ili dvije konjugirano imaginarne točke ili su ta dva dirališta pala zajedno. Konika koja dira apsolutu u dvije različite realne točke zove se *hipercikl*, konika koja dira apsolutu u paru konjugirano imaginarnih točaka zove se *cikl*, a ako imaginarna ili realna apsolutna dirališta padnu u istu točku, hiperbolička se kružnica zove *horicikl* [3], (Slika 1).



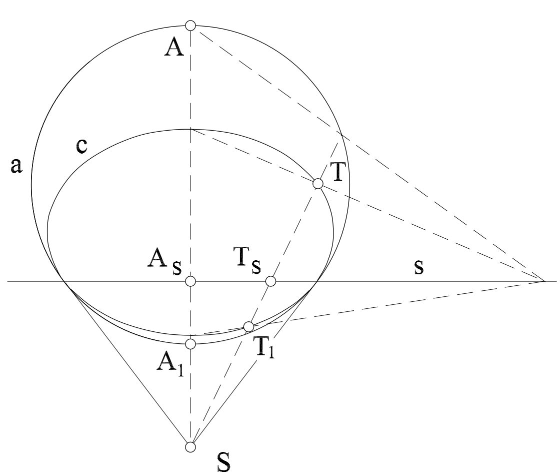
Slika 1

Spojnica *s* apsolutnih dirališta H-kružnice zove se *os* H-kružnice, a apsolutni pol *S* te spojnice njezino je *središte*. Središte hipercikla je neprava točka, a os mu je pravi pravac, središte cikla je prava točka, a os je nepravi pravac. Horicikl ima središte na apsoluti, a os mu je izotropni pravac (tangenta apsolute).

Zbog egzistencije triju vrsta kružnica, u H-ravnini će mnoge geometrijske činjenice biti kompleksnije nego u euklidskoj ravnini, dok će mogućnosti zadavanja nekih geometrijskih figura biti ograničene ili čak neostvarive logikom euklidske ravnine (npr. konstrukcija pravilnih poligona). Kao što je poznato, kružnicu je u euklidskoj ravnini moguće jednoznačno zadati čak na pet načina: trima različitim realnim točkama, jednom jednostrukom i jednom dvostruko brojenom točkom, jednom realnom i parom konjugirano imaginarnih točaka, središtem i točkom, središtem i tangentom. Za razliku od toga hiperboličku kružnicu moguće je jednoznačno zadati na sljedeće načine: točkom i središtem, točkom i osi, tangentom i središtem, tangentom i osi. Trima su točkama, kao i trima tangentama, općenito određene čak četiri H-kružnice! [1], [2].

Bez uvođenja metrike, moguće je na ovom modelu rješavati jednostavne položajne planimetrijske zadatke u vezi s H-kružnicom. U tu je svrhu potrebno uspostaviti jednostavnu linearnu transformaciju, primjerice centralnu kolineaciju ravnine, koja ostavlja apsolutu fiksnom u cjelini.

Neka je zadana involutorna centralna kolineacija H-ravnine kojoj su središte S i os s pol i polara u odnosu na apsolutu, a sjecišta bilo kojeg pravca kroz pol s apsolutom par pridruženih točaka A, A_1 (Slika 2.). Očigledno ova transformacija preslikava apsolutu samu u sebe [3]. Nije teško zaključiti da se svaka H-kružnica sa središtem u središtu takve centralne kolineacije također preslikava sama u sebe, odnosno sve se kružnice koncentričnog pramena H-kružnica s istim središtem S (i istom osi) preslikavaju same u sebe. Pri tome vrijedi: $(SA_s, A_1A) = (ST_s, T_1T) = -1$. Ova se transformacija u hiperboličkoj ravnini zove *osna simetrija*, a točke A_1, A, odnosno T_1, T simetrične su u odnosu na os s.



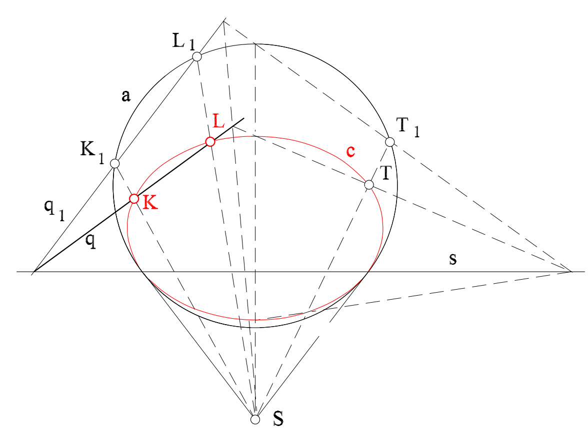
Slika 2

Riješimo ovdje neke elementarne planimetrijske zadatke u vezi s H-kružnicom.

Zadatak 1.
Konstruiraj hipercikl c sa središtem u točki S koji prolazi pravom točkom T, te odredi njegova sjecišta K i L s pravcem q (Slika 3).

Rješenje.

- Središtem S i točkom T jednoznačno je zadan hipercikl c.
- Ranije spomenutom involutornom centralnom kolineacijom taj se hipercikl preslikava sam u sebe. No, valja uočiti da postoji i takva centralna kolineacija, koja ovaj hipercikl preslikava u apsolutu. Ta je kolineacija zadana s istom osi s i središtem S, te parom pridruženih točaka T, $T_1 = ST \cap a$. Hipercikl c konstruira se pomoću ove centralne kolineacije kao kolinearna slika apsolute.
- Kada se ovom kolineacijom preslika i zadani pravac q, njegova će slika q_1 sjeći apsolutu a u dvije točke K_1 i L_1 koje su pridružene traženim sjecištima K i L pravca q s hiperciklom c.



Slika 3

Zadatak 2.
Zadana je os s i jedna tangenta t cikla c. Konstruiraj one tangente cikla koje prolaze zadanom točkom Q (Slika 4.).

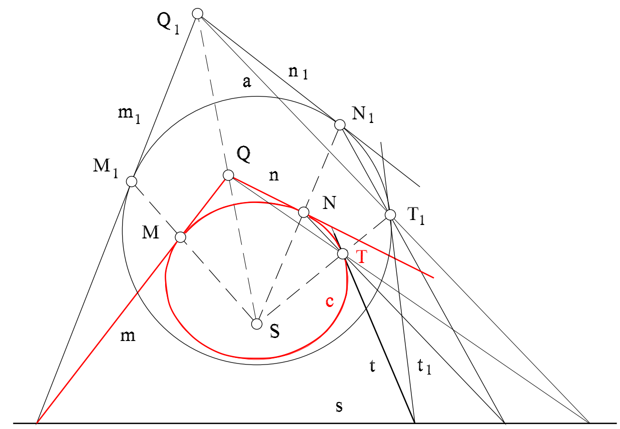
Rješenje.

- Cikl je jednoznačno određen svojom osi s i tangentom t.
- Pol S pravca s u odnosu na apsolutu je središte zadanoga cikla.

- U centralnoj kolineaciji sa središtem S i osi s, kojom se zadani cikl preslikava u apsolutu, zadanoj će tangenti t cikla biti pridružena tangenta t_1 apsolute koja prolazi sjecištem pravca t s osi kolineacije s. Njenom diralištu T_1 s apsolutom pridruženo je dirališta T cikla s tangentom t.

- Pomoću točke T konstruiraju se ostale točke zadanoga cikla kao kolinearne slike apsolutnih točaka.

- Spomenutom se kolineacijom zadana točka Q preslikava u točku Q_1, a tangentama m_1, n_1 apsolute koje prolaze točkom Q_1 kolinearno su tada pridružene tražene tangente m, n cikla c.



Slika 4

Literatura

[1] BABIĆ, I., *Neke kolineacije H-ravnine*, (prihvaćeno za tisak u KoG-u)

[2] SLIEPČEVIĆ, A., BABIĆ, I., *Charakteristische Dreieckpunkte in der projektiv erweiterten hyperbolischen Ebene*, (predano za tisak)

[3] RAJČIĆ, L., *Obrada osnovnih planimetrijskih konstrukcija geometrije Lobačevskog sintetičkim sredstvima*, Glasnik MFA 5, (1950), 57-120.

Ana Sliepčević
Građevinski fakultet Sveučilišta u Zagrebu
Kačićeva 26, 10000 Zagreb
e-mail: anas@grad.hr

Jasna Kos-Modor
Rudarsko-geološko-naftni fakultet Sveučilišta u Zagrebu
Pierottijeva 6, 10000 Zagreb
e-mail: jasna.kos-modor@rgn.hr

Stručni rad
Prihvaćeno 28. 11. 2005.

IVANKA BABIĆ

Neke kolineacije H-ravnine

Some collineations of H-plane

ABSTRACT

On the Klein's model of the hyperbolic plane the harmonic homology is defined. This collineation maps absolute points of the h-plane onto absolute points, real points onto real points and ideal points onto ideal points. It is called line symmetry if the center of collineation is ideal point and point symmetry if the center is real point, because described mappings have equal properties as the analogues mappings in the Euclidean plane. By using point and line symmetries, symmetric images of the lines, points and triangles, bisectors of the angles and perpendicular bisectors of the segments are constructed. At the end one complicated metric problem is solved.

Key words: hyperbolic plane, Klein's model of the hyperbolic plane, central involutory collineation

MSC 2000: 51 M 10, 51 M 15

Neke kolineacije H-ravnine

SAŽETAK

Na Kleinovom modelu hiperboličke ravnine definirana je centralna involutorna kolineacija, koja preslikava granične točke H-ravnine u granične, prave točke u prave, a idealne u idealne. Zovemo ju osna simetrija ukoliko je centar idealna točka, a centralna simetrija ako je centar prava točka, jer imaju sva svojstva istoimenih kolineacija euklidske ravnine. Pomoću osne i centralne simetrije konstruirane su osno-simetrične i centralno-simetrične slike pravaca, točaka i trokuta, simetrale kutova i dužina. Na kraju je riješen jedan složeniji metrički zadatak.

Ključne riječi: hiperbolička ravnina, Kleinov model H-ravnine, centralna involutorna kolineacija.

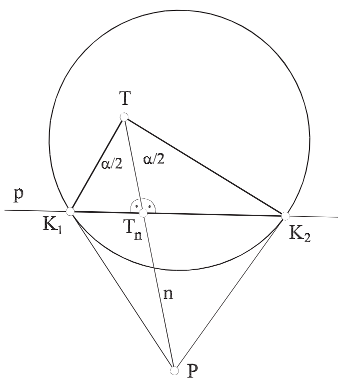
Transformaciju u realnoj projektivnoj ravnini, koja preslikava pravce u pravce, a točke u točke, zovemo *kolineacijom* ravnine. Kolineacija je *perspektivna* ili *centralna* ako postoji pravac na kojemu su sve točke fiksne i jedna istaknuta fiksna točka, koja je izvan ili na fiksnom pravcu. Fiksni pravac zove se *os*, a fiksna točka *centar* ili *središte* centralne kolineacije. *Zrake kolineacije* su pravci koji prolaze centrom kolineacije i na njima se nalaze parovi kolinearno pridruženih točaka. Prema odabiru osi i središta sve se centralne kolineacije dijele na: *homologije* i *elacije*. Homologije su centralne kolineacije u ravnini kod kojih središte ne pripada osi, dok kod elacija središte leži na osi. U homologije ubrajamo *centralne simetrije, osne simetrije, homotetije, perspektivne afinosti u užem smislu, perspektivne kolineacije u užem smislu*, dok u elacije ubrajamo: *translacije, elacije u užem smislu* i *posmike*.

Kolineacije u *H-ravnini* analogno su definirane kao u realnoj projektivnoj ravnini. Posebno su zanimljive centralne kolineacije. Budući da u H-ravnini postoje tri vrste točaka i pravaca, to postoji i više vrsta centralnih kolineacija. U skupu svih centralnih kolineacija H-ravnine posebno se ističu one koje preslikavaju granične (apsolutne) točke u granične (apsolutne). To su *involutorne centralne kolineacije* i one preslikavaju prave točke u prave, a idealne u idealne. Zovu se općenito *zrcaljenja* u koja ubrajamo **osne simetrije** ako je centar neprava točka, a os pravi pravac H-ravnine, odnosno **centralne simetrije H-ravnine** ako je centar u pravoj točki H-ravnine, a os je idealni pravac, jer imaju sva svojstva istoimenih kolineacija euklidske ravnine. (Duljina dužine i veličina kuta dvaju pravaca invarijante su ovakvih transformacija kao i u euklidskom slučaju.), [2]. Korisne su kod rješavanja metričkih zadaća u H-ravnini.

Za konstruktivno rješavanje zadataka vezanih uz preslikavanja točaka H-ravnine na sebe, najpogodniji je Kleinov model H-ravnine, jer su u njemu *H-pravac* i *H-točka* prikazani pravcem i točkom u euklidskom smislu. U tom je modelu apsolutna konika zadana euklidskom kružnicom. Njene točke su *granične točke* H-ravnine. Točke unutar apsolute zovemo *pravim*, a izvan apsolute *nepravim* ili *idealnim točkama* H-ravnine.

Dvije točke bilo koje vrste u H-ravnini određuju jedan i samo jedan pravac, koji zovemo *pravi*, *idealni bez graničnih točaka* ili *idealni s jednom graničnom točkom*, već prema tome da li apsolutnu koniku siječe realno, imaginarno ili ju tangira.

Svakom točkom izvan pravog pravca prolaze dvije paralele sa zadanim pravcem H-ravnine. Figura koju čini pravi pravac i njegove dvije paralele zove se *dvokrajnik*. To je trokut TK_1K_2 kojemu su dva vrha granične točke ili *krajevi*. Poznato je, da je u dvokrajniku simetrala kuta α pri pravom vrhu T okomita na spojnicu graničnih vrhova t.j. stranicu K_1K_2 (Slika 1). Kut $\alpha/2$ pri pravom vrhu zove se *kut paralelnosti*, a visina TT_n određuje udaljenost točke T od stranice K_1K_2 i zove se *distanca paralelnosti*.



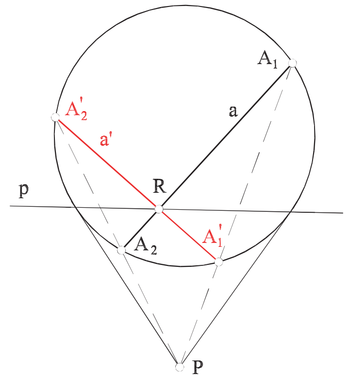
Slika 1

H-okomitost pravaca definirana je apsolutnim polaritetom t.j. pravci su okomiti, ako svaki od njih prolazi apsolutnim polom drugog. Svake dvije apsolutno konjugirane polare bez obzira da li se sijeku u pravoj, graničnoj ili idealnoj točki, međusobno su okomite.

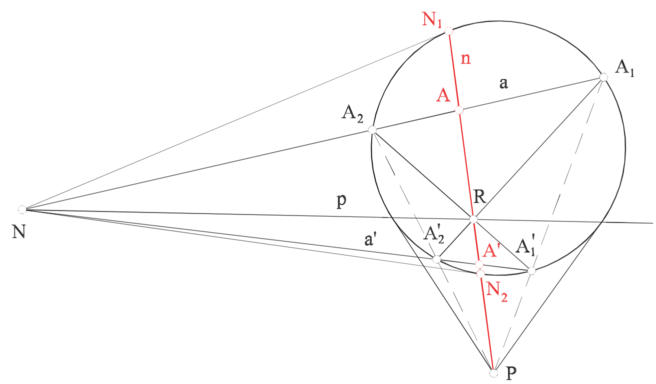
1 Osna i centralna simetrija u H-ravnini

Kolineacija H-ravnine kojoj su centar P i os p pol i polara u odnosu na apsolutnu koniku, a par pridruženih točaka sjecišta zrake i apsolute je **centralna involutorna kolineacija**, koja preslikava apsolutu na sebe tako da njene točke mijenjaju mjesta. Svaku točku (pol) i pridruženu polaru možemo shvatiti kao centar i os jedne centralne involutorne kolineacije.

Neka je centar kolineacije idealna točka P. Svaka zraka točkom P okomita je na os p kolineacije. Ako sjecišta zrake s apsolutom označimo N_1, N_2, a sjecište s osi p s R, tada vrijedi : $(PRN_1N_2) = -1$ (Slika 2b). Na svakoj zraci parovi točaka N_1, N_2 i P, R određuju hiperboličku involuciju, kojoj su točke P i R dvostruke točke. Isto tako za svaki par pridruženih toaka A, A' na zraci n centralne involutorne kolineacije (Slika 2b) vrijedi $(PRAA') = -1$. Slijedi da je $AR = A'R$, t.j. parovi pridruženih točaka stoje simetrično obzirom na os kolineacije, a kako je zraka kolineacije okomita na os, ova je kolineacija analogon osnoj simetriji u euklidskoj ravnini.



Slika 2a



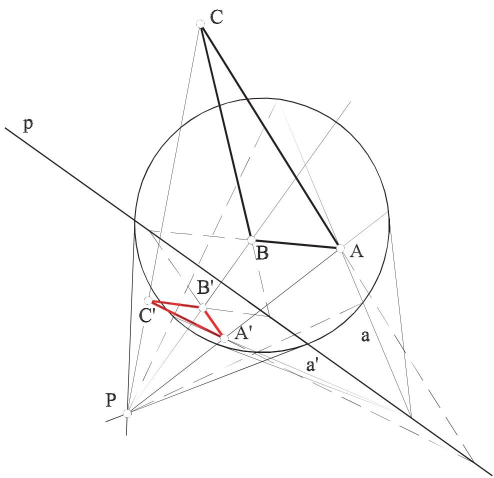
Slika 2b

Centralna involutorna kolineacija preslikava prave točke u prave, granične u granične, a idealne u idealne, prave pravce preslikava u prave, idealne u idealne, a idealne s jednom graničnom točkom u idealne s jednom graničnom točkom. Pri tom je preslikavanju dvoomjer četiriju elemenata (točaka ili pravaca) invarijantan, a kako je metrika u Kleinovom modelu definirana dvoomjerom $[d = 1/2\ln(ABN_2N_1); \varphi = i/2\ln(p_1p_2t_2t_1)]$, znači da ta kolineacija čuva duljinu dužine i veličinu kuta dvaju pravaca [2]. Centralnu involutornu kolineaciju $(\mathbb{P},\mathbb{p})$ zadanu centrom P i njenom polarom kao osi p, zovemo **zrcalenje** i to: **osna simetrija** u slučaju da je centar idealna točka odnosno **centralna simetrija** kada je centar prava točka.

Neka je zadana **osna simetrija** $(\boldsymbol{P},\boldsymbol{p})$ i neki pravac a s graničnim točkama A_1, A_2 (Slika 2a). Osno-simetrična slika toga pravca je pravac a' koji prolazi graničnim točkama A_1', A_2' koje su centralno-involutorno kolinearno pridružene krajevima A_1, A_2. Pravci a i a' sijeku se u točki R na osi kolineacije p. To sjecište može biti prava, granična ili idealna točka. Ukoliko je to idealna točka, kažemo da su pravci a i a' razilazni ili hiperparalelni.

Osno-simetrična slika A' zadane točke A u osnoj simetriji (P,p) može se odrediti na dva načina: a) kao sjecište zrake i zrcalne slike bilo kojeg pravca položenog kroz A; b) pomoću pravca a točkom A koji je okomit na zraku n točke A (Slika 2b). Slika a' pravca a siječe zraku u točki A', a zraka n je zajednička normala pravaca a i a'. Na tom se principu temelji konstrukcija polovišta dužine. Točka R je polovište dužine $\overline{AA'}$.

Na slici 3 zadan je trokut ABC i osna simetrija (P,p). Konstruiran je osno-simetričan trokut $A'B'C'$ tako, da je najprije konstruirana točka A' slika točke A prema konstrukciji na slici 2b., dok su ostale točke konstruirane spomenutom osnom simetrijom pomoću para pridruženih točaka A,A'. Vidljivo je da se pritom prave točke A i B zrcale u prave A' i B', a slika idealnog vrha C trokuta idealna je točka C'.



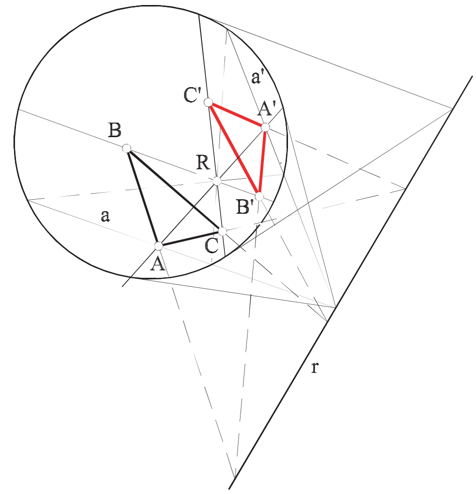
Slika 3

Na slici 4 pravom je točkom R i njenom idealnom polarom r zadana **centralna simetrija** $(\boldsymbol{R},\boldsymbol{r})$. Točki A konstruira se pridružena točka A' na isti način kao u slučaju osne simetrije. Pomoću tog para je određena centralno simetrična slika $A'B'C'$ trokuta ABC.

Teorem 1 *Osna simetrija (S, s), koja pridružuje pravce $a(A_1,A_2)$ i $a'(A_1,A_2')$, pridružuje i njihove polove A i A' i obrnuto, ako pridružuje točke A i A' tada pridružuje i njihove polare $a(A_1,A_2)$ i $a'(A_1,A_2')$.*

Dokaz teorema je evidentan [3].

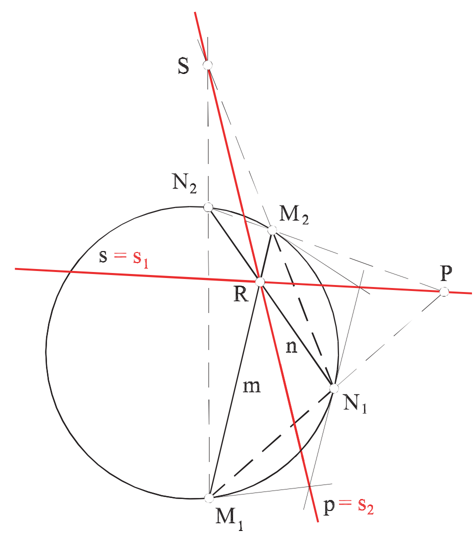
Temeljem ovog teorema moguće je konstruirati osno simetričnu sliku bilo kojeg idealnog pravca zadanom simetrijom (P,p) kao polaru one prave točke, koja je osno simetrična slika pola zadanog pravca. Ova se činjenica koristi kod konstrukcije simetrale kuta idealnih pravaca.



Slika 4

2 Simetrala kuta i simetrala dužine

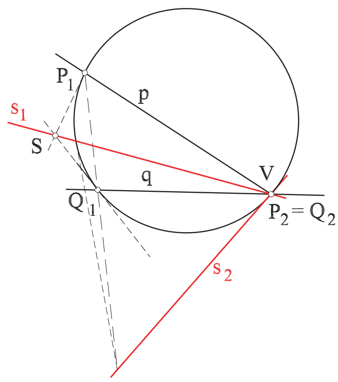
Zadatak 1.
Odredite osnu simetriju koja meusobno preslikava pravce $m(M_1,M_2)$ i $n(N_1,N_2)$ (Slika 5a).



