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Pascal-Brianchon Sets in Pappian Projective Planes

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ABSTRACT

It is well-known that Pascal and Brianchon theorems characterize conics in a Pappian projective plane. But, using these theorems and their modifications we shall show that the notion of a conic (or better a Pascal-Brianchon set) can be defined without any use of theory of projectivities or of polarities as usually.

Key words: conic, Pascal set, Pascal-Brianchon set

MSC 2000: 51A30, 51E15

Pascal-Brianschonovi skupovi u Pappusovim projektivnim ravninama

SAŽETAK

Poznato je da Pascalov i Brianchonov teorem karakteriziraju kivulje 2. reda u Pappusovoj projektivnoj ravnini. Međutim, koristeći te teoreme i njihove modifikacije pokazat ćemo da se pojam krivulje 2. reda (ili bolje: pojam Pascal-Brianchonovog skupa) može definirati bez pomoći projektiviteta ili teorije polariteta, kao što se to obično radi.

Ključne riječi: konika, Pascalov skup, Pascal-Brianchonov skup

1 Introduction

We shall operate in a Pappian projective plane of order at least 5 and characteristic other then 2.

A *simple 6-point* $A_1A_2A_3A_4A_5A_6$ is a set of six points A_1, A_2, A_3, A_4, A_5, A_6 taken in this cyclic order in which any two consecutive points and any other point are non-collinear. We say that this 6-point is a *Pascalian 6-point* and we write $P(A_1,A_2,A_3,A_4,A_5,A_6)$ if $A_1A_2 \cap A_4A_5$, $A_2A_3 \cap A_5A_6$ and $A_3A_4 \cap A_6A_1$ are collinear points.

The Pappus theorem can be stated in the following form:

If A_1, A_3, A_5 resp. A_2, A_4, A_6 are collinear points then $P(A_1,A_2,A_3,A_4,A_5,A_6)$.

Now, we can prove (see [2]):

Theorem 1.1
$P(A_1,A_2,A_3,A_4,A_5,A_6) \implies P(A_{i_1},A_{i_2},A_{i_3},A_{i_4},A_{i_5},A_{i_6})$, where $(i_1,i_2,i_3,i_4,i_5,i_6)$ is any permutation of $\{1,2,3,4,5,6\}$.

It is well-known that Pappus theorem implies the Desargues theorem. More precisely Pappus theorem resp. Desargues theorem is equivalent to the statement of Theorem 1.1 for $(i_1,i_2,i_3,i_4,i_5,i_6) = (1,2,3,4,6,5)$ resp. $(i_1,i_2,i_3,i_4,i_5,i_6) = (1,2,3,6,5,4)$ (see [1], [2]).

By the following definitions we shall generalize the notion of a simple 6-point. Let $\mathtt{I}$ be the relation of incidence.

A *one-fold specialized simple 6-point* $A_1a_1A_1A_2A_3A_4A_5$ is a set of five points A_1, A_2, A_3, A_4, A_5 taken in this cyclic order in which any three points are non-collinear, and of a line a_1 such that $A_i\mathtt{I}a_1$ iff $i = 1$. We say that this 6-point is a *Pascalian one-fold specialized 6-point* and we write $P(A_1a_1A_1,A_2,A_3,A_4,A_5)$ if $a_1 \cap A_3A_4$, $A_1A_2 \cap A_4A_5$, $A_2A_3 \cap A_5A_1$ are collinear points.

A *two-fold specialized simple 6-point* $A_1a_1A_1A_2a_2A_2A_3A_4$ *of type 1* is a set of four points A_1, A_2, A_3, A_4 taken in this cyclic order in which any three points are non-collinear, and of two lines a_1, a_2 such that $A_i\mathtt{I}a_j$ iff $i = j$. We say that this 6-point is a *Pascalian two-fold specialized 6-point of type 1* and we write $P(A_1a_1A_1,A_2a_2A_2,A_3,A_4)$ if $a_1 \cap A_2A_3$, $A_1A_2 \cap A_3A_4$, $a_2 \cap A_4A_1$ are collinear points.