Slika 5a

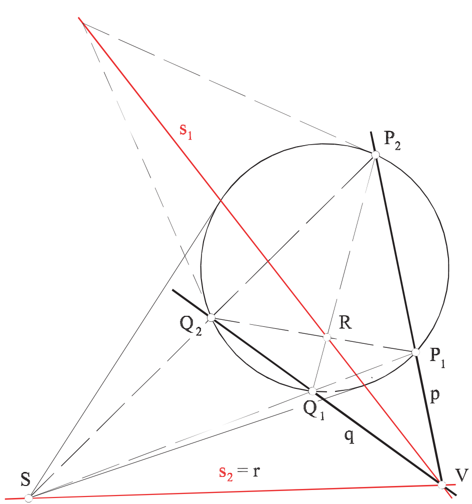
Neka je $R = m \cap n$. Budući da se osnom simetrijom granične točke preslikavaju u granične točke, zrake tražene osne simetrije mogu biti spojnice $\overline{M_1N_1}$ i $\overline{M_2N_2}$ ili $\overline{M_1N_2}$ i $\overline{M_2N_1}$, a njihovo je sjecište centar $P = \overline{M_1N_1} \cap \overline{M_2N_2}$, odnosno $S = \overline{M_1N_2} \cap \overline{M_2N_1}$. Polara p točke P odn. polara s točke S osi su dviju osnih simetrija (P,p) i (S,s) koje preslikavaju pravce m i n. Pravci p i s zapravo su simetrale kutova zadanih pravaca m i n. Iz konstrukcije je vidljivo da su polare p i s apsolutno konjugirane, dakle međusobno okomite.

Općenito se **simetrale kutova** dvaju pravaca mogu konstruirati primjenom na početku spomenutog svojstva dvokrajnika. Konstruira se pravac paralelan s oba kraka zadanog kuta, dakle spojnica krajeva zadanih pravaca (M_2N_1 odn. M_1N_1) na koju tražena simetrala treba biti okomita (Slika 5a). Tražena simetrala prolazi polom ove spojnice i sjecištem R zadanih pravaca. Na taj su način konstruirane simetrale kutova dvaju pravih pravaca, koji se sijeku u pravoj odnosno graničnoj točki (Slike 5a, 5b). Uočimo da se u slučaju sjecišta u graničnoj točki (Slika 5b) jedna od simetrala poklapa s tangentom apsolute.



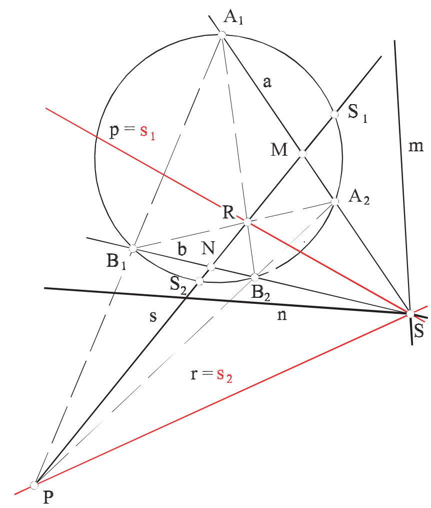
Slika 5b

Ukoliko se pravi pravci p i q sijeku u idealnoj točki V, jednu od simetrala konstruiramo na spomenuti način, a druga je njoj konjugirana (Slika 5c). Tu drugu simetralu moguće je konstruirati i tako, da odredimo centralnu simetriju (R,r) u kojoj su zadani pravci p i q simetrični. Os r te centralne simetrije je druga simetrala kuta zadanih pravaca.



Slika 5c

Konstrukcija simetrala kutova dvaju idealnih pravaca m i n sa sjecištem u idealnoj točki S (Slika 5d) izvodi se primjenom navedenog teorema. Kracima kuta m i n odrede se polovi M i N, čija je spojnica s polara točke S (vrha kuta). Zatim se odrede one osne simetrije, koje preslikavaju točke M i N, a prema teoremu one preslikavaju i njihove polare m i n. Osi r i p tih osnih simetrija su simetrale kutova pravaca m, n.



Slika 5d

Ova konstrukcija vodi na određivanje **simetrale dužine**.

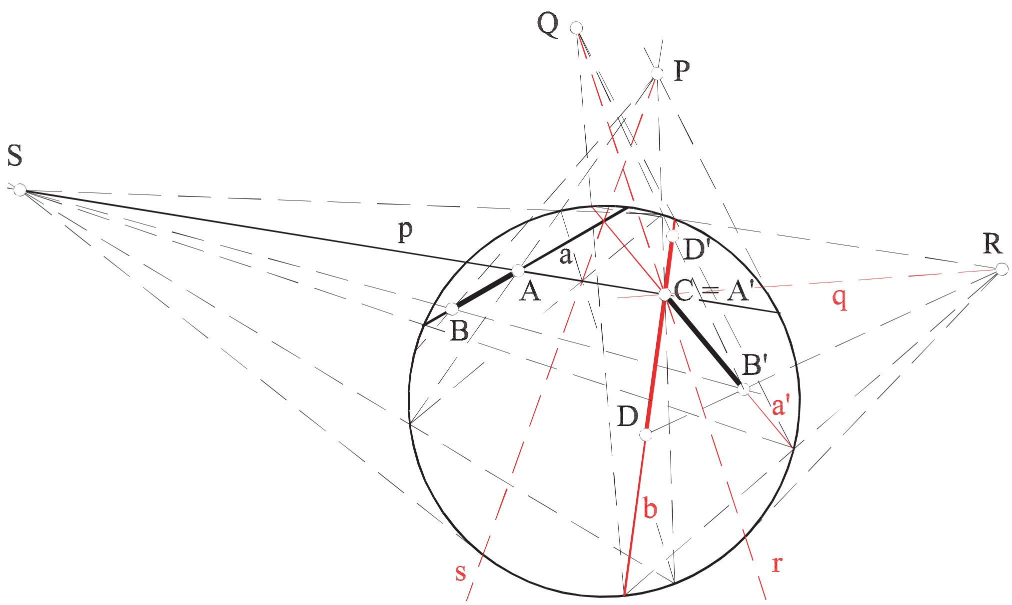
Neka je zadana dužina $\overline{MN}$ na pravcu s s krajevima S_1 i S_2 (Slika 5d). Pretpostavimo da os p tražene osne simetrije (P,p) siječe pravac s u točki R, tada za parove pridruženih točaka vrijedi: $(PRS_1S_2)=-1$ i $(PRMN)=-1$. Ta dva para involutorno-pridruženih točaka određuju na pravcu s hiperboličku involuciju čije dvostruke točke P i R predstavljaju *H-polovišta dužine* $\overline{MN}$ odnosno $\overline{NM}$, [1]. Određivanje polovišta dužine i simetrala te dužine, svodi se na određivanje simetrala kuta, čiji kraci prolaze krajnjim točkama te dužine, a vrh mu je u polu njezinog pravca nosioca. U tu svrhu postavimo pravce a i b točkama M i N okomito na njihovu spojnicu s. Neka su sa A_1, A_2 i B_1, B_2 označeni krajevi ovih pravaca. Prema slici 5c odredimo takve osne simetrije (P,p) i (R,r), koje preslikavaju pravce a i b. One preslikavaju i točke $M\in a$ i $N\in b$. Pravci p i r su tada *H-simetrale dužina* $\overline{MN}$ odnosno $\overline{NM}$, jer prolaze njihovim *H-polovištima* R i P, a okomiti su na pravac s. Uočimo da su ovi pravci ujedno i simetrale kutova koje zatvaraju idealni pravci m i n.

Rješavajući problem konstrukcije simetrala kutova idealnih pravaca riješili smo i problem određivanja simetrala dužine. Zaključujemo da *H-dužina* ima **dvije simetrale** i **dva polovišta** za razliku od dužine u euklidskoj ravnini. Važno je pri tome napomenuti, da se kod H-dužina kojima je jedna krajnja točka granična ili kod onih kojima je jedna krajnja točka prava, a druga idealna, ne može govoriti o polovištima odnosno simetralama dužina.

Koristeći osnu i centralnu simetriju te njihove kompozicije možemo rijeiti niz metričkih zadataka u H-ravnini.

Zadatak 2.
Prenesi dužinu $\overline{AB}$ čiji je nosilac pravac a na pravac b od zadane točke $C \in b$ (Slika 6). Zadatak ćemo riješiti pomoću dvije osne simetrije.

- Prvo odredimo osnu simetriju (S,s) koja preslikava točku A u točku C. Os te simetrije bit će simetrala dužine $\overline{AC}$ koja je određena kao na slici 5d. Tom se osnom simetrijom preslikava pravac a u pravac a', dužina $\overline{AB} \in a$ u dužinu $\overline{A'B'} \in a'$, pri čemu je $A' = C$.
- Druga osna simetrija (R,r), određena prema konstrukciji na slici 5a, preslikava pravac a' u pravac b, a dužinu $\overline{A'B'}$ u dužinu $\overline{CD} \in b$. Uočimo da postoji i druga osna simetrija (Q,q) koja također preslikava pravac a' u b, pa time i dužinu $\overline{A'B'}$ u dužinu $\overline{CD'}$.



Slika 6

Literatura

[1] BABIĆ, I., KUČINIĆ, B., *M-Modell des hyperbolischen H^3-Raums in der Mobius-Ebene*, Rad HAZU 467 (1994), 67-75

[2] KLEIN, F., *Vorlesungen uber nicht-euklidische Geometrie*, Berlin, Springer-Verlag, 1928., Nachdruck 1968.

[3] RAJČIĆ, L., *Obrada osnovnih planimetrijskih konstrukcija geometrije Lobačevskog sintetičkim sredstvima*, Glasnik MFA 5, (1950), 57-120.

Ivanka Babić
Graditeljski odjel Tehničkog Veleučilišta u Zagrebu
Avenija Većeslava Holjevca 15, 10000 Zagreb
e-mail: ibabic@tvz.hr

Professional paper
Accepted 29.11.2005

Milena Stavrić
Albert Wiltsche
Heimo Schimek

New Dimension in Geometrical Education

Methods of Representation Course - New Media, Classical Construction, Geometrical Freehand Drawing

New Dimension in Geometrical Education

ABSTRACT

This paper describes the course "Methods of representation" that has evolved at the Faculty of Architecture in Graz in conjunction with developments in the modern practice of architectural design. We established web sites (http://www.geometrie.tugraz.at/lehre/dm_ue03/ and http://ikg.tugraz.at/dm0/ws04/), which include the introduction in teaching, tutorials and VRLM animations to help students understanding space geometry. The course focuses on classical geometrical representation methods, solid modelling in CAD and geometrical freehand drawing. Each of these parts will be worked out and examples of student exercises will complete the paper.

Key words: descriptive geometry education, CAD, visual communication

MSC 2000: 51N05, 68U07, 97U40

Nova dimenzija u geometrijskom obrazovanju

SAŽETAK

U članku se daje prikaz kolegija "Metode prezentacije", koji se predaje na Arhitektonskom fakultetu u Grazu, a čiji je sadržaj usko povezan sa suvremenim arhitektonskim projektiranjem. Izrađene su web stranice (http://www.geometrie.tugraz.at/lehre/dm_ue03/ i http://ikg.tugraz.at/dm0/ws04/), koje upoznaju studente s predavanjima, vode ih kroz materiju i pružaju VRLM animacije u namjeri da im se pomogne razumjeti geometriju prostora. U kolegiju se stavlja naglasak na klasične geometrijske konstrukcije, 3D modeliranje s CAD podrškom i prostoručno crtanje. Svaki od ovih dijelova se zasebno razrađuje i upotpunjuje s primjerima studentskih programa.

Ključne riječi: izobrazba o deskriptivnoj geometriji, CAD, vizualna komunikacija.

1 Preamble

During the past 15 years new media including new technologies have gradually brought changes in the matter of knowledge absorption on all educational levels. On the one hand new technologies have introduced many facilitating methods and approaches and on the other hand new Medias have made the transfer of information much faster which is at times not in accordance with traditional educational theories. The introduction of CAD technology in technical departments has especially affected the worldwide methods of teaching geometry.

2 The influence of CAD technology on the subject of geometry

Since 1990 CAD technology has been more and more often applied and it is today used at all technical departments. During the period of 1990-2000 different CAD software were mainly implemented at the technical departments in the classes of descriptive geometry. In the new syllabus of the courses in the field of "graphic-visual communication" the content in the field of geometry was reduced - we may even say minimized, while the students were instructed in 2D drawing on PCs. According to Stachel [1] this happened mainly due to a misunderstanding regarding the syllabus of descriptive geometry and its constructive technique by using drawing equipment. Descriptive geometry was neglected although it is the only discipline at the technical departments which teaches future engineers to communicate with one another by means of drawings and also the only discipline which trains visual spatial intelligence. Just a few years after these changes were introduced Field [2] noticed an anecdotic deterioration of the students' ability of spatial visualization.

Reduced instruction in geometry at the universities has initiated a number of enquiries worldwide ([3], [4], [5], [6], [7]). They proved that there was a direct connection between the study of the subject matter of descriptive geometry and the improvement of visual spatial intelligence([8], [9]) - as one of the most important ability for an engineer. The conclusion after this first euphoric computerization of instruction was that two dimensional CAD software were only understood as electronic ruler, ink and pattern, which clearly defined a very low level as the highest achievement of this type of education.

On the basis of the results of the above mentioned enquiries as well as on the basis of the recommendations by UNESCO on the reform of instruction in the next millennium [10] there appeared a second wave of research ([11], [12], [13], [14], [15]) with the aim to theoretically define the optimum syllabus in the field of visual [16] and graphical communication for students of technical universities. During the last five years new courses in the field of visual communication have appeared at the technical universities worldwide ([17], [18], [19], [20], [21], [22], [23], [24], [25]) which have implemented more geometry into their courses.

The inclusion of the CAD technology as a didactical method in the instruction of geometry is necessary today not by means of 2D software applications but by using different 3D applications, methods of animation and simulation [26]. CAD software is a modern tool to perform descriptive and constructive - space - geometry. In this manner the process of knowledge adoption is facilitated and it offers great opportunities for a further development of spatial intelligence.

3 The importance of visual communication for students of architecture

A creative person is one who can process in new ways the information directly at hand - ordinarily sensory data available to all of us. A writer needs words, a musician needs notes, an architect needs visual perception, and all need some knowledge of the techniques of their crafts. Also a creative individual intuitively sees possibilities for transforming ordinary data into a new creation, transcendent over the mere raw materials [27]. Precisely one part of the educational process is to make up the development of creativity of students by helping to develop different skills. We can say that for future architects probably the most important skills are a good ability of perception, visual orientation and the ability to clearly transfer their (three dimensional) imaginations in a two dimensional medium. Another important skill is the ability to read two dimensional drawings fast and in a correct way as well as the possibility to realize interdependencies and connections between numerous two dimensional drawings. The skill of visual communication is developed by studying geometry and we can certainly say that by developing this skill creativity of students will also be developed.

4 Geometry courses at the faculties of architecture

The instruction in the field of geometry at the faculties of architecture is based in teaching students to define architectural three dimensional forms and their representation in a two dimensional medium. At the beginning of the studies it is necessary for students to develop visual skills and to adopt geometrical knowledge from the field of representation of three dimensional objects in a two dimensional medium. After a few semesters and with more knowledge in the field of construction and structure design of architectural forms the students need to study complex forms and their geometrical characteristics.

The use of CAD technology in the process of architectural design has facilitated the generation of geometrical forms and helped to improve the representation of architectural achievements. Generating architectural forms with the help of different 3D CAD software has introduced a new quality to the design process. This quality refers primarily to the possibility of a more creative expression of the designer in the process of generating standard forms as well as allowing him/her enormous freedom in generating new forms. Computer aided architectural design of these complex geometrical forms often leads the designer in an unexpected direction, therefore it leads to a new dimension of creativity through interaction between the architect and the computer. It is safe to say that today with the computer aided design the creativity of the designer is no longer conditioned by the technique of the project representation (as it was the case in the classical design process). Today there are nearly no limits to this interactive creative process as well as the results of this interaction solely depend on the theoretical knowledge of geometrical forms and their characteristics, adequate choice and use of 3D software application. At this place we would like to remind and to point out that this CAD process takes place only in a virtual environment. Today the Rapid Prototyping Technology offers the possibility to manufacture physical objects of complex geometrical forms directly from CAD data sources and to check the extensive virtual design.

On the basis of the aforementioned it is quite clear that the students today could only be prepared for adequate applications of future software and for further developments of the CAD technology by a thorough education in the field of geometry.

5 The syllabus of the course ”Methods of representation”

Starting with the academic year 2002/2003 a new curriculum was introduced for the studies of architecture at the University of Technology in Graz. Besides many revisions which affected most courses also the course ”Geometry” was modified and renamed into ”Methods of representation”. The instruction of the former course ”Geometry” included 3 lectures and 2 exercise classes (per week) in the first semester and 2 exercise classes in the second one. The number of classes in the new course was reduced to 2 lectures while the exercise classes in the first and second semester remained.

The course ”Methods of representation” can only be attended with basic knowledge in geometry. Students learn this ”basic geometry” either in a 2 years course at high school or in an obligatory supplementary ”basic course” at university called ”Ergänzungskurs aus Darstellender Geometrie” which includes 30 lecture and 30 exercise lessons. This course is organised by the Institute of Geometry (http://www.geometrie.tugraz.at/lehre.html), lasts 2 months at the beginning of the first semester and covers almost the whole learning matter of Austrian high schools. Students learn the basic principles of orthogonal projections, with main views and auxiliary views, with basic construction dealing with points, lines and planes in space as well as an elementary constructions in ground and frontal projection of conic sections, spheres, cylinders and cones of rotation and their planar sections. Using the above mentioned knowledge in the field of geometry the students are completely prepared to adopt the contents of the course in ”Methods of representation”.

One innovation of the course ”Methods of representation” concerns the cooperation between the Institute of Geometry from the Faculty of Mathematical and Physical Sciences and the Institute of Architecture and Media, a newly founded institute at the Faculty of Architecture. Another innovation affected the syllabus of the course in the first semester: classical geometric constructions and freehand drawing were combined with the potentials of new media[1].

The syllabus covers the problem of the representation of architecture by means of classical constructional methods and with the help of CAD technology. In the lectures students receive basic theoretical instructions in the field of geometry and CAD technology. In the exercise classes these instructions are applied and practiced by editing and solving ”real” architectural examples and problems.

While defining the syllabus of this course many questions emerged which gave a direction to the further development. Some of these questions can be expressed as follows:

- Which are the needs of an architect to represent his/her ideas?
- Which are the needs of the students in their first year of studies for the representation of their first concepts?
- Which form and art of representation should the students be taught today, taking the fact into account that the current CAD software (AutoCAD, Abisplan 3D, Archi-Cad, Nemetschek,...) support to 99% the process of handling and representing architectural projects in practise?
- What is the difference between a draughtsman and a creative project planner if they both use the same tool for their graphical expressions?
- What can make our course thrilling and interesting for students and how can we motivate them to increase their knowledge?
- How much geometry do we need in our syllabus?
- How do we find a good balance between classical construction design and the handling by means of CAD?
- What makes our course different from all the other courses in this field?

[1] In the second semester the students learn about the field of computer graphics. Students get an introduction in the field of free form surfaces including theory and practical exercises. The intended purpose of the rest of the syllabus is to enable the students to represent future projects.

Of course we cannot present the discussion on the aforementioned topics (including pros and cons) in this work but some specific answers to the questions were offered in the first part of this paper and some will be presented in the following part.

The subject matter of our course consists of three parts

- classical constructions,
- CAD constructions and
- freehand drawing

and is divided into 10 thematic units.

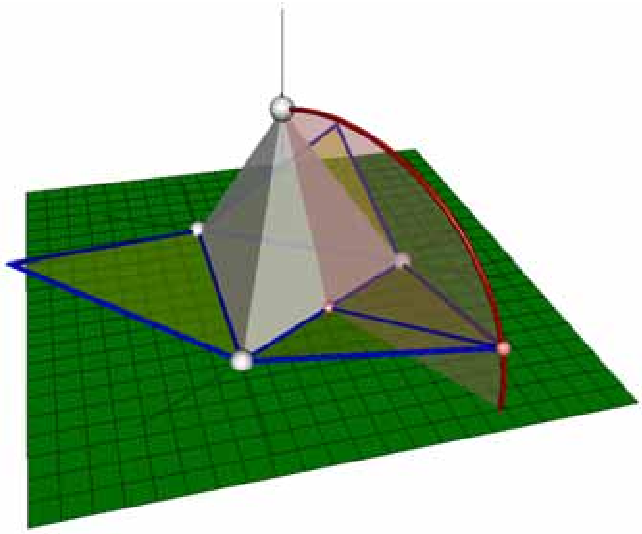


Figure 1 *3D animation*

Classical and CAD exercises units interchange during the semester and in each unit one exercise of freehand drawing is carried out. With this approach the students should be enabled to realize the problem of representation in three different ways. Each exercise course has a tutorial paper and if necessary a 3D animation which facilitates the understanding of the corresponding spatial problems (for example the generation of a tetrahedron by rotating equilateral triangles, see Fig. 1).

During the semester students have to complete five assignments and the examination is divided into two parts. One part of the examination includes the check of knowledge in the field of classical construction design (Fig. 2) and the second part concerns construction designing in CAD (Fig. 3). In the examination examples architecturally effected objects by famous architects are used but only adapted and simplified in order to represent the essential principles (Fig. 4).

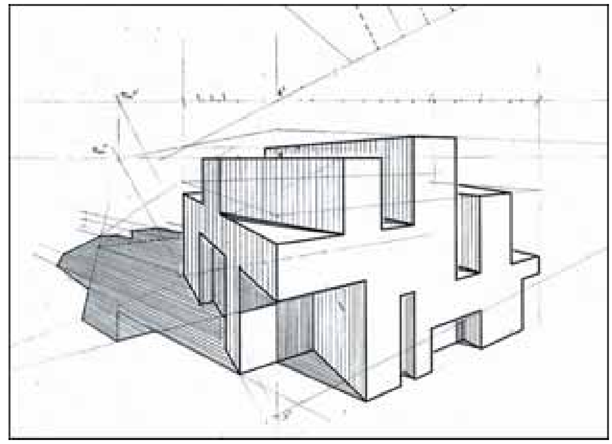


Figure 2 *Examination example - classical construction design*

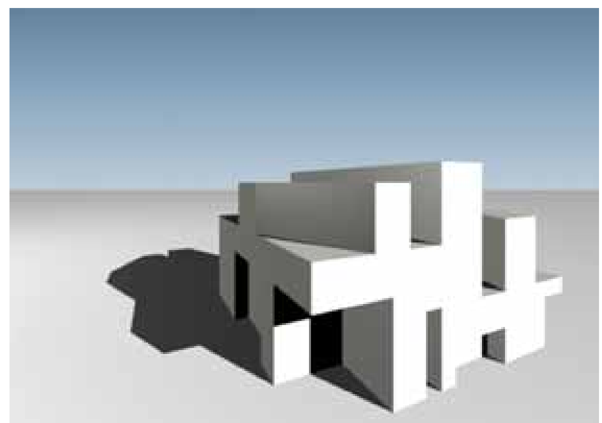


Figure 3 *Examination example - modelling in CAD*

Figure 4 *Original to the figures 2 and 3, Bellevue Art-Museum in Seattle, Architect Steven Holl*

5.1 Classical construction design

The aim of this part of the course is to enable students to represent their objects on a two dimensional medium, i.e. paper with pencil, ruler and compasses - primarily by studying axonometric (Fig. 5) and perspective projection of objects (Fig. 6).