A *two-fold specialized simple 6-point* $A_1a_1A_1A_2A_3a_3A_3A_4$ *of type 2* is a set of four points A_1, A_2, A_3, A_4 taken in this cyclic order in which any three points are non-collinear, and of two lines a_1, a_3 such that $A_i \mathtt{I} a_j$ iff $i = j$. We say that this 6-point is a *Pascalian two-fold specialized 6-point of type 2* and we write $P(A_1a_1A_1, A_2, A_3a_3A_3, A_4)$ if $a_1 \cap a_3$, $A_1A_2 \cap A_3A_4$, $A_2A_3 \cap A_4A_1$ are collinear points.

A *three-fold specialized simple 6-point* $A_1a_1A_1A_2a_2A_2A_3a_3A_3$ is a set of three non-collinear points A_1, A_2, A_3 and of three non-concurrent lines a_1, a_2, a_3 such that $A_i \mathtt{I} a_j$ iff $i = j$.We say that this 6-point is a *Pascalian three-fold specialized 6-point* and we write $P(A_1a_1A_1, A_2a_2A_2, A_3a_3A_3)$ if $a_1 \cap A_2A_3$, $A_1A_2 \cap a_3$, $a_2 \cap A_3A_1$ are collinear points.

Now, we can prove some theorems about Pascalian 6-points.

Theorem 1.2
$P(A_1a_1A_1, A_2, A_3, A_4, A_5) \Longrightarrow P(A_1a_1A_1, A_4, A_3, A_2, A_5)$

Proof. Let $a_1 \cap A_3A_4 = U$, $A_1A_2 \cap A_4A_5 = V$, $A_2A_3 \cap A_5A_1 = W$ be collinear points (Fig. 1). We must prove that the points $a_1 \cap A_3A_2 = U'$, $A_1A_4 \cap A_2A_5 = V'$, $A_4A_3 \cap A_5A_1 = W'$ are collinear. Consider two triangles with the vertices U, A_1, A_4 resp. W, A_2, A_5. As the lines UW, A_1A_2, A_4A_5 pass through the point V, so by Desargues theorem the points $A_1A_4 \cap A_2A_5 = V'$, $A_4U \cap A_5W = W'$, $UA_1 \cap WA_2 = U'$ are collinear.

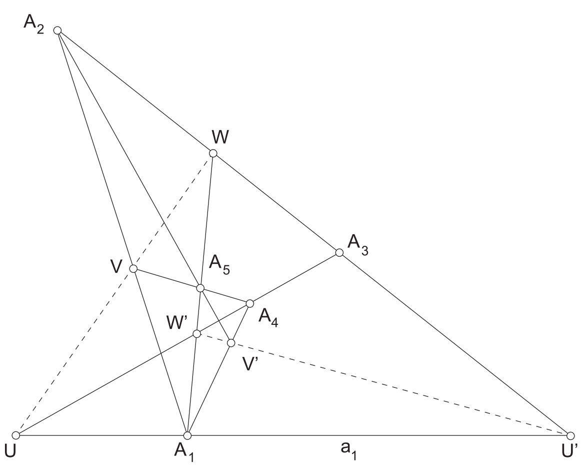

Figure 1

Theorem 1.3
$P(A_1a_1A_1, A_2, A_3, A_4, A_5) \Longrightarrow P(A_1a_1A_1, A_2, A_4, A_3, A_5)$

Proof. We must prove that the collinearity of points $a_1 \cap A_3A_4 = U$, $A_1A_2 \cap A_4A_5 = V$, $A_2A_3 \cap A_5A_1 = W$ implies the collinearity of points $a_1 \cap A_4A_3 = U$, $A_1A_2 \cap A_3A_5 = V'$, $A_2A_4 \cap A_5A_1 = W'$ (Fig. 2). By Pappus theorem we have $P(A_2, A_4, A_3, A_5, W, V)$, i.e. $A_2A_4 \cap A_5W = W'$, $A_4A_3 \cap WV = U$, $A_3A_5 \cap VA_2 = V'$ are collinear points.

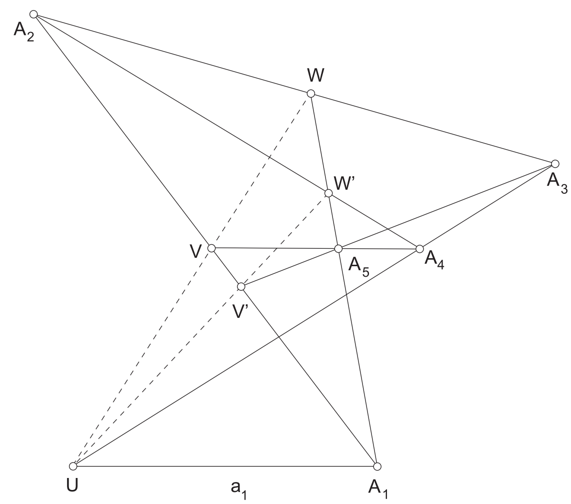

Figure 2

Theorem 1.4
$P(A_1a_1A_1, A_2, A_3, A_4, A_5) \Longrightarrow P(A_1a_1A_1, A_2, A_3, A_5, A_4)$

Proof. $P(A_1a_1A_1, A_2, A_3, A_4, A_5)$ implies by Theorem 1.2 $P(A_1a_1A_1, A_4, A_3, A_2, A_5)$, i.e. $P(A_1a_1A_1, A_5, A_2, A_3, A_4)$. But, Theorem 1.3 implies then $P(A_1a_1A_1, A_5, A_3, A_2, A_4)$ and finally Theorem 1.2 implies $P(A_1a_1A_1, A_2, A_3, A_5, A_4)$.

Obviously, Theorems 1.2, 1.3 and 1.4 imply:

Theorem 1.5
$P(A_1a_1A_1, A_2, A_3, A_4, A_5) \Longrightarrow P(A_1a_1A_1, A_{i_2}, A_{i_3}, A_{i_4}, A_{i_5})$, *where* (i_2, i_3, i_4, i_5) *is any permutation of* $\{2, 3, 4, 5\}$

Further, we have:

Theorem 1.6
$P(A_1a_1A_1, A_2, A_3a_3A_3, A_4) \Longleftrightarrow P(A_1a_1A_1, A_3a_3A_3, A_2, A_4)$.

Proof. We must prove that $a_1 \cap a_3 = U$, $A_1A_2 \cap A_3A_4 = V$, $A_2A_3 \cap A_4A_1 = W$ are collinear points iff $a_1 \cap A_3A_2 = U'$, $A_1A_3 \cap A_2A_4 = V'$, $a_3 \cap A_4A_1 = W'$ are collinear points (Fig. 3). If the points U, V, W are collinear, then the lines A_3A_4, UW, A_1A_2 pass through the point V and according to Desargues theorem the points $UA_1 \cap WA_2 = U'$, $A_1A_3 \cap A_2A_4 = V'$, $A_3U \cap A_4W = W'$ are collinear. Conversely, if U', V', W' are collinear points, then the lines A_2A_4, $U'W'$, A_1A_3 pass through the point V' and Desargues theorem implies the collinearity of points $U'A_1 \cap W'A_3 = U$, $A_1A_2 \cap A_3A_4 = V$, $A_2U' \cap A_4W' = W$.