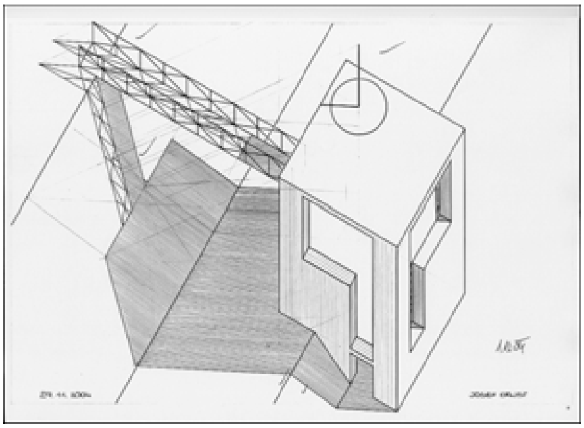


Figure 5 *Axonometric projection*

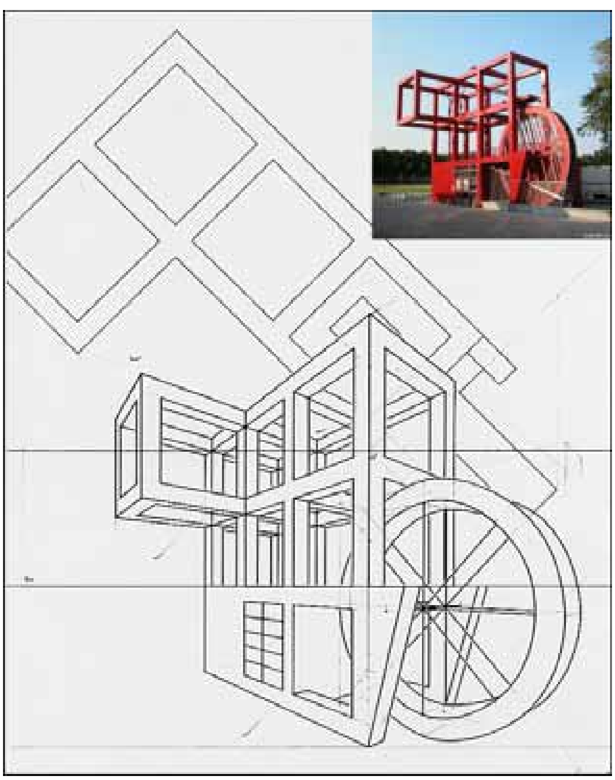


Figure 6 *Perspective projection*

With the classical construction design the rules of construction are taught, strategic thinking is encouraged, a gradual understanding of geometry and also basic components of perceptive skills are developed [28]. In this manner the adopted fundamental knowledge may be applied during freehand drawing or while using software packages. The content consists of horizontal axonometry, perspective, shadow construction by means of parallel source of light in both axonometric (Fig. 5) and perspective projection (Fig. 7) as well as the reconstruction of perspective images (Fig. 8).

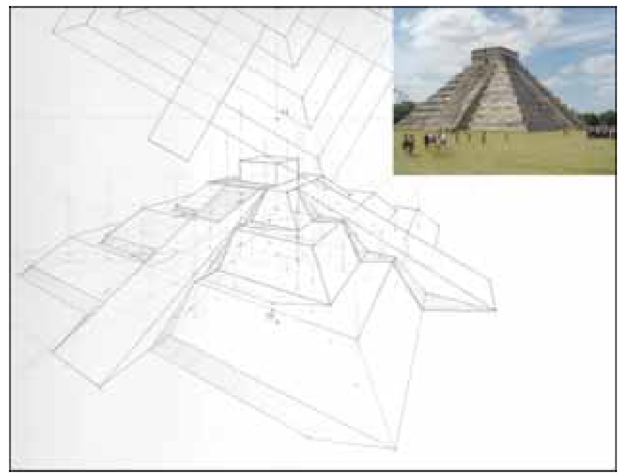


Figure 7 *Shadow construction*

In the lectures students receive general information on different axonometric representations while in exercise classes only horizontal axonometry is practiced - for architects the easiest and the fastest way to represent three dimensional forms by hand.

The basics of perspective representation were introduced into the syllabus for some reasons. The first one is connected to the understanding of parallelism in perspective images (points at infinity, neutral points, etc.) and the application of these rules in freehand drawing. Another reason is directly connected to CAD software and the appropriate use of perspective parameters in CAD software in order to receive desired perspective images and not images by chance. Conversely the knowledge of perspective is essential for the reconstruction of perspective images and for the photomontage of new projects into existing architectural context (Fig. 8 and 17).

One more reason of studying perspective is that perspective drawings (of architecture) by hand exude an own personal touch because of the particular drawing style of the acting person. This is an important matter in architecture in contrast to the often sterilely produced computer presentations.

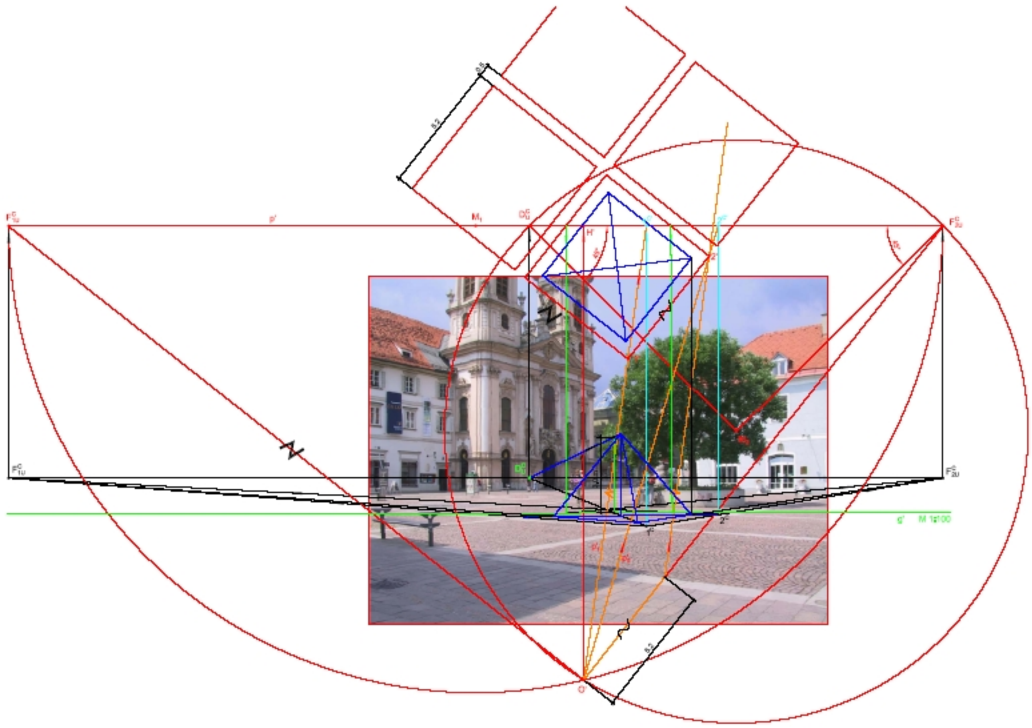


Figure 8 *Reconstruction - Mariahilferplatz, Graz.*

5.2 Freehand drawing

The aim of this part of the course is to enable students to accomplish geometrical freehand drawing ([29], [30], [31]). The content is connected to the syllabus of classical construction design and the design by means of CAD. Mastering this skill starts with the drawing of lines and two dimensional geometrical patterns (Fig. 9) whereby precise drawing is trained as well as fine drawing by hand.

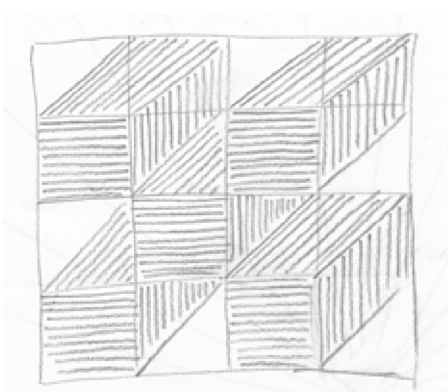


Figure 9 *A two dimensional pattern*

After practising the basic geometrical principles and rules of axonometric projection in classical construction design the students are introduced to axonometric sketching of simple and complex forms (Fig. 10).

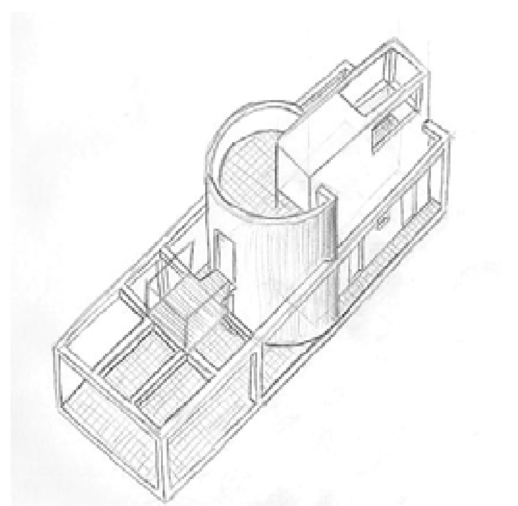


Figure 10 *A complex three dimensional form in axonometric projection*

The first part of perspective sketching is performed by copying perspective images of architectural objects (Fig. 11). The aim of this part of sketching is to recognize basic geometric elements and rules in a perspective image and to imitate the style of the original as well as possible.

Figure 11 *A given 2D perspective image (left) and a student's drawing (right)*

The second part concerns the drawing of a given three-dimensional object (Fig. 12). The task for the students is to recognize basic geometrical bodies in a complex geometrical form and to apply the rules of perspective during sketching. Thus the students practice their perception skills and the skill of precise transformation of the elements of a three dimensional object to a two dimensional medium. This last mentioned skill is directly connected to the modelling process in CAD systems. It trains the process of strategic thinking and of finding solutions for complex problems.

Figure 12 *Drawing of a given 3D object*

In this manner students experience two different ways of perception and its transfer to a two dimensional drawing - copying a 2D image and sketching a 3D object - which they will need in the process of further successful project designing.

5.3 Solid modelling in CAD

The aim of this part of the course is to train students in solid modelling and modification of three dimensional forms ([32], [33], [34]). Students have the possibility to use Auto-CAD but at the same time the concept of the syllabus is made in such a manner that the thematic units may be covered in any 3D CAD software of students' choice. The emphasis is placed on acquiring knowledge of basic geometrical principles in the generation of 3D forms as well as learning a strategic way of thinking while solving geometrical problems in complex architectural forms. The thematic units begin with the drawing of two dimensional geometrical forms. Further on, these forms are used to generate spatial models (Fig. 13) by extrusion.

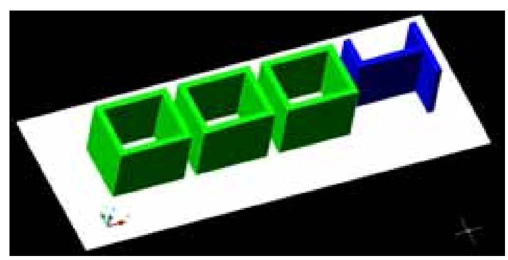


Figure 13 *Extrusion of planar geometrical forms*

Moreover basic geometrical solid models are modified and changed into complex geometrical forms (Fig. 14). They are used to build up hierarchical relations between the participated objects (Fig. 15).

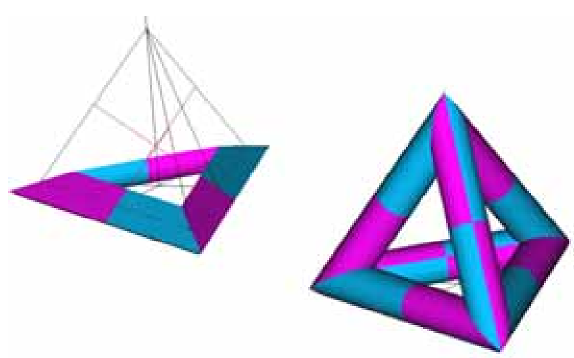


Figure 14 *Complex geometrical forms*

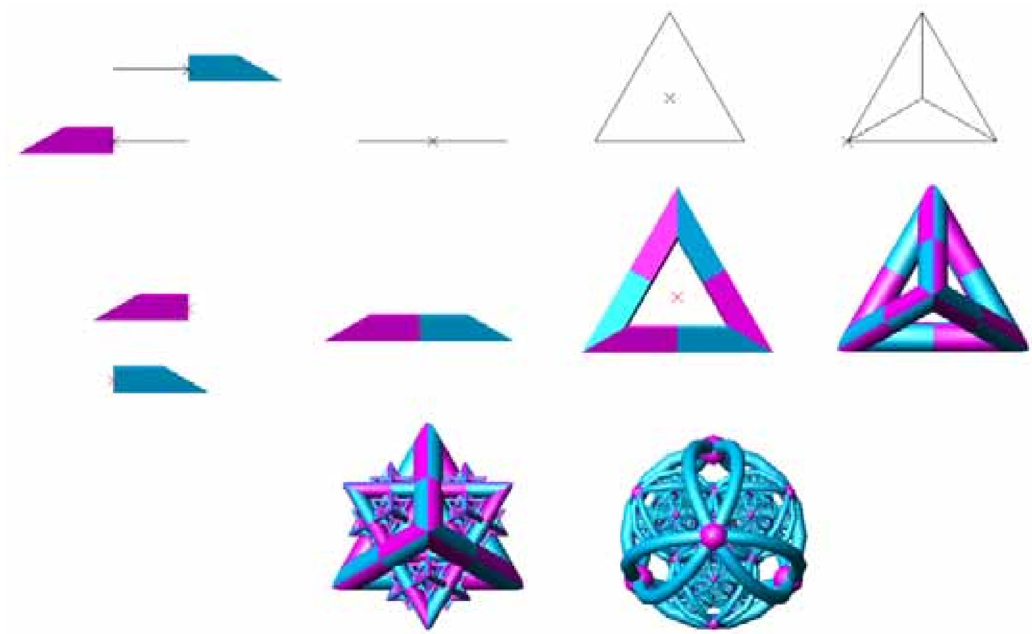


Figure 15 *Hierarchical organized forms*

For this purpose Plato's bodies are constructed and used after analyzing their geometry and rules of their generation.

The course finishes with the modelling of complex forms from architectural practice (Fig. 16).

Figure 16 *Complex architectural forms*

During the course new relative user coordinate systems (UCS) are defined and the work is automated by using blocks. The theoretical knowledge from the field of shadow construction and perspective which was taught in classical construction design is furthermore implemented and practised by the definition of light sources and perspective views from predefined camera and target points in order to apply photo-mounting (Fig. 17).

The chosen thematic units and problems which are handled by means of CAD technology should students enable to clearly see the advantages and restrictions of CAD.

Figure 17 *A modelled fountain inserted in a photo - Mariahilferplatz, Graz.*

6 Evaluation

The course ”Methods of representation” is evaluated every year by ”TUGonline” - a information management system at the University of Technology in Graz. This system supports the communication between the students and the University. Besides the evaluation it includes timetables, the registration of courses, examinations, etc. The students are automatically contacted by the system at the end of the semester and they are asked for a feedback. The outcome of the last years shows that students are very satisfied with the course especially with the diversified content, various interesting examples, the tutorials, the organization of the course and the regular updated web site.

7 Conclusion

In this paper we tried to show how teaching descriptive geometry has changed at the technical departments in the last 15 years by using of new media especially CAD. In our opinion the most important part in handling with CAD packages is a well-founded education in the principles of geometry. The students could only be prepared on the new technology if the traditional geometric education is combined with new media because geometry is the basic science which stays further on and CAD is just an exchangeable tool which will be further developed depending on the state of the art.

In the course ”Methods of representation” at the faculty of architecture we tried to integrate the above mentioned - i.e a combination of classical construction design, computer aided design and geometrical freehand drawing as a link in between. Therefor we used many various examples from different architectural fields but also theoretically geometrical examples. The handling with geometrical motivated examples is the basis for further development of students' creativity. We tried to establish a simple and clear way from the basics to complex geometry with the aim to facilitate the students to manipulate their geometrical knowledge for further work.

References

[1] STACHEL, H., *DG, the Art of Grasping Spatial Relationships*, Proc. 6th ICECGDG, Tokyo, 533-525,1994.

[2] FIELD, B., *A Course in Space Visualisation*, JGG 3(2), 201-210, 1999.

[3] SAITO, T., ET AL, *Relations between Spatial Ability Evaluated by a Mental Cutting Test and Engineering Graphics Education*, Proc.8thICECGDG, Austin, 231-235, 1998.

[4] HAYASAKA, H., ET AL., *Spatial Visualisation Trial Test and Results*, Proc. 8th ICECGDG, Austin, 277-280, 1998.

[5] ABE, H., YOSIDA, K., *Measurement of Interpretation Ability on Architectural Floor Plan*, Proc.8th ICECGDG, Austin, 281-266, 1998.

[6] NAGAE, S., ET AL., *View Variation and Their Effect on Student Solution to Transformation Problems-Effect of Right Side and Left Side View*, Proc.8thICECGDG, Austin, 286-289, 1998.

[7] TAKEYAMA, K., ET AL., *Evaluation of Objective Test Using Pair of Orthographic Projections for Descriptive Geometry Education*, JGG 3(1), 99-109, 1999.

[8] GITTLER, G., *Intelligenzförderung durch Schulunterricht: Darstellende Geometrie und räumliches Vorstellungsvermögen*, In GITTLER G., editor (1994), *Die Seele ist ein weites Land*, Wiener-Universitätsverlag, Wien, 105-122, 1994.

[9] GITTLER, G., GLÜCK, J., *Differential Transfer of Learning: Effects of Instruction in Descriptive Geometry on Spatial Test Performance*, JGG 2(1), 71-84, 1998.

[10] UNESCO, *Learning: The treasure within*, UNESCO Publishing, Paris 1998. TWIGG, C., *The Need for National Learning Infrastructure*, Education Review, Sept/oct. 1994b, 29(5).

[11] OSTROGONAC-ŠEŠERKO, R., ET AL., *Visual Communicaton Curicula for the Global Engineers*, KoG 5, 65-72, 2000/01.

[12] LEOPOLD, C., ET AL., *International Experiences in Developing the Spatial Visualisation Abilities of Engineering Students*, JGG, 5(1),81-91, 2001.

[13] ŠTULIĆ, R., *Nacrtna geometrija i njen značaj za razvoj sposobnosti prostorne vizualizacije*, Prossidings MonGeometrija, Podgorica, Srbija i Crna Gora, 2002.

[14] YE, X., ET AL., *Today‘s students, tomorrow's engineers: an industrial perspektive on CAD education*, CAD 36(14), 1451-1460, 2004.

[15] DANKWORT, W., ET AL., *Engineers' CAx education - it's not only CAD*, CAD 36(14), 1439-1450, 2004.

[16] BERTOLINE, G., *Visual Science: An Emerging Discipline*, JGG 2(2), 181-187, 1998.

[17] PÜTZ, C., SCHMITT F., *Introduction to Computer Aided Design - Concept of a Didactically Funded Course*, JGG 7(1), 111-120, 2003.

[18] ZUO, Z., ET AL, *The Modern Education Mode for Engineering Drawing*, JGG, 7(1), 121-128, 2003.

[19] LEOPOLD, C., *Principles of a Geometry program for Architecture - Experiences, Examples and Evaluations*, JGG 7(1), 101-110, 2003.

[20] POTTMANN, H., *Grundkurs Architektur und Darstellung - Darstellende Geometrie*, Online-Script, Institute of Discrete Mathematics and Geometry, TU Wien.

[21] POTTMANN, H., *CAAD und Geometrie*, Online-Script, Institute of Discrete Mathematics and Geometry, TU Wien.

[22] POTTMANN, H., *Erschließung neuer Geometrien für Architekten*, Online-Script, Institute of Discrete Mathematics and Geometry, TU Wien.

[23] SCHMID-KIRSCH, A., WOLF, R., *Geometrie in der Architekturausbildung- Neue Tendenzen*, IBDG 22(2), 39 - 43, 2003.

[24] BRAKHAGE, K.H, PÜTZ, C., *WinCAG und Einsatz im Fach Darstellende Geometrie für Architekten*, IBDG 22(2), 22 - 30, 2003.

[25] HUSTY, M., *Rhinozeros - Rhino3D*, IBDG 22(1), 36 - 42, 2003.

[26] GLÜCK, J., ET AL., *Geometrie und Raumvorstellung - Psychologische Perspektiven - Studie: Förderung der Raumvorstellung mit Augmented Reality*, IBDG 24(1), 4-11, 2005.

[27] EDWARDS, B., *Drawing on the Right Side of the Brain*, Putnam Publishing Group, New York 1999.

[28] RÖSCHEL, O., *Darstellungsmethoden*, Script, Österreichische Hochschülerschaft TU Graz, 2005.

[29] HOLDER, E., *Design Zeichnen - Lehr und Studienbuch*, Augustus Verlag, München 2000.

[30] HOLDER, E., *Skizzieren und Entwerfen für Einsteiger*, Knaur Ratgeber Verlage, München 2004.

[31] HOLDER, E. PEUKERT, M., *Darstellung und Präsentation, Freihand und mit Computerwerkzeugen gestalten*, Deutsche Verlags-Anstalt, München 2004.

[32] WILTSCHE, A., *Strategien des Konstruierens mit CAD-Paketen*, IBDG, 21(1), 10-14, 2002.

[33] WILTSCHE, A., *Die Kunst ein 3D-CAD-Programm zu erlernen - Teil 1*, IBDG, 22(2), 12-21, 2003.

[34] WILTSCHE, A., *Die Kunst ein 3D-CAD-Programm zu erlernen - Teil 2*, IBDG, 23(1), 21-33, 2004.

Milena Stavrić
Institute of Architecture and Media
email: mstavric@tugraz.at
Albert Wiltsche
Institute of Geometry
email: wiltsche@tugraz.at
Heimo Schimek
Institute of Architecture and Media
email: schimek@tugraz.at

Stručni rad
Prihvaćeno 1. 12. 2005.

NIKOLETA SUDETA
MARIJA ŠIMIĆ

Zrcalne slike u perspektivi

Reflections in Perspective

ABSTRACT

The paper gives an overview of the constructions of reflections in perpective with horizontal line of sight. A few examples are represented depending on the position of the reflecting plane Σ with respect to the horizontal plane and picture plane Π. Reflecting surface can be perpendicular to the picture plane Π or inclined to it. The solutions of those examples are reached by applying the basic rule of geometrical optics and equivalent angles.

Key words: reflections in perspective, normals to the reflecting plane, vanishing points, equivalent angles

MSC 2000: 51N05

Zrcalne slike u perspektivi

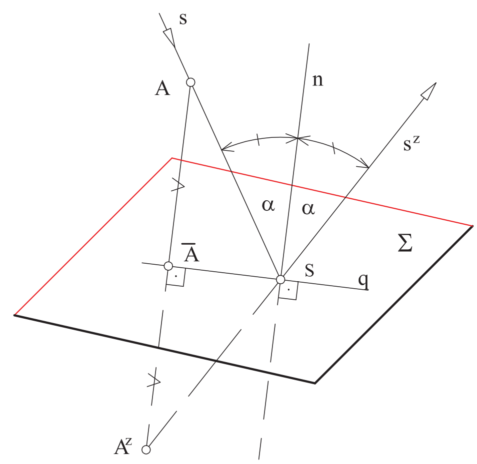
SAŽETAK

Članak daje pregled konstrukcija zrcalnih slika u perspektivi s horizontalnom osi pogleda. Promatraju se slučajevi ovisno o položaju ravnine zrcala Σ prema horizontalnoj ravnini i ravnini slike Π. Ravnina zrcala je ili okomita na ravninu slike Π ili je u općem položaju prema njoj. Primjenom osnovnog pravila geometrijske optike te korištenjem jednakosti kuteva dana su konstruktivna rješenja pojedinih slučajeva.

Ključne riječi: zrcalne slike u perspektivi, okomice na ravninu zrcala, nedogledi, jednakost kuteva

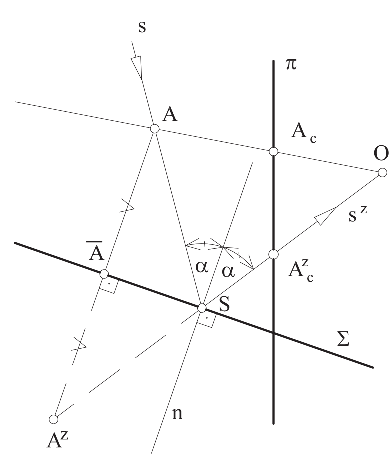
Perspektivna slika neke građevine često se upotpunjuje zrcalnim slikama ako se građevina nalazi u neposrednoj blizini vodene površine ili ispred vertikalne staklene fasade susjedne zgrade. To se javlja i u interijeru u kojem se nalaze vertikalno ili koso postavljena zrcala, glatke podne ili stropne obloge.

Princip konstrukcije zrcalne slike na nekoj ravnini temelji se na zakonu geometrijske optike. Naime, probada li neki pravac s ravninu Σ u točki S, tada njegova zrcalna slika s^z prolazi točkom S, a pravci s i s^z zatvaraju s okomicom n na ravninu Σ u točki S jednake kuteve. Pravci s, s^z i n pripadaju istoj ravnini Δ koja je okomita na Σ. Nekoj točki A zrcalna slika u odnosu na Σ je točka A^z. Polovište $\overline{A}$ dužine $\overline{AA^z}$ je u ravnini Σ (slika 1a). Svaka točka ravnine zrcala Σ podudara se, dakako, sa svojom zrcalnom slikom. Zrcalna slika bilo kojeg pravca p, usporednog s ravninom Σ, je pravac p^z usporedan s pravcem p.



Slika 1a

Zrcalna ravnina Σ, općeg položaja prema ravnini slike Π, okomito projicirana u smjeru presječnice ravnina Σ i Π, prikazana je na slici 1b. Centralna projekcija točke A označena je s A_c, a centralna projekcija njezine zrcalne slike A^z s $A_c{}^z$. Tada se pravac s može tumačiti kao zraka svjetlosti koja se prolazeći točkom A odbija od zrcalne ravnine Σ i tek tada kao s^z dolazi u očište O.



Slika 1b

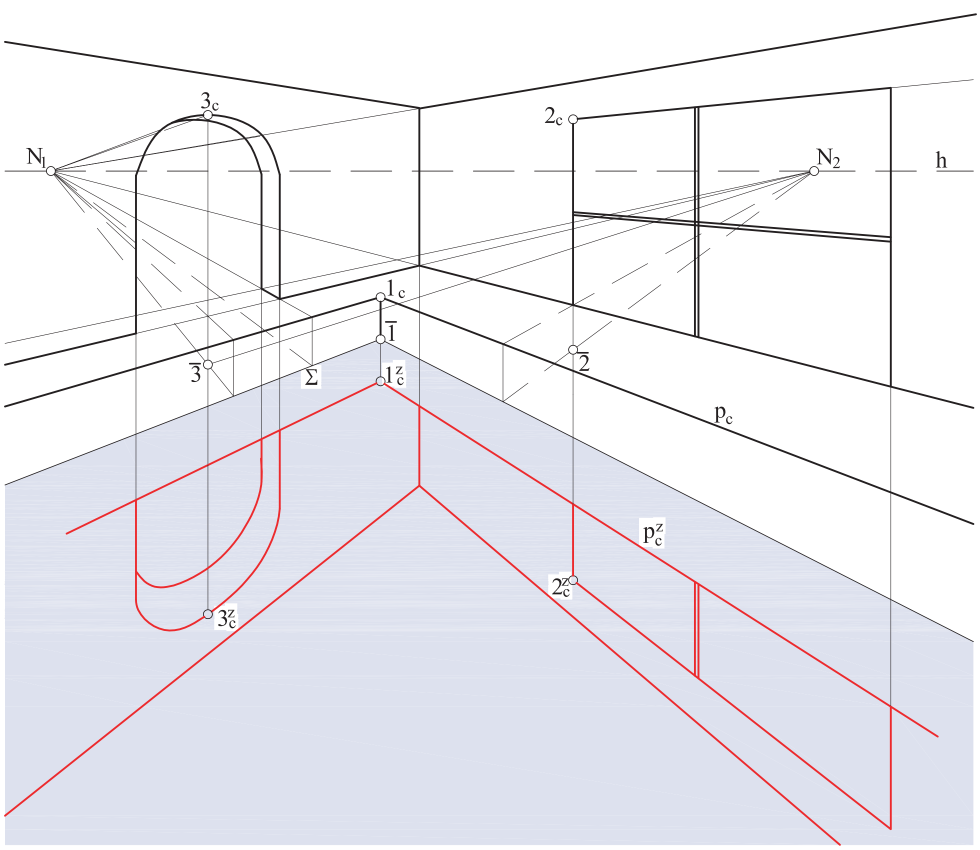
1 Horizontalna zrcala

Na slici 2. prikazana je u perspektivi s horizontalnom osi pogleda unutrašnjost jednog dijela prostorije i njena zrcalna slika na glatkoj površini vode u bazenu. U takvoj su perspektivi konstrukcije zrcaljenja najjednostavnije, jer su okomice na zrcalnu ravninu usporedne s ravninom slike Π, pa se dužine na takvim pravcima zrcale bez promjena duljina, a polovišta ostaju sačuvana. Tako je na slici 2. dužina $\overline{1_c{}^z\overline{1}}$ zrcalna slika dužine $\overline{1_c\overline{1}}$, jer točka $\overline{1}$ leži u zrcalnoj ravnini Σ. Analogna je konstrukcija ostalih točaka, samo što se na vertikali koja prolazi točkom 2_c (odnosno 3_c) treba prethodno odrediti točka $\overline{2}$ (odnosno $\overline{3}$) u razini ravnine Σ. Pravac p (rub bazena) je usporedan s ravninom Σ, pa je i njegova zrcalna slika p^z usporedna s p. Stoga njihove perspektivne slike p_c i $p_c{}^z$ imaju zajednički nedogled N_1. Analogno vrijedi za sve horizontalne pravce na slici 2. Ista se konstrukcija primjenjuje i za bokocrtno zrcalo jer su okomice na ravninu zrcala i u tom slučaju paralelne s ravninom slike Π.

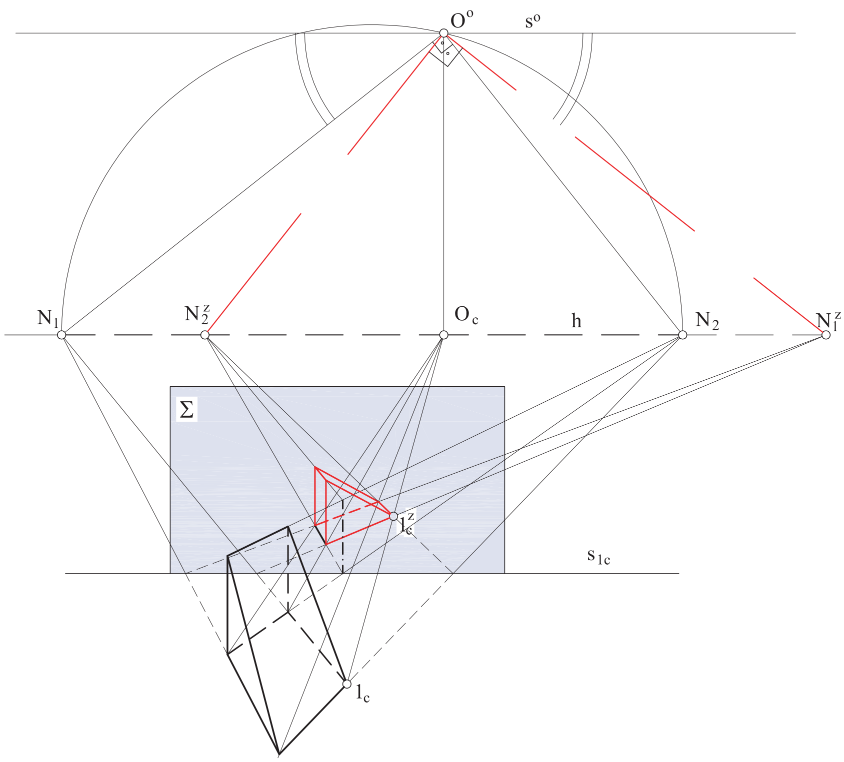
2 Vertikalna zrcala

Neka je zadana perspektivna slika uspravne trostrane prizme i frontalno zrcalo Σ usporedno s ravninom slike Π (slika 3). Nedogledi međusobno okomitih smjerova horizontalnih bridova prizme su N_1 i N_2. Okomice na zrcalnu ravninu okomite su i na ravninu Π pa im je nedogled glavna točka O_c. Na takvim se pravcima ne čuvaju omjeri pa se koristi svojstvo da pravac i njegov zrcalni pravac zatvaraju s ravninom zrcala isti kut. Prave veličine kuteva konstruiraju se u rotiranom položaju pa je $\angle(N_1O^o,s^o)=\angle(s^o,O^oN_1{}^z)$ i analogno $\angle(N_2O^o,s^o)=\angle(s^o,O^oN_2{}^z)$. Zbog toga se zrcalni nedogledi $N_1{}^z$ i $N_2{}^z$ nalaze na horizontu h i simetrični su s obzirom na O_c nedogledima N_1 i N_2. Točka 1_c i njezina zrcalna slika $1_c{}^z$ leže na okomici na ravninu Σ, dakle, na pravcu s nedogledom O_c. Pravac N_11_c i njegova zrcalna slika $N_1{}^z1_c{}^z$ sijeku se u točki zrcalne ravnine na njenom prvom tragu s_{1c}. Na taj je način uz konstrukciju visine prizme dobivena zrcalna slika zadane prizme.

Na slici 4b prikazano je zrcaljenje prizme na vertikalnom zrcalu Σ kosom prema ravnini slike Π. Na tlocrtnom prikazu (slika 4a) točki $1'$ konstruirana zrcalna slika $1^{z\prime}$ nalazi se na okomici n' s obzirom na trag s_1 tako da vrijedi $\overline{1_s1'}=\overline{1_s1^{z\prime}}$.

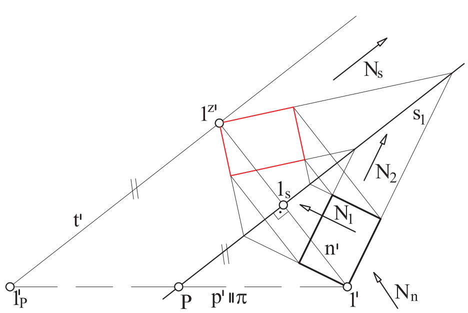


Slika 2

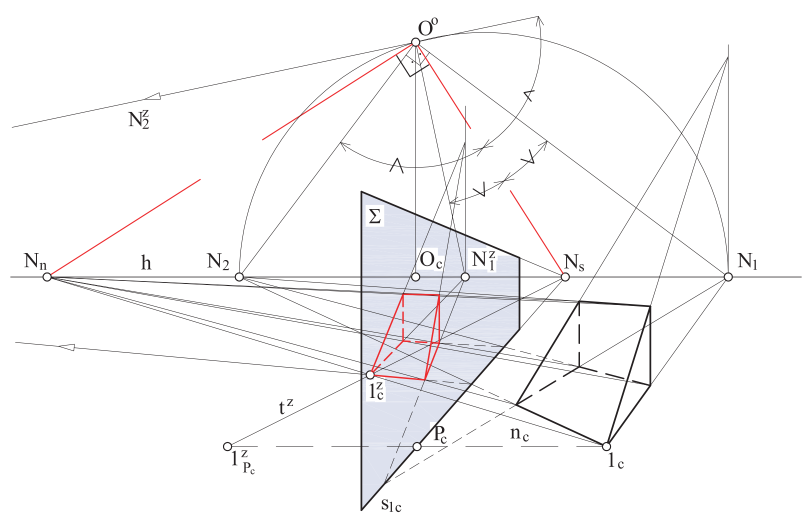


Slika 3

Budući da ta jednakost u perspektivi ne ostaje sačuvana, osim na pravcima paralelnim s ravninom slike, položen je točkom $1'$ pomoćni pravac p' usporedan s Π, na kojem je $\overline{1'P} = \overline{P1_p{}'}$. Pravac t' točkom $1_p{}'$ paralelan s prvim tragom zrcalne ravnine s_1, siječe okomicu n' u zrcalnoj točki $1^{z\prime}$. Ova se konstrukcija koristi u perspektivi. Budući da je točka N_s nedogled svih horizontalnih bridova ravnine zrcala, a točka N_n nedogled svih okomica na ravninu zrcala Σ, vrijedi $N_sO^o \perp O^oN_n$. Postupkom primijenjenim u slučaju frontalnog zrcala tj. izjednačavanjem kuteva $\angle N_1O^oN_s = \angle N_sO^oN_1{}^z$ i $\angle N_2O^oN_s = \angle N_sO^oN_2{}^z$ dobiju se nedogledi zrcalnih slika okomitih smjerova horizontalnih bridova prizme. Analogno prethodnom slučaju konstruira se zrcalna slika cijele prizme.



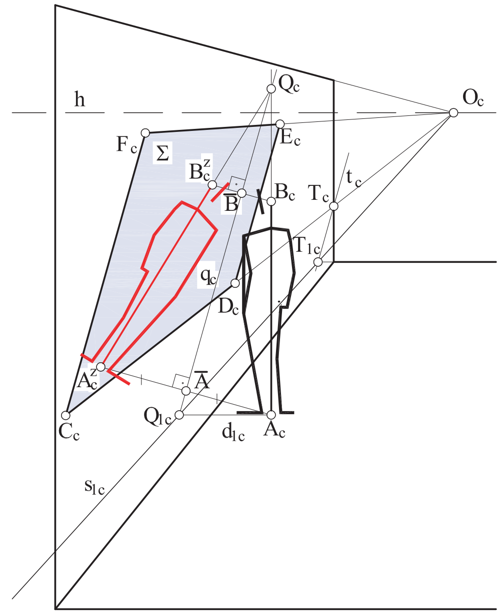
Slika 4a



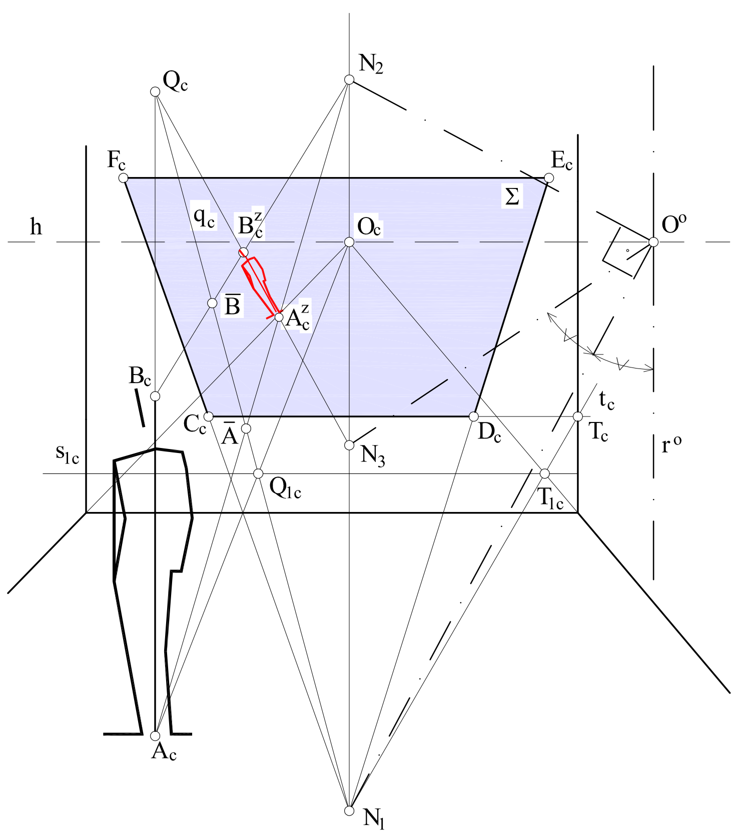
Slika 4b

3 Zrcala nagnuta prema horizontalnoj ravnini

Na bokocrtni zid sobe, koja je u frontalnom položaju, naslonjeno je pravokutno zrcalo $CDEF$ svojim horizontalnim bridom $\underline{CD}$ (slika 5). Nedogled horizontalnih stranica zrcala $\overline{C_cD_c}$ i $\overline{E_cF_c}$, okomitih na ravninu slike Π, je glavna točka O_c. Stranice $\overline{CF}$ i $\overline{DE}$ su paralelne s ravninom slike pa su i njihove perspektivne slike međusobno paralelne. Prvi trag s_{1c} ravnine zrcala Σ je spojnica točaka O_c i T_{1c}, prvog probodišta presječnice t_c ravnine zrcala i frontalnog zida sobe. Zbog toga je presječnica t_c paralelna s bridom $\overline{D_cE_c}$ i prolazi točkom T_c. Na takvom zrcalu zrcali se vertikalna dužina $\overline{AB}$ (visina figure). Vertikalni pravac AB i njegova zrcalna slika A^zB^z nalaze se u ravnini Δ okomitoj na Σ. Ona je također vertikalna (okomita na horizontalnu ravninu) i paralelna s ravninom slike Π. Presječnica ravnine Δ i ravnine Σ je pravac q_c paralelan s C_cF_c. Prvo probodište tog pravca je točka Q_{1c}, konstruirana kao sjecište prvog traga s_{1c} ravnine Σ i prvog traga d_{1c} ravnine Δ. Točka A_c i njena zrcalna slika $A_c{}^z$ leže na pravcu okomitom na ravninu zrcala Σ. Budući da je taj pravac paralelan s Π vrijedi $\overline{A_c\overline{A}} = \overline{\overline{A}A_c{}^z}$. Analogno vrijedi $\overline{B_c\overline{B}} = \overline{\overline{B}B_c{}^z}$. Pravci A_cB_c i $A_c{}^zB_c{}^z$ moraju se sjeći u točki Q_c presječnice q_c.



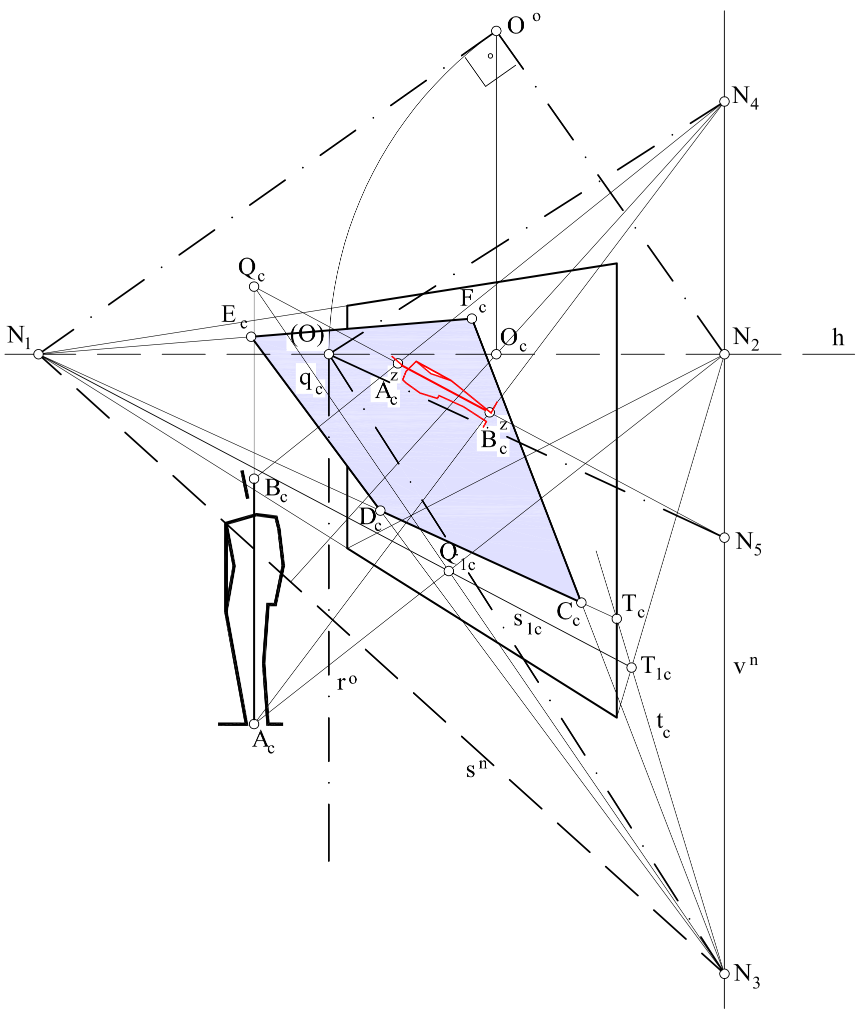
Slika 5



Slika 6

Na slici 6. prikazano je pravokutno zrcalo $CDEF$ donjim horizontalnim bridom pričvršćeno za frontalni zid sobe. Ovaj slučaj je malo složeniji od prethodnog. Nedogled N_1 stranica $\overline{C_cF_c}$ i $\overline{D_cE_c}$ nalazi se na vertikali točkom O_c, tj. nedoglednici svih vertikalnih ravnina okomitih na ravninu slike. Ravnina Δ postavljena pravcem AB okomito na ravninu zrcala okomita je i na horizontalnu ravninu i na ravninu slike. Prvi trag s_{1c} ravnine zrcala Σ i njena presječnica q_c s ravninom Δ određeni su kao na slici 5. Nedogled N_2 svih pravaca okomitih na ravninu zrcala Σ konstruira se na poznati način ($\angle N_1O^oN_2 = 90^\circ$). Točka N_3, nedogled pravca $A_c{}^zB_c{}^z$, dobivena je jednakošću kuteva $\angle(r^o, O^oN_1) = \angle(N_1O^o, O^oN_3)$. Zrcalna slika $A_c{}^z$ (odnosno $B_c{}^z$) točke A_c (odnosno B_c) konstruirana je kao sjecište okomice točkom A_c (odnosno B_c) na ravninu Σ i pravca Q_cN_3.