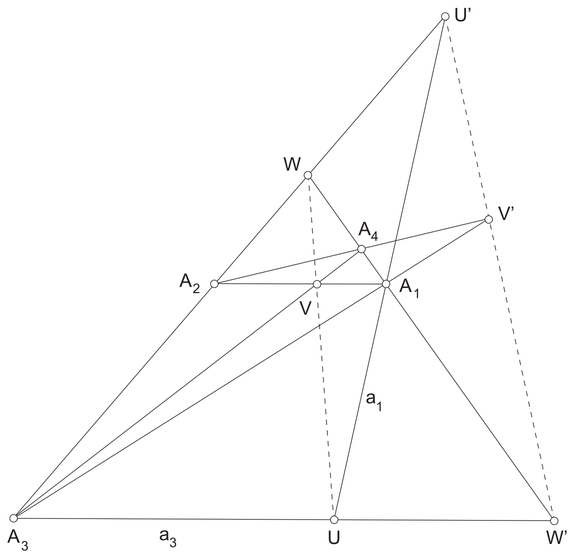

Figure 3

Theorem 1.7
$P(A_1a_1A_1,A_2a_2A_2,A_3,A_4) \Longrightarrow P(A_1a_1A_1,A_2a_2A_2,A_4,A_3)$.

Proof. According to Theorem 1.6 we have $P(A_1a_1A_1,A_3,A_2a_2A_2,A_4)$, i.e. $P(A_1a_1A_1,A_4,A_2a_2A_2,A_3)$ and then Theorem 1.6 implies $P(A_1a_1A_1,A_2a_2A_2,A_4,A_3)$.

2 Ordinary Pascal sets

Let A_1, A_2, A_3, A_4, A_5 be five points such that any three of them are non-collinear. An *ordinary Pascal set* determined by A_1, A_2, A_3, A_4, A_5 is the set of points $p(A_1,A_2,A_3,A_4,A_5) = \{A_1,A_2,A_3,A_4,A_5\} \cup \{X \mid P(A_1,A_2,A_3,A_4,A_5,X)\}$.

In virtue of Theorem 1.1 we have $p(A_1,A_2,A_3,A_4,A_5) = p(A_{i_1},A_{i_2},A_{i_3},A_{i_4},A_{i_5})$, where (i_1,i_2,i_3,i_4,i_5) is any permutation of $\{1,2,3,4,5\}$.

Now, we have a theorem proved in [2].

Theorem 2.1
$p(A_1,A_2,A_3,A_4,A_5) = p(A_{1'},A_{2'},A_{3'},A_{4'},A_{5'})$ for any different points $A_{1'},A_{2'},A_{3'},A_{4'},A_{5'} \in p(A_1,A_2,A_3,A_4,A_5)$, i.e. an ordinary Pascal set is uniquely determined by any five different of its points.

Theorem 2.1 and the definition of ordinary Pascal set imply that any three different points of an ordinary Pascal set are non-collinear.

A line a_1 such that $P(A_1a_1A_1,A_2,A_3,A_4,A_5)$ holds is said to be a *tangent of the ordinary Pascal set $p(A_1,A_2,A_3,A_4,A_5)$ at its point A_1*. According to Theorem 1.5 a_1 is a tangent of $p(A_1,A_{i_2},A_{i_3},A_{i_4},A_{i_5})$ at the point A_1, where (i_2,i_3,i_4,i_5) is any permutation of $\{2,3,4,5\}$.

Let us prove:

Theorem 2.2
There is one and only one tangent of $p(A_1,A_2,A_3,A_4,A_5)$ at the point A_1.

Proof. Let $V = A_1A_2 \cap A_4A_5$, $W = A_2A_3 \cap A_5A_1$. A line a_1 is a tangent of $p(A_1,A_2,A_3,A_4,A_5)$ at the point A_1 iff $P(A_1a_1A_1,A_2,A_3,A_4,A_5)$ holds, i.e. iff $A_1 \mathtt{I} a_1$ and iff the points $U = a_1 \cap A_3A_4$, V, W are collinear, i.e. iff $a_1 = A_1U$, where $U = A_3A_4 \cap VW$ (Fig. 1).