U posljednjem slučaju pravokutno zrcalo $CDEF$ pričvršćeno je na vertikalnu ravninu, općeg položaja prema ravnini slike Π (slika 7.). Točka N_1 je nedogled horizontalnih pravaca vertikalne ravnine, a N_2 svih pravaca okomitih na tu ravninu. Nedoglednica v^n svih vertikalnih ravnina okomitih na ravninu zrcala je vertikala točkom N_2. Neka je N_3 nedogled stranica $\overline{D_cE_c}$ i $\overline{C_cF_c}$. On mora ležati na nedoglednici v^n. Ravnina Δ, postavljena pravcem AB, okomita je na ravninu zrcala. Prvi trag s_{1c} i presječnica q_c konstruirani su kao i u prethodnim slučajevima. Pravac s^n, tj. spojnica N_1N_3 je nedoglednica ravnine zrcala Σ. Nedogled N_4 svih okomica na tu ravninu dobije se konstrukcijom pravog kuta u rotiranom položaju. Također se može konstruirati i kao nedogled svih okomica ravnine Σ. Kao i dosad, pravac A_cB_c i njegova zrcalna slika $A_c{}^zB_c{}^z$ sijeku se u točki Q_c zrcalne ravnine Σ. Jednakost kuteva $\angle(r^o,(O)N_3) = \angle N_3(O)N_5$ daje nedogled N_5 pravca $A_c{}^zB_c{}^z$. Dakle, točka A_c^z (odnosno $B_c{}^z$) je sjecište pravaca A_cN_4 i Q_cN_5 (odnosno B_cN_4 i Q_cN_5).
Primjenom opisanih postupaka mogu se konstruirati zrcalne slike u perspektivi, s horizontalnom osi pogleda, objekata postavljenih na horizontalnu ravninu.



Slika 7

Literatura

[1] ABBOTT, W., *Theory and Practice of Perspective*, Blackie & Son Limited, London and Glasgow, 1964.

[2] BOŽIČEVIĆ, J., *Linearna perspektiva*, Zagreb, 1942.

[3] KURILJ, P., SUDETA, N., ŠIMIĆ, M., *Perspektiva*, Golden marketing-Tehnička knjiga, Zagreb, Arhitektonski fakultet Sveučilišta u Zagrebu, 2005.

[4] NIČE, V., *Perspektiva*, Školska knjiga, Zagreb, 1978.

Nikoleta Sudeta
e-mail: nikoleta.sudeta@arhitekt.hr

Marija Šimić
e-mail: marija.simic@arhitekt.hr

Arhitektonski fakultet Sveučilišta u Zagrebu
Kačićeva 26, 10000 Zagreb

Stručni rad
Prihvaćeno 5. 12. 2005.

PREDRAG LONČAR

O invarijantama polinoma četvrtog stupnja

O invarijantama polinoma četvrtog stupnja

ABSTRACT

In this paper invariants of the polynomial of the fourth degree over the field of real numbers (discriminant and some others) are studied. As a consequence, some elementary inequalities are obtained and some characteristic graphs of these polynomials are drawn.

Key words: polynomial, invariants

MSC 2000: 11C08, 11D25

O invarijantama polinoma četvrtog stupnja

SAŽETAK

U ovom radu se proučavaju invarijante polinoma četvrtog stupnja nad poljem realnih brojeva, diskriminanta i neke druge. Kao posljedica dobivaju se neke elementarne nejednakosti i crtaju neki karakteristični grafovi tih polinoma.

Ključne riječi: polinom, invarijante

1 Uvod

Ovaj rad je motiviran radom [5]. Glavni cilj rada je podrobnije izučavanje invarijanata polinoma četvrtog stupnja. Neka je zadan polinom četvrtog stupnja

$$P_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \tag{1}$$

gdje je ($a_i \in R$, $i = 0,1,2,3,4$) i $a_4 \neq 0$.
Neka je $P(x)$ *normirani* polinom četvrtog stupnja:

$$P(x) = x^4 + ax^3 + bx^2 + cx + d, \tag{2}$$

pri čemu je

$$a = \frac{a_3}{a_4}, \quad b = \frac{a_2}{a_4}, \quad c = \frac{a_1}{a_4}, \quad d = \frac{a_0}{a_4}. \tag{3}$$

Nultočke polinoma (1) odnosno (2) označavat ćemo uvijek s x_1, x_2, x_3 i x_4. Dakle,

$$P_4(x) = a_4(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

Derivacije polinoma označit ćemo s $P'(x)$, $P''(x)$... i zvati prva, druga... derivacija od $P(x)$. Vrijedi

$$P'(x) = 4x^3 + 3ax^2 + 2bx + c,$$

$$P''(x) = 12x^2 + 6ax + 2b.$$

U daljnjem ćemo koristiti rezultate iz rada [5], a posebice ovu podjelu:
1. slučaj. $P_4(x)$ ima dvije realne i dvije kompleksne nultočke,
2. slučaj. $P_4(x)$ ima sve nultočke kompleksne i
3. slučaj. $P_4(x)$ ima sve nultočke realne.
U daljnjem tekstu ćemo ih zvati: prvi slučaj, drugi slučaj i treći slučaj. Drugi slučaj nastupa tada i samo tada kada je $a_4 \cdot P_4(x) > 0$ za sve realne x.
Taylorov razvoj polinoma $P(x)$ u okolini točke $x_0 = h$ je

$$P(x) = (x-h)^4 + \frac{P'''(h)}{3!}(x-h)^3 + \frac{P''(h)}{2!}(x-h)^2 + \frac{P'(h)}{1!}(x-h) + P(h). \tag{4}$$

Supstitucijom $t = x - h$ dobivamo

$$Q(t) = P(t+h) = t^4 + \frac{P'''(h)}{3!}t^3 + \frac{P''(h)}{2!}t^2 + \frac{P'(h)}{1!}t + P(h), \tag{5}$$

pri čemu je za sve h uvijek $P''''(h) = 4! = 24$. Nultočke polinoma $Q(t)$ su $t_i = x_i - h$, $i \in \{1,2,3,4\}$. Zapis (2) polinoma $P(x)$ je Taylorov razvoj toga polinoma oko točke $x_0 = 0$ i stoga je

$$\frac{P''''(0)}{4!} = 1, \quad \frac{P'''(0)}{3!} = a, \quad \frac{P''(0)}{2!} = b, \quad \frac{P'(0)}{1!} = c, \quad P(0) = d. \tag{6}$$

Radi lakšeg izučavanja polinoma $P(x)$ uvodimo supstituciju $t = x - H$ i biramo H tako da član uz treću potenciju nestane, tj. $\frac{P'''(H)}{3!} = 0$.

Sada $P'''(H)=0$ povlači $H=-\frac{a}{4}$. U novoj varijabli $t=x+\frac{a}{4}$ dobivamo kanonski oblik polinoma četvrtog stupnja

$$S(t)=t^4+pt^2+qt+r, \tag{7}$$

pri čemu je

$$p=-\frac{3a^2-8b}{8}, \quad q=\frac{a^3-4ab+8c}{8},$$
$$r=\frac{256d-64ac+16a^2b-3a^4}{256}. \tag{8}$$

Uvedemo li oznake

$$N=3a^2-8b, \qquad Q=8c-4ab+a^3,$$
$$R=256d-64ac+16a^2b-3a^4, \tag{9}$$

možemo pisati

$$S(t)=t^4-\frac{N}{8}t^2+\frac{Q}{8}t+\frac{R}{256}, \tag{10}$$

pri čemu vrijedi

$$N=-8p, \quad Q=8q, \quad R=256r. \tag{11}$$

Stoga kanonski oblik (7) možemo pisati:

$$S(t)=t^4-\frac{N}{8}t^2+\frac{Q}{8}t+\frac{R}{256}. \tag{12}$$

Nultočke polinoma $S(t)$ su $t_i=x_i+\frac{a}{4}$, $i\in\{1,2,3,4\}$ i kako je koeficijent od t^3 jednak nuli, po Vietovim formulama imamo $t_1+t_2+t_3+t_4=0$. Dakle, kanonski oblik $S(t)$ je onaj jedinstveni oblik polinoma četvrtog stupnja u kojem je težište $t_t=\frac{t_1+t_2+t_3+t_4}{4}$ nultočaka jednako nuli i stoga je taj oblik uvijek isti bez obzira iz kojeg Taylorovog razvoja (4) krenemo (sjetimo se da iz jednog Taylorovog razvoja prelazimo uvijek u drugi nekom supstitucijom $x=X+l$). Stoga p, q i r možemo računati tako da u formulu (8) stavimo umjesto a, b, c, d izraze $\frac{P'''(h)}{3!}, \frac{P''(h)}{2!}, \frac{P'(h)}{1!}, P(h)$ (usporediti sa formulom (6)). Time dobijemo:

$$\begin{aligned}
96p &= 48P''(h)-(P'''(h))^2,\\
1728q &= (P'''(h))^3-72P''(h)P'''(h)+1728P'(h),\\
110592r &= 110592P(h)-4608P'(h)P'''(h)+\\
&\quad +96P''(h)(P'''(h))^2-(P'''(h))^4,
\end{aligned} \tag{13}$$

pri čemu je h proizvoljan realan broj.

Da izrazi na desnoj strani formula ne ovise o h uvjeravamo se i tako da te izraze deriviramo po h i koristeći $P''''(x)=24$ pokažemo da su sve njihove prve derivacije jednake 0, tj. ti su izrazi konstante. Kako su formule (13) valjane za $h=0$ (zbog formule (8)), slijedi da one vrijede i za bilo koji h.

Napomena 1 *Polinom $S(t)$ u prvom slučaju ima realne nultočke t_1, t_2 i konjugirano kompleksne $-\frac{t_1+t_2}{2}\pm Li$, $L>0$, pa glasi*

$$t^4+\left[L^2-\frac{3t_1^2+2t_1t_2+3t_2^2}{4}\right]t^2$$
$$-(t_1+t_2)\left[L^2+\frac{(t_1-t_2)^2}{4}\right]t+t_1t_2\left[L^2+\frac{(t_1+t_2)^2}{4}\right].$$

Polinom $S(t)$ u drugom slučaju ima kompleksne nultočke $-w\pm Li$ i $w\pm Mi$, $L\neq 0$, $M\neq 0$, pa glasi

$$t^4+(L^2+M^2-2w^2)t^2+2w(M^2-L^2)t+(L^2+w^2)(M^2+w^2).$$

Polinom $S(t)$ u trećem slučaju ima realne nultočke t_1, t_2, t_3 i $t_4=-t_1-t_2-t_3$, pa glasi

$$t^4-(\Sigma^2-S)t^2+(S\Sigma-\Pi)t-\Sigma\Pi,$$

gdje je

$$\Sigma=t_1+t_2+t_3, \ S=t_1t_2+t_2t_3+t_3t_1, \ \Pi=t_1t_2t_3.$$

Iz napomene 1 vidimo da u drugom slučaju vrijedi $r=(L^2+w^2)(M^2+w^2)>0$, što izlazi i iz $r=t_1\overline{t_1}t_3\overline{t_3}=|t_1|^2|t_3|^2>0$.

2 Invarijante polinoma $P(x)$

Definicija 1 *Simetrične, homogene (isti broj članova) i izobarične (zbroj produkta težine članova i njegove potencije je isti za sve članove) izraze od koeficijenata polinoma (1) koji se ne mijenjaju prilikom zamjene $x=t-h$ zovemo invarijantama polinoma (1). Invarijantu podijeljene sa a_4^h gdje je h stupanj homogenosti invarijante zvati ćemo apsolutnom invarijantom (i njen stupanj homogenosti je 0).*

Prilikom supstitucije $t=kx+l$ invarijanta se množi s nekom potencijom od k. Važno je da u tim izrazima uključimo sve koeficijente a_4, a_3, a_2, a_1 i a_0 polinoma koje smatramo da su težina 0, 1, 2, 3 i 4. Tako npr. izraz za a_4^2N glasi

$$a_4^2N=3a_3^2-8a_2a_4 \tag{14}$$

i to je invarijanta stupnja homogenosti 2 i stupnja izobaričnosti 2. Izraz

$$N=\frac{3a_3^2-8a_2a_4}{a_4^2} \tag{15}$$

je po definiciji apsolutna invarijanta. Apsolutne invarijante polinoma su zapravo izrazi koji su homogene i simetrične funkcije razlika $x_i - x_j$ polaznog polinoma (1) pa onda simetrične funkcije razlika $t_i - t_j$ bilo kojeg polinoma dobivenog pomicanjem polinoma (1) (uočite da je $x_i - x_j = (x_i - h) - (x_j - h)$ za sve h). Tako npr. izraz za N glasi

$$N = (x_1-x_2)^2 + (x_1-x_3)^2 + (x_1-x_4)^2 + (x_2-x_3)^2 + \\ +(x_2-x_4)^2 + (x_3-x_4)^2 \tag{16}$$

i očito je nenegetivan ako su sve nultočke x_i realne i jednak 0 ako i samo ako je $x_1 = x_2 = x_3 = x_4$ tj. onda i samo onda kada je je $p = q = r = 0$. Izraz N zvat ćemo *Newtonovski izraz* polinoma (1). Apsolutne invarijante su izrazi sastavljeni od p, q i r, tj. od N, Q i R, s time da p, q i r, odnosno N, Q i R zapišemo, uz pomoć formula (8) i (3), pomoću koeficijenata a_4, a_3, a_2, a_1 i a_0. Jedna apsolutna invarijanta je izraz $p^2 - 4r$, odnosno izraz $N^2 - R$, iz [5] koji nakon sređivanja glasi,

$$p^2 - 4r = \frac{16a_4^2(a_2^2 + a_1a_3 - 4a_0a_4) - 16a_2a_3^2a_4 + 3a_3^4}{16a_4^4}. \tag{17}$$

3 Diskriminanta

Jedna od najvažnijih invarijanata polinoma, apsolutno ireducibilna po svim svojim keficijentima [4, str. 450.-453.] i [6, str. 126.-127.] je tzv. *diskriminanta D* polinoma definirana sa $D = a_4^6\Pi_{i<j}(x_i - x_j)^2$. Kao izraz od koeficijenata ona se dobiva preko Vietovih formula. U sljedećem teoremu dajemo niz korisnih izraza za diskriminantu. Uvedimo oznake:

$$A_1 = a_2^2 - 3a_1a_3 + 12a_0a_4 = \frac{A_2}{16}, \tag{18}$$

$$B_1 = 27a_1^2a_4 + 27a_0a_3^2 + 2a_2^3 - 72a_0a_2a_4 - 9a_1a_2a_3, \tag{19}$$

$$N_1 = 3a_3^2 - 8a_2a_4 = a_4^2N, \tag{20}$$

$$K_1 = 4a_4^2(9a_1^2 - 32a_0a_2) - 4a_1a_2a_4a_3 + \\ +(48a_0a_4 + a_2^2)a_3^2 - 3a_1a_3^3 = 4a_4^2K, \tag{21}$$

$$G = a_1^2a_4 + a_3^2a_0 - 4a_0a_2a_4, \quad H = a_1a_3 - 4a_0a_4, \tag{22}$$

$$P = 27Q^2, \tag{23}$$

gdje su Q i R su dani formulama (11), (8) i (3).

Teorem 1 *Diskriminanta polinoma $P_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, odnosno, ako je $a_0 \neq 0$, polinoma $R_4(x) = x^4P_4(\frac{1}{x}) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$, dana je uz oznake (18) do (23) izrazima*

$$\begin{aligned} D \;=\; & 256a_0^3a_4^3 - 192a_0^2a_1a_3a_4^2 - 128a_0^2a_2^2a_4^2 - \\ & -6a_0a_1^2a_3^2a_4 - 80a_0a_1a_2^2a_3a_4 + 16a_0a_2^4a_4 - \\ & -4a_1^3a_3^3 + a_1^2a_2^2a_3^2 + 144a_0^2a_2a_3^2a_4 + \\ & +144a_0a_1^2a_2a_4^2 - 27a_0^2a_3^4 - 27a_1^4a_4^2 \\ & -4a_0a_2^3a_3^2 - 4a_1^2a_2^3a_4 + 18a_0a_1a_2a_3^3 + \\ & +18a_1^3a_2a_3a_4, \end{aligned} \tag{24}$$

$$27D = 4A_1^3 - B_1^2, \tag{25}$$

$$144a_4^2D = -3K_1^2 + 2N_1A_1 \cdot K_1 + a_4^4RA_1^2, \tag{26}$$

$$D = -27G^2 - 2a_2(2A_1 - 3H)G + (A_1 - H)H^2, \tag{27}$$

$$2^{12} \cdot 3^3D = -P^2 + 2N_1(N_1^2 - 3A_2)P - \\ -(N_1^2 - 4A_2)(N_1^2 - A_2)^2. \tag{28}$$

Ako je $P_4(x)$ oblika $S(x) = x^4 + px^2 + qx + r$, onda se veličine A_1 i B_1 reduciraju na veličine $A = p^2 + 12r$ i $B = 27q^2 + 2p^3 - 72pr$, a veličina K glasi $K = 9q^2 - 32pr$. Uz oznake

$$\Delta = p^2 - 4r \quad i \quad D_d = -\frac{1}{2}(8p^3 + 27q^2),$$

diskriminanta polinoma $S(x)$ dana je izrazima:

$$D = 16rp^4 - 4q^2p^3 - 128r^2p^2 + 144rq^2p - 27q^4 + 256r^3, \tag{29}$$

$$D = 256r^3 - 128p^2r^2 + 16p(p^3 + 9q^2)r - q^2(27q^2 + 4p^3), \tag{30}$$

$$27D = 4A^3 - B^2, \tag{31}$$

$$D = -27q^4 + 4p(36r - p^2)q^2 + 16r(p^2 - 4r)^2, \tag{32}$$

$$D = -4\Delta^3 + 4p^2\Delta^2 - 36pq^2\Delta + (32p^3 - 27q^2)q^2, \tag{33}$$

$$D = 16r\Delta^2 - 36pq^2\Delta + (32p^3 - 27q^2)q^2, \tag{34}$$

$$D = -27(q^2 - 4pr)^2 - 4p(p^2 + 18r)(q^2 - 4pr) + \\ +16r^2(p^2 + 16r), \tag{35}$$

$$9D = -3K^2 - 4pAK + 16rA^2, \tag{36}$$

$$\frac{27}{4}D = -D_d^2 - 6pAD_d - 4(2p^2 - 3r)A^2, \tag{37}$$

$$3D = 4(p^2 + 12r)\left[(p^2 - 4r)^2 + 6pq^2\right] - \\ -\left[2p(p^2 - 4r) + 9q^2\right]^2, \tag{38}$$

$$9D = -3(B - 2K)^2 + 8p(p^2 + 12r)(B - 2K) - \\ -4(p^2 - 4r)(p^2 + 12r)^2. \tag{39}$$

Dokaz. Izraz (25) dokazan je u [7, (15), str. 231.], a izraz (29) slijedi odmah iz (24) zamjenom a_0 sa r, a_1 sa q, a_2 sa p, a_3 sa 0 i a_4 sa 1. Koristeći formule (11), (8) i (3) iz (25) i (29) dobivamo sve ostale izraze za diskriminantu $P_4(x)$. Iz definicije diskriminante polinoma pomoću nultočaka polinoma i iz Vietove formule za produkt nultočaka lako izlazi da su diskriminante polinoma $P_4(x)$ i $R_4(x) = x^4 P_4(\frac{1}{x})$ jednake (uočiti da ako 0 nije nultočka polinoma $P_4(x)$, onda su nultočke polinoma $R_4(x)$ dane sa $\frac{1}{x_i}$, $i = 1,2,\ldots,4$.). Ako je 0 nultočka polinoma $P_4(x)$, onda je $P_4(x) = x \cdot P_3(x)$ i diskriminanta polinoma $P_4(x)$ jednaka je $[P_3(0)]^2 \cdot$ diskriminanta od $P_3(x)$ i gornje formule tada daju ispravnu diskriminantu kubnog polinoma $R_3(x) = x^4 P_4(\frac{1}{x})$ pomnoženu s $[P_3(0)]^2$ tj. s a_1^2.

Napomena 2 *U radu [5] diskriminanta Descartesove kubne rezolvente polinoma $S(t)$ označena je s D_1. Usporedbom izraza za D_1 u radu [5] i izraza za diskriminantu D u Teoremu 1 vidimo da vrijedi $D = -108D_1$. Izrazi za diskriminantu u Teoremu 1 napisani su kao kvadratne i kubne funkcije nekih invarijanata polinoma $P_4(x)$, pa možemo gledati njihovu diskriminantu po tim invarijantama. Kratko ćemo reći da u izrazu (25) diskriminanta kvadratnog trinoma po B_1 je $16A_1^3$, u izrazu (26) diskriminanta po K_1 je $64A_1$, u izrazu (27) diskriminanta po G je $4A_1^3$, a u izrazu (28) diskriminanta po P je $16^4 A_1^3$. Isto tako u izrazu (30) diskriminanta kvadratnog trinoma po varijabli r je $3 \cdot 2^{19} q^2 D_d^3$, u izrazu (33) diskriminanta po Δ je $384 q^2 D_d^3$, a u izrazu (35) diskriminanta po $q^2 - 4pr$ je A^3. Zapamtimo da u izrazu (36) diskriminanta po varijabli K iznosi $16A^3$, a u izrazu (37) diskriminanta po D_d iznosi $4A^3$.*

Navedimo glavni teorem dokazan u radu [5, str. 14., teorem 2]:

Teorem 2 *Neka je $P(x) = x^4 + px^2 + qx + r$. Tada smo u prvom slučaju onda i samo onda kada je $D < 0$ ili ($D = 0$ i ($p^2 - 4r < 0$ ili ($p^2 - 4r > 0$ i $p > 0$))) ili ($D = 0$ i $p^2 - 4r = 0$ i $p > 0$ i $q \neq 0$), u drugom slučaju onda i samo onda kada je ($D > 0$ i ($p^2 - 4r < 0$ ili ($p^2 - 4r \geq 0$ i $p > 0$))) ili ($q = 0$ i $p^2 - 4r = 0$ i $p > 0$), u trećem slučaju onda i samo onda kada je $D \geq 0$ i $p^2 - 4r \geq 0$ i $p \leq 0$.*

U slučaju $q = 0$ imamo po formuli (34) iz Teorema 1 $D = 16r\Delta^2$ i ovu posljedicu, koje dokaz prepuštamo čitaocu.

Korolar 1 *Neka je $P_4(x) = x^4 + px^2 + r = 0$ bikvadratna jednadžba. U prvom slučaju smo tada i samo tada kada je $r < 0$ ili ($r = 0$ i $p > 0$). U drugom slučaju smo onda i samo onda ako je ($p^2 - 4r < 0$) ili ($p^2 - 4r \geq 0$ i $p > 0$ i $r > 0$). U trećem smo slučaju ako i samo ako je ($p^2 - 4r \geq 0$ i $p \leq 0$ i $r \geq 0$).*

Izreka Teorema 2 u slučaju realnih različitih nultočaka može se poboljšati.

Lema 1 *Polinom $P(x)$ ima četiri realne različite nultočke ako i samo ako je*

$$D > 0 \text{ i } p < 0 \text{ i } p^2 - 4r > 0.$$

Dokaz. Promatrajmo **Eulerovu kubnu rezolventu**

$$z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0 \tag{40}$$

uvedenu u dokazu Teorema 2 iz rada [5] uz pomoć koje je **L. Euler** riješio jednadžbu četvrtog stupnja. Lako se provjeri da je diskriminanta od (40) upravo diskriminanta polinoma $S(t)$, dakle i od $P(x)$, što je pokazano formulom (38) iz Teorema 1. (Podsjetimo se da je diskriminanta polinoma $b_3x^3 + b_2x^2 + b_1x + b_0$ data izrazom $4(b_2^2 - 3b_1b_3)(b_1^2 - 3b_0b_2) - (b_1b_2 - 9b_0b_3)^2$.) Poznato je [3, str. 67., zadatak 2.] da polinom $P(x)$ ima četiri realne nultočke onda i samo onda ako su sve nultočke od (40) realne i nenegativne. Nadalje, polinom $P(x)$ ima četiri realne različite nultočke onda i samo onda ako su sve nultočke od (40) realne i nenegativne i međusobno različite. No, nultočke od (40) su realne i nenegativne i međusobno različite onda i samo onda ako je $D > 0$ i $p < 0$ i $p^2 - 4r > 0$. Dokažimo to, čime ujedno imamo dokaz Leme 1. Ako su nultočke od (40) realne i pozitivne i međusobno različite, onda je po kriteriju realnosti nultočaka kubne jednadžbe $D > 0$, a po Vietovim formulama $p < 0$ i $p^2 - 4r > 0$. Ako pak je $D > 0$ i $p < 0$ i $p^2 - 4r > 0$, onda su zbog $D > 0$ sve nultočke od (40) realne i međusobno različite. Pretpostavimo da je $q \neq 0$. Onda po Descartesovom teoremu o broju negativnih nultočaka izlazi da (53) nema negativnih nultočaka (broj promjena predznaka u nizu -1, $2p$, $-(p^2 - 4r)$, $-q^2$ je nula!), a kako je $q \neq 0$, 0 nije nultočka od (53), i jer su sve nultočke od (40) realne, one su sve pozitivne. Neka je je $q = 0$. Onda je $r > 0$ (koristiti formulu (34) za diskriminantu iz Teorema 1). U tom slučaju (53) glasi $z(z^2 + 2pz + p^2 - 4r)$ i njene nultočke su 0 i dvije nultočke kvadratne jednadžbe $z^2 + 2pz + p^2 - 4r$, koje su realne i različite (jer je diskriminanta od $z^2 + 2pz + p^2 - 4r$ jednaka $16r > 0$), a po pretpostavkama $p < 0$ i $p^2 - 4r > 0$ i po Vietovim formulama obje su pozitivne.

Zadatak 1.
Pretpostavimo da smo u trećem slučaju i da su t_i, $i \in \{1,2,3,4\}$ realne nultočke polinoma $S(t)$. Dokažite da je tada $p = -\frac{1}{2}(t_1^2 + t_2^2 + t_3^2 + t_4^2) \leq 0$, i da je $p = 0$ onda i samo onda ako je $S(t) = t^4$. Dokažite na temelju nejednakosti između kvadratne i geometrijske sredine brojeva $|t_i|$, $i \in \{1,2,3,4\}$ da tada vrijedi $p^2 - 4r \geq 0$ i da je u trećem slučaju $p^2 - 4r = 0$ onda i samo onda kada je $S(t) = (t^2 - t_1^2)^2$ za neki realni t_1.

Lema 2 *U trećem slučaju vrijedi nejednakost* $2p(p^2-4r)+9q^2 \le 0$.

Dokaz. Pretpostavimo da je u trećem slučaju $2p(p^2-4r)+9q^2>0$. Znamo da je u trećem slučaju $D\ge 0$ i $p\le 0$ i $p^2-4r\ge 0$ po Lemi 1, i da je po Korolaru 2 $A=p^2+12r\ge 0$. Onda bi po formuli (39) za diskriminantu iz Teorema 1 imali $D\le -\frac{1}{3}(B-2K)^2<0$, protivno tvrdnji Leme 1. Dakle je $B-2K=2p(p^2-4r)+9q^2\le 0$.

Analizirajmo sada diskriminantu iz Teorema 1. Iz formule (25) u Teoremu 1 slijedi da $A_1<0$ povlači $D<0$ i da $A_1\le 0$ povlači $D\le 0$. Odavde slijedi da $D\ge 0$ povlači $A_1\ge 0$ i da $D>0$ povlači $A_1>0$. Dakle, ako je u trećem slučaju $A_1=0$, tada je $D=0$, i iz izraza (25) i $B_1=0$, što ćemo kasnije u Propoziciji 4 karakterizirati kao slučaj barem trostruke realne nultočke. U drugom slučaju za polinom $S(t)$ vrijedi $S(t)>0$ za sve $t\in R$, pa stoga $S(0)=R>0$, tj. $r>0$ i $A_1=\dfrac{3}{64}R+\left(\dfrac{N}{8}\right)^2>0$. Osim toga je $A_1>\left(\dfrac{N}{8}\right)^2$ tj. $|N|<8\sqrt{A_1}$.
Neka je $A_1\ge 0$, tj. $A_2\ge 0$. Iz izraza (28), zbog $P\ge 0$, vidimo da pretpostavke $2N_1(N_1^2-3A_2)\le 0$ i $N_1^2-4A_2>0$ povlače $D<0$. No, $2N_1(N_1^2-3A_2)\le 0$ i $N_1^2-4A_2>0$ je ekvivalentno s $N_1\le 0$ i $N_1^2-4A_2>0$, odnosno s $A_1\ge 0$ i $N_1<-8\sqrt{A_1}\le 0$, odnosno s $N_1<0$ i $R<0$. Dakle, $D<0$ ako je $N_1<0$ i $R<0$. Za kanonski oblik $S(t)$, lako vidimo i iz izraza (29) da $p>0$ i $r<0$ povlače $D<0$. Iz izraza (29) ujedno vidimo da $p\ge 0$, $r<0$ povlače $D<0$ (prvi slučaj) i da $p\ge 0$, $r\le 0$ i $q\ne 0$ isto povlače $D<0$ (prvi slučaj).
Neka su p i r zadani. Proučimo diskriminantu u obliku (32) kao kvadratnu jednadžbu po q^2. U drugom i trećem slučaju imamo $p^2+12r\ge 0$ po gore dokazanom, pa $D\ge 0$ povlači

$$\frac{2}{27}\left[p(36r-p^2)-\sqrt{p^2+12r}^{\,3}\right]\le q^2\le$$
$$\le\frac{2}{27}\left[p(36r-p^2)+\sqrt{p^2+12r}^{\,3}\right]. \qquad (41)$$

Neka smo u prvom slučaju. Ako je $p^2+12r\le 0$, imamo, po gore dokazanom, da je za bilo koji q ispunjeno $D<0$. Neka vrijedi $p^2+12r>0$. Lako vidimo da je $p(36r-p^2)+\sqrt{p^2+12r}^{\,3}\le 0$ onda i samo onda kada je $r\le 0$, $p\ge 0$, pa u slučaju $(r\le 0,\ p\ge 0)$ nema nikakvih uvjeta na q. Analizom $p(36r-p^2)<\sqrt{p^2+12r}^{\,3}$, koja povlači nejednakost $r(p^2-4r)^2>0$, izlazi da u slučaju ($r>0$ ili $r=0$, $p<0$) vrijedi samo nejednakost

$$\frac{2}{27}\left[p(36r-p^2)+\sqrt{p^2+12r}^{\,3}\right]\le q^2.$$

Ako je ($r<0$ i $p<0$), onda vrijedi nejednakost

$$q^2\le\frac{2}{27}\left[p(36r-p^2)-\sqrt{p^2+12r}^{\,3}\right]$$

ili nejednakost

$$\frac{2}{27}\left[p(36r-p^2)+\sqrt{p^2+12r}^{\,3}\right]\le q^2.$$

Napomena 3 *Pretpostavimo da su p i q zadani i da je diskriminanta $D(\Delta)$ je zapisana kao kubni polinom od $\Delta=p^2-4r$ (formula (33) iz Teorema 1). Imamo $D(\frac{4}{3}p^2)=-\frac{4}{27}D_d^2$, $D'(\Delta)=-4(3\Delta^2-2p^2\Delta+9pq^2)$, $D'(0)=-36pq^2$, $D'(\frac{4}{3}p^2)=\frac{8}{3}pD_d$. Pretpostavimo $p>0$. Tada po Teoremu 2 ne može nastupiti treći slučaj i vrijedi $D_d<0$. Iz izraza za $D'(\Delta)$ vidimo da je $D(\Delta)$ strogo padajuća za $\Delta<0$, i za $\Delta>\frac{2}{3}p^2$ i da je $D'(0)\le 0$. Neka je $p>0$ i $q\ne 0$. Kako diskriminanta $D(\Delta)$ po Δ iznosi $q^2D_d^3$ (napomena 2), i kako je $q^2D_d^3<0$, jednadžba $D(\Delta)=0$ ima točno jedno jednostruko realno rješenje (i dva konjugirano kompleksna rješenja). Dakle postoji realno δ tako da $\Delta<\delta$ povlači $D>0$ (drugi slučaj po Teoremu 2) i $\Delta>\delta$ povlači $D<0$ (prvi slučaj). Kako je $\lim_{\Delta\to-\infty}D(\Delta)=\infty$, $D(0)=(32p^3-27q^2)q^2$ i $\lim_{\Delta\to\infty}D(\Delta)=-\infty$, vrijedi $\delta<0$ ako je $32p^3-27q^2<0$, $\delta=0$ ako je $32p^3-27q^2=0$ i $\delta>0$ ako je $32p^3-27q^2>0$. Isti zaključak dobivamo koristeći Descartesov teorem o broju pozitivnih nultočaka polinoma $D(-\Delta)$, odnosno $D(\Delta)$, [1, p. 178., Descartesovo pravilo predznaka].*

Napomena 4 *Neka je $p>0$, $D\ge 0$ i $\Delta\ge 0$. Dokažimo da je tada $32p^3-27q^2\ge 0$. Pretpostavimo suprotno, tj. da je $p>0$, $D\ge 0$ i $\Delta\ge 0$, i $32p^3-27q^2<0$. Uočimo da je tada $q\ne 0$. No tada po napomeni 3 jednadžba $D(\Delta)=0$ ima točno jedno jednostruko negativno rješenje δ u kojem $D(\Delta)$ mijenja znak pa je za sve $\Delta\ge 0$ ispunjeno $D(\Delta)<0$. No to je u proturječju s pretpostavkom $D\ge 0$ i $\Delta\ge 0$. Analogno bi dokazali da ako je $p>0$, $D\ge 0$ i $\Delta>0$, tada vrijedi $32p^3-27q^2>0$. Primijenimo obje dokazane tvrdnje na slučaj $p>0$, $q\ne 0$, $D=0$ i $\Delta>0$. Po Teoremu 2 u prvom smo slučaju. Dalje koristimo napomenu 1 i njene oznake. Zbog $D=0$ imamo $t_1=t_2=t\ne 0$. Stavimo li $t^2=T$ i $L^2=N$, imamo $p=N-2T$ i $\Delta=N\cdot(N-8T)$. Nejednakost $32p^3-27q^2>0$ postaje $8(N-2T)^3\ge 27TN^2$ za sve $N\ge 8T>0$. Jednakost vrijedi ako i samo ako je $\Delta=0$, tj. ako i samo ako je $N=8T$.*

Napomena 5 *Neka smo u prvom slučaju i neka je $\Delta=0$. Tada je po Teoremu 2, $\Delta=0$, $D<0$, ili $\Delta=0$, $D=0$, $q\ne 0$. Ako je $\Delta=0$, $D<0$, onda, zbog $D(0)=(32p^3-27q^2)q^2$, vrijedi $q\ne 0$ i $32p^3-27q^2<0$. Ako je $\Delta=0$, $D=0$,*

$q \neq 0$, imamo, zbog $D(0) = (32p^3 - 27q^2)q^2$, jednakost $32p^3 = 27q^2 > 0$, dakle $p > 0$, u skladu s Teoremom 2. Dvostruku realnu nultočku polinoma $S(t)$ u slučaju $\Delta = 0$, $D = 0$, $q \neq 0$ računamo,

$$t_1 = t_2 = -(signq)\cdot\sqrt{\frac{p}{6}} = -\sqrt[3]{\frac{q}{16}}.$$

Prema tome, vrijedi $32p^3 - 27q^2 \leq 0$. Ako nastupi znak jednakosti, onda je $D = 0$ i stoga $t_1 = t_2$ (vidjeti napomenu 1). Uvedimo oznake $S = t_1 + t_2$ i $P = t_1 t_2$. Izrazimo li Δ pomoću S i T i riješimo li kvadratnu jednadžbu $\Delta = 0$ po $4L^2$ imamo

$$4L^2 = 3S^2 + 4P \pm 8\sqrt{P\cdot S^2}. \tag{42}$$

Iz formule (42) vidimo da je ovdje $P \geq 0$ i da bez smanjenja općenitosti možemo uzeti $t_1 \geq 0$, $t_2 \geq 0$. Uvedimo oznake $\sqrt{t_1} = \lambda$, $\sqrt{t_2} = \mu$ ($\lambda \geq 0$, $\mu \geq 0$) i izaberimo predznak $+$ u formuli (42). Onda po napomeni 1 imamo $p = 2\lambda\mu(\lambda^2 + \lambda\mu + \mu^2)$ i $q = -\left[(\lambda+\mu)(\lambda^2+\mu^2)\right]^2$. Nejednakost $32p^3 - 27q^2 \leq 0$ nakon sređivanja postaje nejednakost

$$\left[\frac{\lambda\mu(\lambda^2+\lambda\mu+\mu^2)}{3}\right]^3 \leq \left[\frac{(\lambda+\mu)\left(\lambda^2+\mu^2\right)}{4}\right]^4, \tag{43}$$

za sve nenegativne λ i μ. Znak jednakosti vrijedi tada i samo tada kada je $\lambda = \mu$.

4 Invarijante A, B i K

Pogledajmo izraz (25) u Teoremu 1. Zbog ireducibilnosti diskriminante naslućujemo da bi izrazi

$$A = \frac{A_1}{a_4^2} = \frac{a_2^2 - 3a_1a_3 + 12a_0a_4}{a_4^2} \tag{44}$$

i

$$B = \frac{B_1}{a_4^3} = \tag{45}$$

$$= \frac{27a_1^2a_4 + 27a_0a_3^2 + 2a_2^3 - 72a_0a_2a_4 - 9a_1a_2a_3}{a_4^3}$$

trebali biti invarijante. Zaista, supstitucija $t = x - h$ u $P_4(x)$ daje, nakon sređivanja

$$a_2^2(h) - 3a_1(h)a_3(h) + 12a_0(h)a_4(h) = a_2^2 - 3a_1a_3 + 12a_0a_4$$

i isto za drugi izraz. U kanonskom obliku za polinom $S(t)$ te veoma važne invarijante glase

$$\begin{aligned} A &= p^2 + 12r \\ B &= 27q^2 + 2p^3 - 72pr \end{aligned}$$

Nadalje vrijede relacije

$$64A = N^2 + 3R \tag{46}$$

$$256B = 108Q^2 - N^3 + 9NR.$$

Kako je

$$3R = 64A - N^2, \tag{47}$$

kanonski oblik polinoma $S(t)$ glasi

$$S(t) = t^4 - \frac{N}{8}t^2 + \frac{Q}{8}t + \frac{64A - N^2}{768}. \tag{48}$$

Invarijante A i B mogu se zapisati pomoću nultočaka x_i, $i = 1,2,3,4$. U tu svrhu promatrajmo Ferarrijevu kubnu rezolventu polinoma $P(x)$ (vidjeti zapis 2),

$$Y^3 - bY^2 + (ac - 4d)Y - (c^2 + a^2d - 4bd) = 0$$

i njene korijene Y_i, $i = 1,2,3$. Kao što je poznato, vrijedi $Y_1 = x_1x_2 + x_3x_4$; $Y_2 = x_1x_3 + x_2x_4$; $Y_3 = x_1x_4 + x_2x_3$, [2, str. 117, zadatak 814 a)]. Definirajmo sada $U = Y_2 - Y_3 = (x_1 - x_2)(x_3 - x_4)$; $V = Y_3 - Y_1 = (x_1 - x_3)(x_4 - x_2)$; $W = Y_1 - Y_2 = (x_1 - x_4)(x_2 - x_3)$ pri čemu je $U + V + W = 0$. Sada imamo pomoću Vietovih formula za polinom $P(x)$:

$$A = b^2 - 3(ac - 4d) =$$

$$= Y_1^2 + Y_2^2 + Y_3^2 - (Y_1Y_2 + Y_2Y_3 + Y_3Y_1), \tag{49}$$

$$A = \frac{1}{2}(U^2 + V^2 + W^2), \tag{50}$$

i

$$B = 27(a^2d + c^2 - 4bd) - 9b(ac - 4d), \tag{51}$$

$$B = U^2(V - W) + V^2(W - U) + W^2(U - V) =$$

$$= (W - U)(V - U)(U - W). \tag{52}$$

Iz jednakosti (50) i (52) vidimo da su po definiciji 1 A i B i apsolutne invarijante.

Promotrimo još jednu invarijantu koja je vezana uz vrijednosti polinoma $P_4(x)$ u svim točkama ekstrema ili točki horizontalne infleksije (jednoj, ako ona postoji), x_e. Vrijedi $P_4'(x_e) = 0$, odakle imamo

$$x_e^3 = -\frac{3a_3x_e^2 + 2a_2x_e + a_1}{4a_4}.$$

Odavde dobijemo

$$4P_4(x_e) = \frac{-(3a_3^2 - 8a_2a_4)x_e^2 + 2(6a_1a_4 - a_2a_3)x_e}{4a_4} +$$

$$+\frac{(16a_0a_4-a_1a_3)}{4a_4}. \tag{53}$$

Označimo sa K diskriminantu desne strane jednakosti (53):

$$K=\frac{4a_4^2(9a_1^2-32a_0a_2)-4a_1a_2a_3a_4+48a_0a_3^2a_4}{4a_4^2}+$$

$$+\frac{a_2^2a_3^2-3a_1a_3^3}{4a_4^2}. \tag{54}$$

Ako je $a_3=0$ i $a_4=1$, onda je $K=9q^2-32pr$. Diskriminanta kubnog polinoma $S(t)$, kažimo D_d, je invarijanta dana do na pozitivni faktor izrazom

$$2^7D_d=N^3-27Q^2, \tag{55}$$

odnosno, po formulama (11) s

$$2D_d=-(8p^3+27q^2). \tag{56}$$

Veze između invarijanata D_d (formule (55) i (56)), B i K za polinom $P_4(x)$ su:

$$2D_d=9K-4B=N\cdot A-3K, \tag{57}$$

$$12K=4B+N\cdot A, \tag{58}$$

i

$$64B=-N^3+48N\cdot A+27Q^2.$$

Iz formule (58) lako slijedi da je K invarijanta jer su to N, A i B. Iz formule (25) u Teoremu 1, i iz formule (58) odmah izlazi sljedeća propozicija

Propozicija 1 *Ako je $(A<0)$, ili $(A=0$ i $B\neq 0)$, ili $(A=0$ i $K\neq 0)$, tada je $D<0$ i polinom $P_4(x)$ ima dvije realne različite, i dvije kompleksne nultočke.*

Relacije (31) i (36) povlače sljedeće dvije posljedice

Korolar 2 *Ako je $D\geq 0$ (drugi ili treći slučaj), onda je $(A>0)$ ili $(A=B=D=0)$.*

Korolar 3 *Ako je $D\geq 0$ (drugi ili treći slučaj), onda je $(A>0)$ ili $(A=K=D=0)$.*

Propozicija 2 *Pretpostavimo da je za polinom $P_4(x)$ invarijanta $K<0$ i time $p\neq 0$. Ako je $N<0$, onda $P_4(x)$ ima sve nultočke kompleksne i samo jedan ekstrem i vrijedi $D_d<0$ i $D\geq 0$, a ako je $N>0$, onda $P_4(x)$ ima dvije realne različite i dvije kompleksne nultočke i vrijedi $D<0$.*

Dokaz. Ako za sve x_e vrijedi $a_4P_4(x_e)>0$, (vidjeti formulu (53)), onda $P_4(x)$ ima sve nultočke kompleksne, a ako za sve x_e vrijedi $a_4P_4(x_e)<0$, onda $P_4(x)$ ima dvije realne i dvije kompleksne nultočke, jer je $\lim_{x\to\pm\infty}a_4P_4(x)=+\infty$. To će se desiti u slučaju $K<0$, (kada je $p\neq 0$), jer tada kvadratni trinom na desnoj strani formule (53) prima vrijednosti uvijek istog predznaka (suprotnog od predznaka od N).

Propozicija 3 *Ako su sve nultočke polinoma $P_4(x)$ realne, vrijedi $K\geq 0$. Ako su sve nultočke polinoma $P_4(x)$ realne i različite, vrijedi $K>0$.*

Dokaz. Dokažimo drugi dio propozicije. Pretpostavimo da je $a_4=1$ i $a_3=0$ i da su sve nultočke polinoma $P_4(x)$ realne i različite. Ako je $r=0$, onda je $q\neq 0$ jer su sve nultočke polinoma $P_4(x)$ realne i različite, pa stoga sve proste (jednostruke). No tada je $K=9q^2>0$. Neka je $r\neq 0$, što povlači da 0 nije nultočka polinoma $P(x)$. Onda su i sve nultočke polinoma $R(y)=y^4P(\frac{1}{y})=ry^4+qy^3+ry^2+1=0$ opet realne i različite. Po Rolleovom teoremu i nultočke derivacije $R'(y)=y(4ry^2+3qy+2r)$ su sve realne i različite, pa diskriminanta polinoma $4ry^2+3qy+2r$, a to je upravo broj K, mora biti pozitivna. Prvi dio propozicije dokazuje se posve analogno i dovoljno ga je dokazati samo za slučaj $r\neq 0$.