Theorem 2.3
Let $A_{5'} \in p(A_1,A_2,A_3,A_4,A_5) \setminus \{A_1,A_2,A_3,A_4\}$. A line a_1 is the tangent of $p(A_1,A_2,A_3,A_4,A_5)$ at the point A_1 iff a_1 is the tangent of $p(A_1,A_2,A_3,A_4,A_{5'})$ at the point A_1.

Proof. The statment is trivial if $A_{5'} = A_5$.
Let further $A_{5'} \neq A_5$. In virtue of Theorem 1.1 $A_{5'} \in p(A_1,A_2,A_3,A_4,A_5) \setminus \{A_1,A_2,A_3,A_4\}$ implies $A_5 \in p(A_1,A_2,A_3,A_4,A_{5'}) \setminus \{A_1,A_2,A_3,A_4\}$ and we have $P(A_1,A_2,A_{5'},A_4,A_5,A_3)$, i.e. the points $A_1A_2 \cap A_4A_5 = U$, $A_2A_{5'} \cap A_5A_3 = V$, $A_{5'}A_4 \cap A_3A_1 = W$ are collinear (Fig. 4).

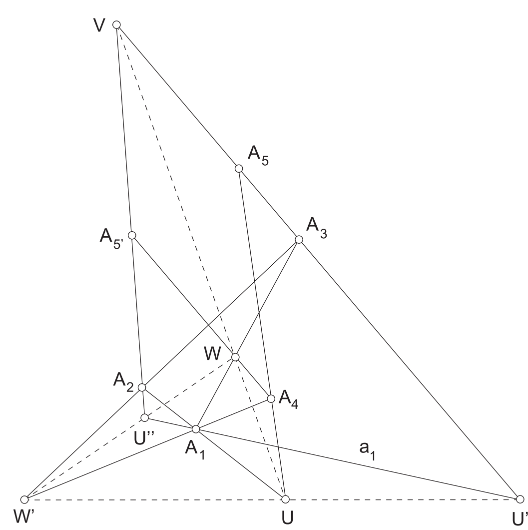

Figure 4

Let a_1 be the tangent of $p(A_1,A_2,A_3,A_4,A_5)$ at the point A_1, i.e. let $P(A_1a_1A_1,A_2,A_3,A_4,A_5)$ holds. Then by Theorem 1.4 we have $P(A_1a_1A_1,A_2,A_3,A_5,A_4)$, i.e. $a_1\cap A_3A_5=U'$, $A_1A_2\cap A_5A_4=U$, $A_2A_3\cap A_4A_1=W'$ are collinear points. By Pappus theorem we have $P(U',A_1,A_3,A_2,V,U)$, i.e. $U'A_1\cap A_2V=U''$, $A_1A_3\cap VU=W$, $A_3A_2\cap UU'=W'$ are collinear points. But, this means that $a_1\cap A_2A_{5'}=U''$, $A_1A_3\cap A_{5'}A_4=W$, $A_3A_2\cap A_4A_1=W'$ are collinear points, i.e. we have $P(A_1a_1A_1,A_3,A_2,A_{5'},A_4)$, wherefrom by Theorem 1.5 $P(A_1a_1A_1,A_2,A_3,A_4,A_{5'})$ follows, i.e. a_1 is the tangent of $p(A_1,A_2,A_3,A_4,A_{5'})$ at the point A_1. The proof of the converse follows by the substitution $A_5\leftrightarrow A_{5'}$.

On the basis of Theorem 2.3 we can prove:

Theorem 2.4

Let $A_{2'},A_{3'},A_{4'},A_{5'}\in p(A_1,A_2,A_3,A_4,A_5)\setminus\{A_1\}$ be four different points. A line a_1 is the tangent of $p(A_1,A_2,A_3,A_4,A_5)$ at the point A_1 iff a_1 is the tangent of $p(A_1,A_{2'},A_{3'},A_{4'},A_{5'})$ at the point A_1, i.e. the tangent of an ordinary Pascal set at anyone of its points is uniquely determined.