Zadatak 2.
Pretpostavimo da smo u trećem slučaju. Zapišite tada za polinom $S(t)$ dokazane nejednakosti $A=p^2+12r\geq 0$, $K=9q^2-32pr\geq 0$ i $D_d=-\frac{1}{2}(8p^3+27q^2)\geq 0$ pomoću njegovih nultočaka t_i, $i\in\{1,2,3,4\}$, odnosno pomoću veličina Σ, S i Π iz napomene 1. Dokažite da u trećem slučaju $A=p^2+12r=0$ tada i samo tada ako je $\{t_1,t_2,t_3,t_4\}=\{t,t,t,-3t\}$ za neko realno t (slučaj realne, barem trostruke nultočke). Dokažite da je u trećem slučaju $K=9q^2-32pr=0$ i $D=0$ tada i samo tada ako je $\{t_1,t_2,t_3,t_4\}=\{t,t,t,-3t\}$ za neko realno t ili $\{t_1,t_2,t_3,t_4\}=\{0,0,r,-r\}$ za neko realno r. Dokažite da ako su sve nultočke t_i, $i\in\{1,2,3,4\}$ realne i međusobno različite, vrijedi $A>0$ i $D_d>0$.

Zadatak 3.
Pretpostavimo da smo u trećem slučaju. Dokažite da tada za polinom $S(t)$ vrijedi **Newtonovska nejednakost** $\left(\frac{q}{\binom{4}{1}}\right)^2\geq\left(\frac{p}{\binom{4}{2}}\right)r$, tj. $3q^2\geq 8pr$, koja je slabija od $K=9q^2-32pr\geq 0$. Kada vrijedi znak jednakosti?
Uputa: Dovoljno je dokazati da za $r\neq 0$ vrijedi $3q^2>8pr$. Gledajte polinom $R(z)=\frac{1}{r}z^4\cdot S(\frac{1}{z})$ i zapišite $\frac{q}{r}$ i $\frac{p}{r}$ pomoću njegovih realnih nultočaka z_i, $i\in\{1,2,3,4\}$. Potom nejednakost $3\left(\frac{q}{r}\right)^2>8\frac{p}{r}$ svedite na nejednakost $N>0$ (vidjeti formulu (16)).

Napomena 6 *Pretpostavimo da smo trećem slučaju i da je $p^2+q^2+r^2 \neq 0$. Koristeći izreku Korolara 2, Leme 1, Leme 2 i Propozicije 3, imamo tada ove nejednakosti*

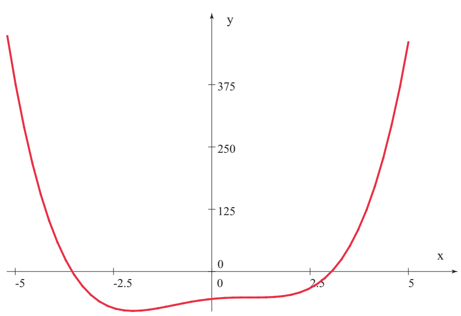
$$0 \leq -\frac{9q^2}{2p} \leq p^2-4r \leq p^2-\frac{9q^2}{8p}, \tag{59}$$

ili ekvivalentno

$$0 \leq \frac{8p^3+27q^2}{8p} \leq p^2+12r \leq \frac{8p^3+27q^2}{2p}. \tag{60}$$

5 Grafovi

Na temelju proučenih invarijanata opišimo grafove polinoma $P(x)$, koji je sveden na kanonski oblik $S(t)$ (formula (48)) i nacrtajmo samo neke posebne slučajeve. Uzmimo da $P(x)$ ima točku horizontalne infleksije x_i. Tada $P'(x)$, tj. $S'(t)$ ima dvostruku, ali ne i trostruku nultočku, što je ekvivalentno s $D_d = 0$ i $N > 0$.



Slika 1.

Slika 1. prikazuje slučaj točke horizontalne infleksije za polinom $P(x) = x^4-6x^2+8x-55 = (x-1)^3(x+3)-52$. Slučaj horizontalne infleksije nastupa onda i samo onda kada je $D_d = 0$ i $p < 0$ ili ekvivalentno kada je $p = -\frac{3}{2}\sqrt[3]{q}^2 < 0$. Kako je $N = -8p$, i N invarijanta na pomake imamo $N > 0$ u slučaju horizontalne infleksije. Važne točke grafa su ekstrem $E(-\sqrt[3]{q}, -\frac{3}{2}\sqrt[3]{q}^4+r)$, točka kose infleksije $K(-\frac{\sqrt[3]{q}}{2}, -\frac{13}{16}\sqrt[3]{q}^4+r)$ i točka horizontalne infleksije $H(\frac{\sqrt[3]{q}}{2}, \frac{3}{16}\sqrt[3]{q}^4+r)$. Kanonski oblik polinoma u slučaju horizontalne točke infleksije je

$$S(t) = t^4-\frac{3\sqrt[3]{q^2}}{2}t^2+qt+r.$$

Zadatak 4.
Dokažite da u slučaju horizontalne točke infleksije pomakom koordinatnog sustava $x = X+h$, $y = Y$ polinom $P(x)$ možemo pisati kao $\widetilde{P}(X) = X^4+A_3X^3+A_0$.

Napomena 7 *Gledajmo diskriminantu D polinoma $S(x)$ kao funkciju od Δ (formula (33) iz Teorema 1). Iz napomene 3 znamo da je $D(\frac{4}{3}p^2) = -\frac{4}{27}D_d^2$, $D'(\frac{4}{3}p^2) = \frac{8}{3}pD_d$. Odavde vidimo da slučaj horizontalne infleksije nastupa tada i samo tada kada $D(\Delta)$ ima jednu dvostruku realnu nultočku $\Delta_1 = \Delta_2 = \frac{4}{3}p^2$ i jednu od nje različitu jednostruku nultočku $\Delta_3 = -\frac{5}{3}p^2$. Slučaj $S(t) = t^4+r$ nastupa onda i samo onda kada $D(\Delta)$ ima trostruku realnu nultočku $\Delta_1 = \Delta_2 = \Delta_3 = 0$.*

Neka je ekstrem ili točka horizontalne infleksije ujedno i nultočka i to ekstrem nultočka parnog reda, a točka horizontalne infleksije nultočka neparnog reda (ovo je npr. slučaj ako $S(t)$ ima trostruku ili četverostruku nultočku). Onda je $D = 0$ i po formuli (53) $16a_4P(x_e) = 0$, i $K \geq 0$.

Propozicija 4 *Polinom $P_4(x)$ ima točno trostruku realnu nultočku onda i samo onda kada je $A = 0$, $K = 0$ i $N > 0$ ili ekvivalentno $A = 0$, $B = 0$ i $N > 0$. U tom slučaju je njegov kanonski oblik*

$$S(t) = t^4-3\frac{\sqrt[3]{q}^2}{2}t^2+qt-3\frac{\sqrt[3]{q}^4}{16}, \tag{61}$$

ili

$$S(t) = (t+\frac{3q}{4p})^3(t-\frac{9q}{4p}) = (t-\frac{\sqrt[3]{q}}{2})^3(t+\frac{3\cdot\sqrt[3]{q}}{2}). \tag{62}$$

Polinom $P_4(x)$ četverostruku realnu nultočku tada i samo tada kada je $A = 0$, $B = 0$ i $N = 0$.

Dokaz. Neka polinom $P_4(x)$ ima barem trostruku nultočku x_t. Onda je $D = 0$, $D_d = 0$, što povlači iz (57) da je $3K = N\cdot A$, te iz formule (16) slijedi da je $N \geq 0$ (a $N > 0$ samo ako je nultočka točno trostruka realna). Onda je ta nultočka ili jedini ekstem ili jedina točka horizontalne infleksije. Kako su B, A, N i K invarijante možemo, bez smanjenja općenitosti, uzeti $x_t = 0$, i $P_4(x) = a_4x^4+a_3x^3$, dakle $a_2 = a_1 = a_0 = 0$. Sada su A i B po formulama (44) i (45) nula, a zbog $3K = N\cdot A$ je i $K = 0$. Po formuli (14) je $N = 3a_3^2$. Kako je $N \neq 0$, slijedi $N > 0$ i $a_3 \neq 0$. To je upravo slučaj trostruke realne nultočke. Obratno, neka je $K = A = 0$, odnosno, po formuli (58) neka je ekvivalentno $B = A = 0$. Kako su B, A, N i K invarijante, uzmimo kanonski oblik (7) polinoma $P_4(x)$. Onda je $D_d = 0$ po formuli (57) i $D = 0$ po formuli (31) iz Teorema 1 pa imamo barem dvostruku nultočku x_d. No $D_d = 0$ povlači $N \geq 0$. Uvažimo li relaciju $D_d = 0$, tj. $N^3 = 27Q^2$, kanonski oblik (48) polinoma $S(t)$ glasi:

$$S(t) = t^4-\frac{3\sqrt[3]{Q^2}}{8}t^2+\frac{Q}{8}t-\frac{9\sqrt[3]{Q^4}}{768}.$$

Odavde, i iz $Q = 8q$, slijedi formula (61). Ako je $N = 0$, onda je i $Q = 0$, i imamo četverostruku realnu nultočku. Ako je $N > 0$, imamo trostruku realnu nultočku. Formulu $x_t = -\dfrac{3q}{4p} = \dfrac{\sqrt[3]{q}}{2} = \dfrac{\sqrt[3]{Q}}{4}$ lako dobijemo iz Vietovih formula, pa imamo i formulu (62).

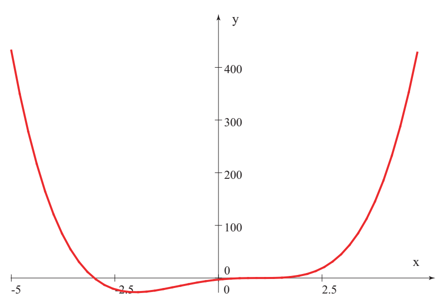
Zadatak 5.
Dokažite da slučaj trostruke realne nultočke nastupa onda i samo onda kada je $D_d = 0$, $A = 0$ i $N > 0$, tj. tada i samo tada kada je $8p^3 + 27q^2 = 0$, $p^2 + 12r = 0$ i $p < 0$.

Upotrebom supstitucije $x = \dfrac{1}{z}$ i Propozicije 4, dokazuje se ova posljedica

Korolar 4 *Svaki polinom $P_4(x)$ može se svesti na oblik*

$$a_4x^4 + c_3x^3 + c_2x^2 + c_0 = 0, \tag{63}$$

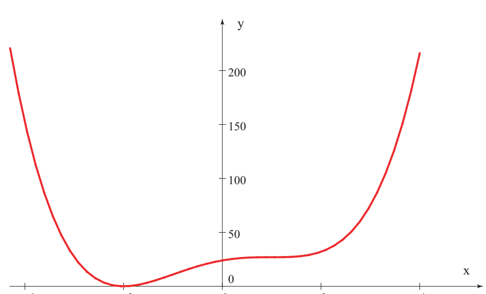
gdje je $a_4 \neq 0$, a c_3, c_2 i c_1 su realni brojevi. Polinom (63) ima pri tom točno trostruku realnu nultočku akko je ($c_3 \neq 0$, $c_2 = c_0 = 0$) ili ($c_0 \neq 0$ i $c_2^2 + 12a_4c_0 = 0$ i $9c_3^2 - 32a_4c_2 = 0$).



Slika 2

Slika 2 prikazuje slučaj trostruke realne nultočke za polinom $P(x) = x^4 - 6x^2 + 8x - 3 = (x-1)^3(x+1)$.

Sljedeća slika prikazuje jedini preostali slučaj horizontalne infleksije u kojem je $D = 0$, ali koji nije slučaj trostruke realne nultočke. To je slučaj $D_d = D = 0$, $A \neq 0$ i $N > 0$. Preciznije, to je slučaj $D_d = D = 0$, $A > 0$ i $N > 0$.



Slika 3

Slika 3 prikazuje slučaj horizontalne infleksije s dvostrukom realnom nultočkom i dvije konjugirano kompleksne nultočke. Na gornjoj slici je nacrtan graf polinoma $P(x) = x^4 - 6x^2 + 8x + 24 = (x+2)^2\left[(x-2)^2 + 2\right]$.
Iz formule (37) u Teoremu 1 vidimo da taj slučaj nastupa onda i samo onda kada je $8p^3 + 27q^2 = 0$, $2p^2 - 3r = 0$ i $p < 0$, ili ekvivalentno onda i samo onda kada je $N^3 = 27Q^2$ i $9N^2 = 64A$. Tada je po formuli (46) $3R = 8N^2$. Kanonski oblik polinoma $S(t)$, uz korištenje formula (11) glasi:

$$S(t) = t^4 - \frac{3\sqrt[3]{q^2}}{2}t^2 + qt + \frac{3\sqrt[3]{q^4}}{2}$$

ili

$$S(t) = (t + \sqrt[3]{q})^2\left[(t - \sqrt[3]{q})^2 + \frac{1}{2}\sqrt[3]{q}^2\right].$$

Napomena 8 *Neka je točka t_e koja je ekstrem ili točka horizontalne infleksije, ujedno i nultočka polinoma $S(t)$ i to ekstrem nultočka parnog reda, ovdje reda dva ili četiri, a točka horizontalne infleksije nultočka neparnog (ali većeg od jedan) reda, dakle ovdje reda tri. Onda po formuli (53) imamo $S(t_e) = 0$ i $K \geq 0$. Primijetimo da $S(t_e) = 0$ povlači $D = 0$ (jer je t_e višestruka nultočka) pa po formuli (31) iz Teorema 1 slijedi $A \geq 0$, a da $K = 0$ povlači $9D = 16r \cdot A^2 = \dfrac{4}{3}(A - p^2) \cdot A^2$ po formuli (36) iz Teorema 1. Neka je $P(x_e) = 0$ i $K = 0$. Onda je ili 1°($D = 0$ i $r = 0$ i $A > 0$) i imamo **prvi** ili **treći slučaj**, ili 2°($D = 0$ i $r \neq 0$ i $A = 0$) ili 3°($D = 0$ i $r = A = 0$). Slučaj 2° je slučaj trostruke realne nultočke po Propoziciji 4. U tom slučaju je, opet po Propoziciji 4, $D_d = 0$ i $B = 0$, a kako je $A = p^2 + 12r$, to $A = 0$ povlači $r = -\dfrac{1}{12}p^2 < 0$, i time $p \neq 0$. Osim toga iz $D_d = 0$ slijedi $8p^3 = -27q^2$ pa je $p < 0$, tj. $N > 0$. Kanonski oblik polinoma $S(t)$ za slučaj trostruke realne nultočke time glasi ($Q = 8q$),*

$$S(t) = t^4 - \frac{3\sqrt[3]{Q^2}}{8}t^2 + \frac{Q}{8}t - \frac{9\sqrt[3]{Q^4}}{768}.$$

U slučaju 3°, tj. ako je $D = 0$ i $r = A = 0$, tada je $p = q = D_d = 0$, i $B = 0$, pa imamo $P(x) = x^4$ i četverostruku realnu nultočku. Ako je $D_d = 0$, onda za r dovoljno vrijedi $S(t) > 0$ i imamo drugi slučaj. Dakle, ako je $D_d = 0$ moguća su sva tri slučaja, jedino u trećem slučaju ne mogu tada sve realne nultočke biti različite.

Neka je $D_d > 0$. Onda je $N > 0$ i graf ima tri ekstrema i dvije točke infleksije, po jednu između dva susjedna ekstrema. Moguća su sva tri slučaja nultočaka od $P_4(x)$.

Ako je $D_d < 0$, graf polinoma $P(x)$ ima samo jedan ekstrem, nema točke horizontalne infleksije i mogu nastupiti samo prvi i drugi slučaj. Dvije točke infleksije pojavljuju se onda i samo onda kada je $N > 0$, obje desno od ekstrema ako je $Q > 0$ i obje lijevo ako je $Q < 0$. Ako je $D_d < 0$ i $N = 0$, imamo slučaj točke dodira višeg reda u $t = 0$ i kanonski oblik polinoma $S(t)$ tada glasi $S(t) = t^4 + qt + r$. Razmotrimo pobliže treći, realni slučaj. Po Rolleovom teoremu $P'(x)$ tada ima tri realne nultočke, a $P'(x)$ dvije realne nultočke. Ako $P(x)$ ima sve realne nultočke i kratnost svake je najviše dva, onda opet po Rolleovom teoremu $P'(x)$ ima tri realne različite i prema tome jednostruke (proste) nultočke pa je $D_d > 0$, a $P'(x)$ dvije realne različite i proste nultočke. Dakle, graf od $P(x)$ ima tri različita ekstrema i dvije različite točke kose infleksije između dva susjedna ekstrema. Po formuli (16) je tada $N > 0$. Ako $P(x)$ ima sve realne nultočke i jednu točno trostruku nultočku, opet po Rolleovom teoremu, $P'(x)$ ima sve realne nultočke, i to jednu kratnosti jedan, koja je ekstrem i jednu kratnosti dva, koja je točka horizontalne infleksije i stoga vrijedi $D_d = 0$. Ako $P(x)$ ima jednu četverostruku realnu nultočku, ona je ekstrem. Tada $P'(x)$ ima jednu trostruku realnu nultočku pa je $D_d = 0$. Po formuli (16) sada je $N = 0$. Dakle, ako $P(x)$ ima sve realne nultočke i jednu barem trostruku nultočku, onda $P(x)$ ima jedan ekstrem i $D_d = 0$. Vidimo da $P'(x)$ ima u trećem slučaju uvijek tri realne nultočke, pa vrijedi $D_d \geq 0$ što povlači, po formuli (55), $N \geq 0$. Iz analize slijedi da u trećem slučaju $D_d = 0$ onda i samo onda kada $P(x)$ ima trostruku ili četverostruku realnu nultočku, a $N = 0$ onda i samo onda kada je $P(x) = (x - x_0)^4$.

U slučaju $q = 0$, **graf bikvadratne funkcije** $P_b(x) = x^4 + px^2 + r$ je zbog parnosti funkcije, simetričan s obzirom na y-os i ovisi o p. Ako je $p \geq 0$, graf ima jedan ekstrem, i to lokalni minimum u $x = 0$ i nema točaka infleksije. Ako je $p < 0$, graf ima tri ekstrema (dva lokalna minimuma i jedan lokalni maksimum u $x = 0$) i dvije točke infleksije. Pri tom se između dvaju susjednih ekstrema, od kojih je jedan lokalni minimum, a drugi lokalni maksimum, nalazi jedna točka infleksije.

Literatura

[1] DEMIDOVICH B. P., MARON I. A., *Computational Mathematics*, Mir, 1976.

[2] FADDEEV D. K., SOMINSKIJ I. S., *Sbornik zadač po vissej algebre*, Nauka, Moskva, 1968.

[3] RADIĆ M., *Rješivost algebarskih jednadžbi*, Zagreb, 1966.

[4] REDEI L., *Algebra 1.*, Akademiai Kiado, Budapest, 1967.

[5] VIHER R. *Posljedice Descartesove metode za faktorizaciju polinoma 4. stupnja*, KoG 5, (2000/01), 11-15.

[6] WAERDEN B. L. VAN DER, *Algebra I i II*, Springer-Verlag, 1967.

[7] WEBER H., *Lehrbuch der Algebra*, *I*, Vieweg & Sohn, 1912.

Predrag Lončar

Geotehnički fakultet Sveučilišta u Zagrebu

Hallerova aleja 7, 42000 Varaždin

e-mail: ivan.loncar1@vz.htnet.hr

Professional paper
Accepted 29. 12. 2005.

Aleksandar Čučaković

Constructive Procedure for Transformation of Collinear Spaces

Constructive Procedure for Transformation of Collinear Spaces

ABSTRACT

This paper offers the constructive solution for transformation of points between two general collinear spaces Σ and $\overline{\Sigma}$ by using Monge's projection. The constructive procedure is based on the fact that each straight line of space Σ which is parallel to vanishing plane $\mathbf{R} \in \Sigma$ and one side of autocollinear tetrahedron $D_1D_2D_3D_4$, is transformed into the straight line of space $\overline{\Sigma}$ which is parallel to vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$ and the same side of autocollinear tetrahedron.

Key words: collinear spaces, autocollinear tetrahedron, vanishing plane, perspectivity of the pencils of lines

MSC 2000: 51N05

Konstruktivni postupak za transformaciju kolinearnih prostora

SAŽETAK

U radu se daje konstruktivno rješenje za preslikavanje točaka između dva opće kolinearna prostora Σ i $\overline{\Sigma}$, u Mongeovoj projekciji. Konstrukcija se temelji na činjenici da se svaki pravac prostora Σ koji je paralelan s izbježnom ravninom $\mathbf{R} \in \Sigma$ i jednom od stranica autokolinearnog tetraedra $D_1D_2D_3D_4$, preslikava u pravac prostora $\overline{\Sigma}$ koji je paralelan s izbježnom ravninom $\overline{\mathbf{Q}} \in \overline{\Sigma}$ i istom stranicom autokolinearnog tetraedra.

Ključne riječi: kolinearni prostori, autokilinearni tetraedar, izbježna ravnina, perspektivitet pramenova pravaca

1 Introduction

In the transformation of two general collinear spaces Σ and $\overline{\Sigma}$, one has to take into consideration that the composition of projections and types of projections, necessary for a space representation, are transforming into new composition of quite different types of projections in another space. Therefore, the applied method of transformation of two general collinear spaces must be based on their common invariants.

The constructive procedure in Monge's projection must have common real autocollinear tetrahedron and uses two basic invariants for the transformation of two collinear spaces.

- The first invariant consists of the following: all planes parallel to vanishing plane $\mathbf{R} \in \Sigma$ correspond to the planes parallel to vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$. [2, p. 10]
- The second invariant consists of the following: each straight line which is parallel to vanishing plane $\mathbf{R} \in \Sigma$ and one side of autocollinear tetrahedron corresponds to the straight line which is parallel to the mentioned side of tetrahedron and vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$. [2, p. 17]

2 Some properties of collinear spaces

Two collinear spaces Σ and $\overline{\Sigma}$, located in the same projective space, always have four fixed points which are the vertexes of one tetrahedron $D_1D_2D_3D_4$. Six edges $D_1D_2 = d_3$, $D_2D_3 = d_1$, $D_1D_3 = d_2$, $D_1D_4 = d_4$, $D_2D_4 = d_5$, $D_3D_4 = d_6$ and four sides $D_1D_2D_3 = \Delta_4$, $D_1D_2D_4 = \Delta_3$, $D_2D_3D_4 = \Delta_1$, $D_1D_3D_4 = \Delta_2$ of the tetrahedron are autocollinear, i.e. they are twofold lines and planes of the collineation with fixed points at the related vertexes D_1, D_2, D_3 and D_4. The collineation between Σ and $\overline{\Sigma}$ is uniquely determined with autocollinear tetrahedron and the pair of corresponding points or the pair of corresponding planes. [3, p. 93]

2.1 Vanishing planes, lines and points

Let collinear spaces Σ and $\overline{\Sigma}$ be located in the same real projective space. The planes, lines and points which correspond with the plane, lines and points at infinity are the *vanishing elements* of the collineation. The collineation between Σ and $\overline{\Sigma}$ can be determined with autocollinear tetrahedron $D_1D_2D_3D_4$ and vanishing plane $\mathbf{R} \in \Sigma$.

Let side $\Delta_4 = D_1D_2D_3$ of autocollinear tetrahedron lie in the horizontal plane and (r_1, r_2) are the traces of vanishing plane $\mathbf{R} \in \Sigma$ (Fig1).

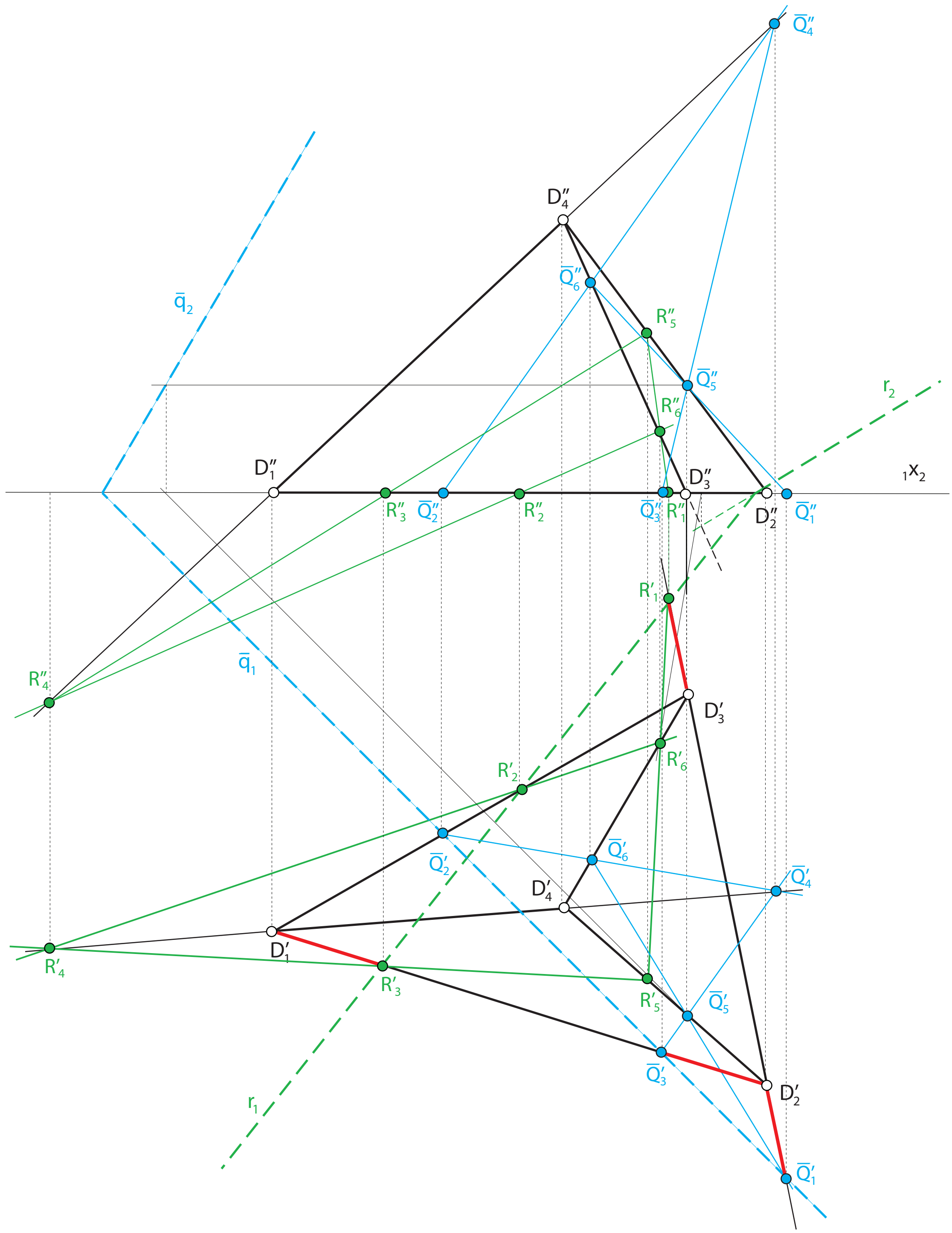


Figure 1

The points of intersection of six lines ($d_1 - d_6$) and vanishing plane $\mathbf{R}$ defines six vanishing points ($R_1 - R_6$) on autocollinear edges. The lines of intersection of four sides ($\Delta_1 - \Delta_4$) and vanishing plane $\mathbf{R}$ are four vanishing lines in autocollinear planes. Each of them joins three vanishing points: $\Delta_1 \cap \mathbf{R}$ joins R_1, R_5, R_6; $\Delta_2 \cap \mathbf{R}$ joins R_2, R_4, R_6; $\Delta_3 \cap \mathbf{R}$ joins R_3, R_4, R_5; $\Delta_4 \cap \mathbf{R}$ joins R_1, R_2, R_3. The construction is simple because three vanishing points (R_1, R_2, R_3) lie on the first trace r_1 and it is neccessary to construct only one other vanishing point as the intersection of autocollinear edge and vanishing plane (in Fig 1 it is point R_6).

The construction of vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$, six vanishing points ($\overline{Q}_1 - \overline{Q}_6$) on autocollinear edges ($d_1 - d_6$) and four related vanishing lines in autocollinear sides ($\Delta_1 - \Delta_4$) is based on the following statement: *The vanishing points of two projective ranges of points which are located on the same line, are always equidistanced from the fixed points of the projectivity*. [3, p. 17]

It enables very simple construction of vanishing points $\overline{Q}_1 - \overline{Q}_6$ on lines $d_1 - d_6$, related vanishing lines and the traces ($\overline{q}_1, \overline{q}_2$) of vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$. For example, in Fig 1 trace $\overline{q}_1$ joins points $\overline{Q}_1$ and $\overline{Q}_2$, where $d(\overline{Q}_1, D_2) = d(\overline{R}_1, D_3)$ and $d(\overline{Q}_3, D_2) = d(\overline{R}_3, D_1)$.

2.2 Perspectivities of pencils of lines in horizontal plane

For two colliner fields, located in the same real projective plane, the pencils of corresponding lines with vertexes in the pair of corresponding points on the edges of autocollinear triange $D_1D_2D_3$, are perspectivelly ajusted. The axes of that perspectivities always pass through the opposite vertexes of autocollinear triangle. If one of the vertexes of pencils of lines is the point at infinity, the axis of perspectivity is parallel to a vanishing line. [1, p. 2].

In Fig 2 this property is shown for pencils of lines (S) and $(\overline{S}^{\infty})$, where S and $\overline{S}^{\infty}$ are the pair of corresponding points on edge D_2D_3. The axis of perspectivity $(S) \overline{\overline{\wedge}} (\overline{S}^{\infty})$ is parallel to vanishing line v and passes through point D_1.

This statement enables the construction of corresponding points in all autocolinear planes of collinear spaces Σ and $\overline{\Sigma}$. In Fig 3 this construction is shown in the horizontal plane from Fig 1. Perspectivities $(R_1) \overline{\overline{\wedge}} (\overline{R}_1^{\infty})$ and $(R_2) \overline{\overline{\wedge}} (\overline{R}_2^{\infty})$ are used for the construction of corresponding points X and $\overline{X}$ which are in general position to twofold elements.

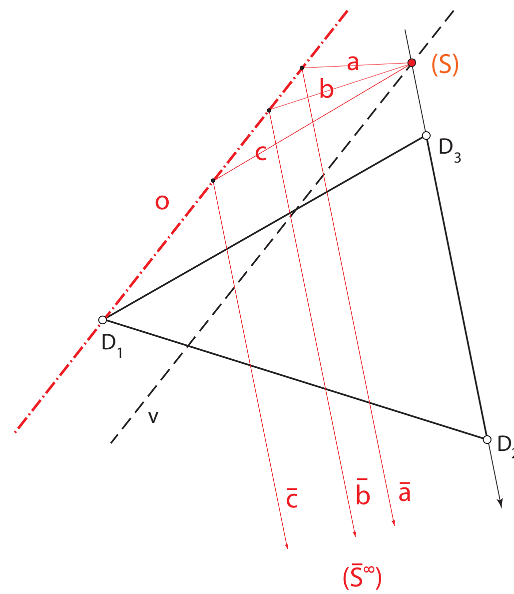


Figure 2

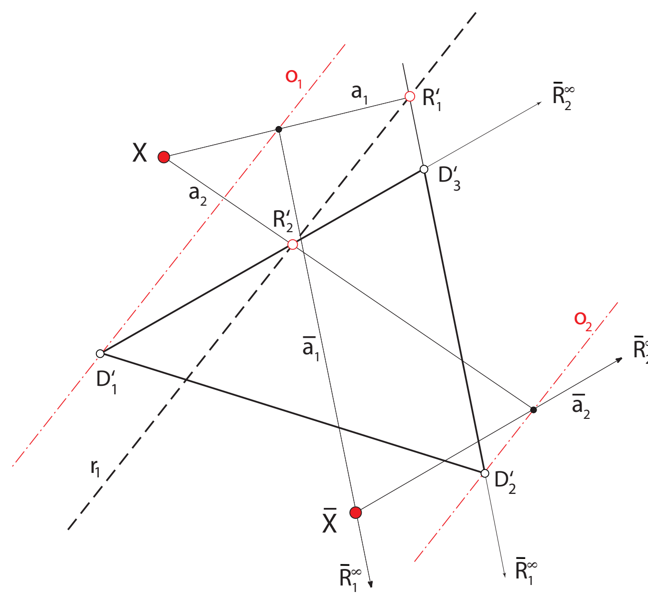


Figure 3

3 Construction of corresponding points for collinear spaces in Monge's projection

The constructive procedure is based on the fact that each straight line of space Σ, which is parallel to vanishing plane $\mathbf{R} \in \Sigma$ and one side of autocollinear tetrahedron $D_1D_2D_3D_4$, is transformed into the straight line of space $\overline{\Sigma}$ which is parallel to vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$ and the same side of autocollinear tetrahedron.

Let the collineation between Σ and $\overline{\Sigma}$ be defined with autocollinear tetrahedron $D_1D_2D_3D_4$ and vanishing plane $\mathbf{R} \in \Sigma$, and let them be positioned as in Fig 1. Point A (see Fig 4) is in general position to autocollinear elements and point $\overline{A}$ is constructed in the following way:

1. Traces $(\overline{q}_1, \overline{q}_2)$ of vanishing plane $\overline{\mathbf{Q}}$ and vanishing lines R_1R_6, R_2R_6, $\overline{Q}_1\overline{Q}_6$, $\overline{Q}_2\overline{Q}_6$ are constructed as in Fig 1.
2. Lines a_1, a_2 which pass through point A and are parallel to R_1R_6 and R_2R_6, respectively, are constructed. They lie in plane $\mathbf{T} \parallel \mathbf{R}$. Points A_1, A_2 are the intersections of lines a_1, a_2 and horizontal plane.
3. Since points A_1 and A_2 lie in autocollinear plane (horizontal plane) we can construct corresponding points $\overline{A}_1$ and $\overline{A}_2$ as in Fig 3, by using perspectivities $(R_1)\,\overline{\overline{\wedge}}\,(\overline{R}_1^{\infty})$ and $(R_2)\,\overline{\overline{\wedge}}\,(\overline{R}_2^{\infty})$. For the construction of point $\overline{A}_2$, line $\overline{t}_1$ (the first trace of plane $\overline{\mathbf{T}} \parallel \overline{\mathbf{Q}}$) is also used.
4. According to the statement mentioned in the introduction: the lines through points $\overline{A}_1$ and $\overline{A}_2$, parallel to $\overline{Q}_1\overline{Q}_6$ and $\overline{Q}_2\overline{Q}_6$, respectively, are corresponding lines of a_1 and a_2, i.e. they are $\overline{a}_1$ and $\overline{a}_2$.

   Point $\overline{A}$ is the intersection of lines $\overline{a}_1$ and $\overline{a}_2$.

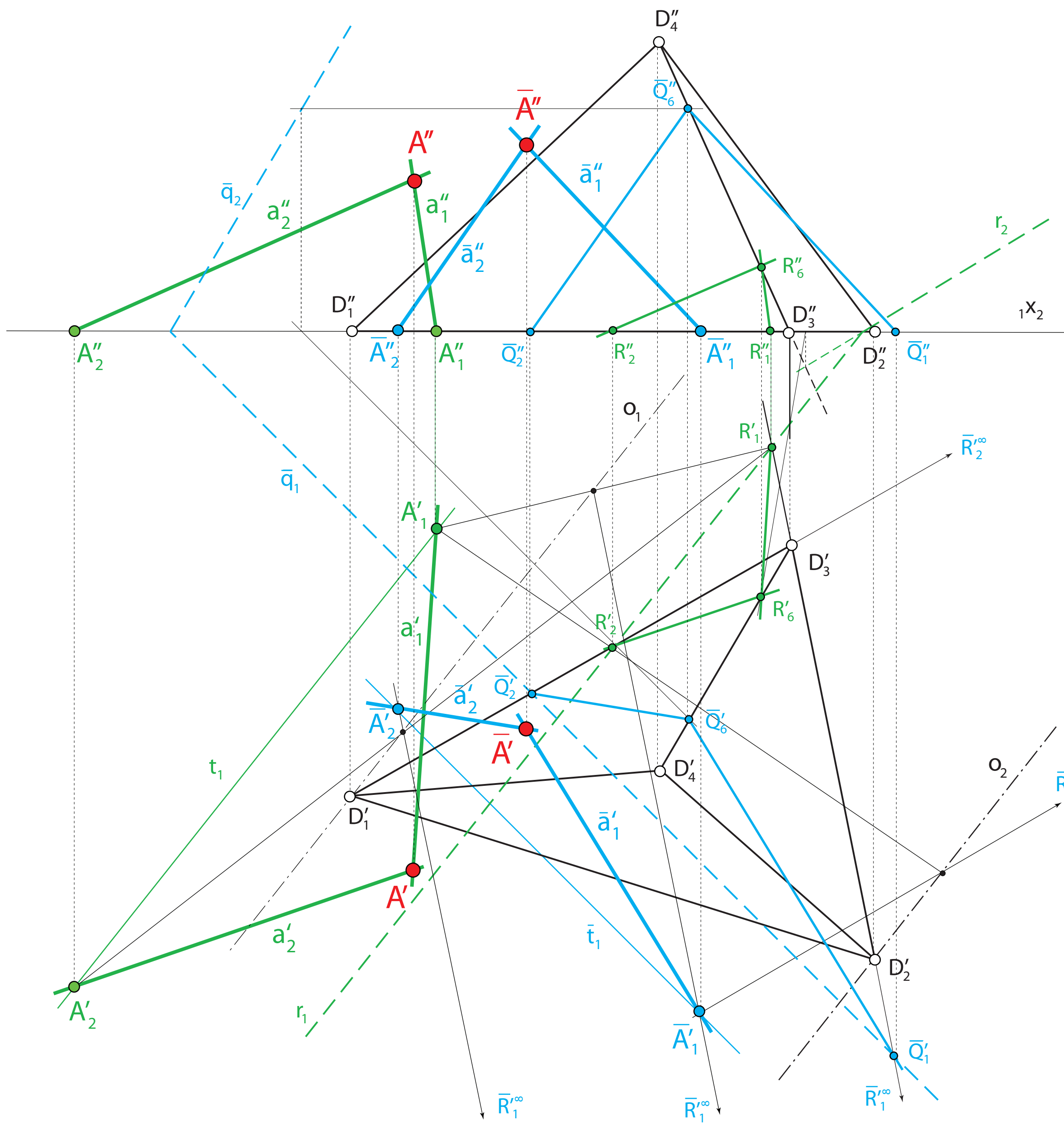


Figure 4

References

[1] ČUČAKOVIĆ, A., *Invariables of conformity and coordinatization of general collinear and of general affine planes and spaces*, Ph.D. thesis, Belgrade 1992.

[2] GAGIĆ, LJ., *Graphic Transformation of Collinear Spaces*, Ph.D. thesis, Belgrade 1977.

[3] NIČE, V., *Deskriptivna geometrija I*, Školska knjiga, Zagreb, 1979.

Aleksandar Čučaković
Faculty of Civil Engineering, University of Belgrade, Belgrade
e-mail: cucak@grf.bg.ac.yu

GUIDE FOR AUTHORS

SCOPE. "KoG" publishes scientific and professional papers from the fields of geometry, appplied geometry and computer graphics.

SUBMISSION. Scientific papers submited to this journal should be written in English or German, professional papers should be written in Croatian, English or German. Only unpublished material can be accepted.

Two single-side copies of the manuscript with wide margins and double spaced should be sent to the one of the editors:

Sonja Gorjanc	Jelena Beban - Brkić
sgorjanc@grad.hr	jbeban@geof.hr
Faculty of Civil Engineering	Faculty of Geodesy

Kačićeva 26, 10000 Zagreb, Croatia

The first page should contain the article title, author and coautor names, affilation, a short abstract in English, a list of keywords and the Mathematical subject classification.

ELECTRONIC FORMATS. Accepted papers should be sent by electronic mail as ASCII files (LaTeX format is recommanded) to the address: sgorjanc@grad.hr

OFFPRINTS. The total of 20 reprints of each contribution will be sent to its first mentioned author (if not otherwise desired) free of charge.

UPUTE ZA AUTORE

PODRUČJE. "KoG" objavljuje znanstvene i stručne radove iz područja geometrije, primijenjene geometrije i kompjutorske grafike.

UPUTSTVA ZA PREDAJU RADA. Znanstveni radovi trebaju biti napisani na engleskom ili njemačkom jeziku, a stručni na hrvatskom, engleskom ili njemačkom. Rad treba biti neobjavljen.

Dva primjerka rukopisa sa širokim marginama i dvostrukim proredom treba poslati na adresu jedne od urednica:

Sonja Gorjanc	Jelena Beban - Brkić
sgorjanc@grad.hr	jbeban@geof.hr
Građevinski fakultet	Geodetski fakultet

Kačićeva 26, 10000 Zagreb

Prva stranica treba sadržavati naslov rada, podatke o autoru i koautorima, sažetak na hrvatskom i engleskom, ključne riječi i MSC broj.

ELEKTRONIČKI FORMATI. Prihvaćene radove autori dostavljaju elektronskom poštom kao ASCII datoteke (preporučuje se LaTeX format) na adresu: sgorjanc@grad.hr

POSEBNI OTISCI. Autori dobivaju 20 otisaka svog rada koji se šalju prvom koautoru, ukoliko nije dukčije dogovoreno.

How to get KoG?

The easiest way to get your copy of KoG is by contacting the editor's office:

Nikoleta Sudeta, nsudeta@arhitekt.hr
Faculty of Architecture
Kačićeva 26, 10 000 Zagreb, Croatia
Tel: (+385 1) 4561 219, Fax: (+385 1) 4828 079

The price of the issue is €15 + mailing expenses €5 for European countries and €10 for other parts of the world.

The amount is payable to:

ACCOUNT NAME:	Hrvatsko društvo za konstruktivnu geometriju i kompjutorsku grafiku Kačićeva 26, 10000 Zagreb, Croatia
ACCOUNT NO.:	2421809001
BANK NAME:	Zagrebačka banka
SWIFT-CODE:	ZABA HR 2X

Kako nabaviti KoG?

KoG je najjednostavnije nabaviti u uredništvu časopisa:

Nikoleta Sudeta, nsudeta@arhitekt.hr
Arhitektonski fakultet
Kačićeva 26, 10 000 Zagreb
Tel: (01) 4561 219, Fax: (01) 4828 079

Za Hrvatsku je cijena primjerka 100 KN + 10 KN za poštarinu.

Nakon uplate za:
HDKGIKG (za KoG), Kačićeva 26, 10000 Zagreb
žiro račun broj **2360000-1101517436**
poslati ćemo časopis na Vašu adresu.

Ako Vas zanima tematika časopisa i rad našega društva, preporučamo Vam da postanete članom HDKGIKG (godišnja članarina iznosi 100 KN). Za članove društva časopis je besplatan.

Ove akademske godine tiskana su dva nova sveučilišna udžbenika koji su znatno doprinijeli osvježenju nastave geometrije na tehničkim fakultetima u Hrvatskoj. Autorice su članice Hrvatskog društva za konstruktivnu geometriju i kompjutorsku grafiku (HDKGIKG).

dr. sc. VLASTA SZIROVICZA
mr. sc. EMA JURKIN

Deskriptivna Geometrija

Nakladnici: HDKGIKG i Građevinski fakultet Sveučilišta u Zagrebu

Deskriptivna geometrija je prvi digitalni udžbenik iz područja geometrije u Hrvatskoj. Izrađen je pomoću programa *Microsoft PowerPoint* što omogućuje njegovo korištenje pod najraširenijim operativnim sustavima (*Windows 2000, Windows XP*).

Primjena tako razvijenog prezentacijskog programa omogućila je autoricama izradu vrlo kvalitetnog edukacijskog materijala. Korištenjem *Slide Show* tehnike korisniku je omogućen jasan pregled konstruktivnog postupka (s mogućnošću vraćanja na prethodne korake), a pomoću *linkova* sadržaj se stalno smisleno povezuje. Vrlo spretno korištenje nove tehnologije i dobro odabrani primjeri znatno olakšavaju razumijevanje i usvajanje složenih prostornih odnosa kao i ravninskih konstrukcija.

Knjiga je namijenjena prvenstveno studentima, ali ju mogu koristiti i učenici srednjih škola. Ima 12 poglavlja. Prva su tri uvodna i obrađuju neke ravninske konstrukcije i geometrijske transformacije. Mongeova metoda projiciranja na dvije i više ravnina, s riješenim primjerima, obrađena je u sljedeća četiri poglavlja. Zatim slijede aksonometrijske metode, presjeci, prodori, kotirana projekcija te njena primjena metodom slojnica na topografskim plohama. Na kraju udžbenika nalazimo animaciju iz nagrađenog studentskog rada.

mr. sc. PAULA KURILJ
mr. sc. NIKOLETA SUDETA
mr. sc. MARIJA ŠIMIĆ

Perspektiva

Nakladnik: Golden marketing – Tehnička knjiga, Zagreb

Perspektiva je popunila prazninu u ovom području geometrije budući da je posljednje izdanje knjige istog naziva, autora Vilka Ničea, tiskano davne 1978. godine. Udžbenik je namijenjen u prvom redu studentima arhitekture i dizajna za predmete *Nacrtna geometrija i perspektiva* te *Geometrija u graditeljstvu*, ali i svima onima koje - poznavajući osnove nacrtne geometrije - zanima njezin sadržaj i primjena. Knjiga može poslužiti i likovnim umjetnicima.
Vizualno jasne konstrukcije uz dovoljno dug, ali ne preopsežni popratni tekst, dojmljiv prijelom, kvalitetni tisak u tvrdom uvezu te lijepi grafički prilozi, među kojima se posebno ističu crteži Branke Kaminski (jednom od crteža časopis Architectural Review dodijelio je drugo mjesto u izboru za arhitektonski crtež XX stoljeća), karakteristike su ove knjige.

Sadržaj knjige podijeljen je u šest poglavlja. Prvo poglavlje sadrži osnovne pojmove centralnog projiciranja. U drugom su poglavlju predstavljene dvije osnovne metode konstrukcije perspektivnih slika s horizontalnom osi pogleda. Konstrukcija kružnih lukova predstavljena je u trećem poglavlju. Preostala tri poglavlja - perspektiva s nagnutom osi pogleda, osnove fotogrametrije, konstrukcija sjena i zrcalne slike - specifična su i pogodna za izučavanje u izbornim kolegijima studija.

ISSN 1331-1611

9 771331 161005