Proof. By Theorem 2.1 $p(A_1,A_2,A_3,A_4,A_5)=P(A_1,A_{2'},A_{3'},A_{4'},A_{5'})$. At least one of the points $A_{2'}$, $A_{3'}$, $A_{4'}$, $A_{5'}$ is different from the points A_2, A_3, A_4. Let be e.g. $A_{5'}\neq A_2,A_3,A_4$. From $A_{5'}\in p(A_1,A_2,A_3,A_4,A_5)\setminus\{A_1,A_2,A_3,A_4\}$ by Theorem 2.1 $p(A_1,A_2,A_3,A_4,A_5)=p(A_1,A_{5'},A_2,A_3,A_4)$ follows and by Theorem 2.3 a_1 is the tangent of $p(A_1,A_2,A_3,A_4,A_5)$ at the point A_1 iff a_1 is the tangent of $p(A_1,A_{5'},A_2,A_3,A_4)$ at the point A_1. At least one of the points $A_{2'}$, $A_{3'}$, $A_{4'}$ is different from the points A_2, A_3. Let be e.g. $A_{4'}\neq A_2,A_3$. From $A_{4'}\in p(A_1,A_{5'},A_2,A_3,A_4)\setminus\{A_1,A_{5'},A_2,A_3\}$ by Theorem 2.1 $p(A_1,A_{5'},A_2,A_3,A_4)=p(A_1,A_{4'},A_{5'},A_2,A_3)$ follows and by Theorem 2.3 a_1 is the tangent of $p(A_1,A_{5'},A_2,A_3,A_4)$ at the point A_1 iff a_1 is the tangent of $p(A_1,A_{4'},A_{5'},A_2,A_3)$ at the point A_1. At least one of the points $A_{2'}$, $A_{3'}$ is different from the point A_2. Let be e.g. $A_{3'}\neq A_2$. From $A_{3'}\in p(A_1,A_{4'},A_{5'},A_2,A_3)\setminus\{A_1,A_{4'},A_{5'},A_2\}$ by Theorem 2.1 $p(A_1,A_{4'},A_{5'},A_2,A_3)=p(A_1,A_{3'},A_{4'},A_{5'},A_2)$ follows and by Theorem 2.3 a_1 is the tangent of $p(A_1,A_{4'},A_{5'},A_2,A_3)$ at the point A_1 iff a_1 is the tangent of $p(A_1,A_{3'},A_{4'},A_{5'},A_2)$ at the point A_1. Finally, from $A_{2'}\in p(A_1,A_{3'},A_{4'},A_{5'},A_2)\setminus\{A_1,A_{3'},A_{4'},A_{5'}\}$ by Theorem 2.3 follows that a_1 is the tangent of $p(A_1,A_{3'},A_{4'},A_{5'},A_2)$ at the point A_1 iff a_1 is the tangent of $p(A_1,A_{2'},A_{3'},A_{4'},A_{5'})$ at the point A_1.

If a is the tangent of an ordinary Pascal set p at its point A, then we say that AaA is a *flag* of p.

Theorem 2.5

If $A_1a_1A_1$ is a flag of an ordinary Pascal set p, then A_1 is the unique point such that $A_1\in p$ and $A_1\mathtt{I}a_1$.

Proof. Suppose that there is a point A_2 such that $A_2\neq A_1$; $A_2\mathtt{I}a_1$ and $A_2\in p$. But p contains at least five different points and there are three different points $A_3,A_4,A_5\in p\setminus\{A_1,A_2\}$. Then we have $P(A_1a_1A_1,A_2,A_3,A_4,A_5)$ which contradicts with $A_2\mathtt{I}a_1$.

3 One-fold specialized Pascal sets

Let A_1, A_2, A_3, A_4 be four points such that any three of them are non-collinear and let a_1 be a line such that $A_i\mathtt{I}a_1$ iff $i=1$. An *one-fold specialized Pascal set* determined by the flag $A_1a_1A_1$ and the points A_2, A_3, A_4 is the set of points $p(A_1a_1A_1,A_2,A_3,A_4)=\{A_1,A_2,A_3,A_4\}\cup\{X\mid P(A_1a_1A_1,A_2,A_3,A_4,X)\}$.

According to Theorem 1.5 we have $p(A_1a_1A_1,A_2,A_3,A_4)=p(A_1a_1A_1,A_{i_2},A_{i_3},A_{i_4})$, where (i_2,i_3,i_4) is any permutation of $\{2,3,4\}$.

Theorem 3.1

$p(A_1a_1A_1,A_2,A_3,A_4)=p(A_1a_1A_1,A_2,A_3,A_{4'})$ for any point $A_{4'}\in p(A_1a_1A_1,A_2,A_3,A_4)\setminus\{A_1,A_2,A_3\}$.

Proof. If $A_{4'}=A_4$, the statement is trivial. Let be further $A_{4'}\neq A_4$. As $A_{4'}\in p(A_1a_1A_1,A_2,A_3,A_4)\setminus\{A_1,A_2,A_3,A_4\}$, so we have $P(A_1a_1A_1,A_2,A_3,A_4,A_{4'})$, wherefrom by Theorem 1.4 $P(A_1a_1A_1,A_2,A_3,A_{4'},A_4)$ follows, i.e. $A_4\in p(A_1a_1A_1,A_2,A_3,A_{4'})\setminus\{A_1,A_2,A_3,A_{4'}\}$ holds. Let $X\in p(A_1a_1A_1,A_2,A_3,A_4)\setminus\{A_1,A_2,A_3,A_4\}$, i.e, let $P(A_1a_1A_1,A_2,A_3,A_4,X)$ holds, and let $X\neq A_{4'}$. It is necessary to prove $X\in p(A_1a_1A_1,A_2,A_3,A_{4'})\setminus\{A_1,A_2,A_3,A_{4'}\}$, i.e. $P(A_1a_1A_1,A_2,A_3,A_{4'},X)$. Therefore, because of Theorem 1.5 we must prove that $P(A_1a_1A_1,A_2,A_4,A_3,A_{4'})$, $P(A_1a_1A_1,A_2,A_4,A_3,X)$ and $A_{4'}\neq X$ imply $P(A_1a_1A_1,A_2,X,A_3,A_{4'})$. But, the first two hypotheses mean that $a_1\cap A_4A_3=U$, $A_1A_2\cap A_3A_{4'}=V$, $A_2A_4\cap A_{4'}A_1=W$ resp. $a_1\cap A_4A_3=U$, $A_1A_2\cap A_3X=V'$, $A_2A_4\cap XA_1=W'$ are collinear points (Fig. 5). Consider two triangles with the vertices W,A_1, U resp. A_2, X, V'. As the lines WA_2, A_1X, UV' pass through the point W' so by Desargues theorem $A_1U\cap XV'=U''$, $UW\cap V'A_2=V$, $WA_1\cap A_2X=W''$ are collinear points. But, $U''=a_1\cap XA_3$, $V=A_1A_2\cap A_3A_{4'}$, $W''=A_2X\cap A_{4'}A_1$ and we have $P(A_1a_1A_1,A_2,X,A_3,A_{4'})$. On the same manner (by the substitution $A_4\leftrightarrow A_{4'}$) we can prove that $X\in p(A_1a_1A_1,A_2,A_3,A_{4'})\setminus\{A_1,A_2,A_3,A_{4'}\}$ and $X\neq A_4$ imply $X\in p(A_1a_1A_1,A_2,A_3,A_4)\setminus\{A_1,A_2,A_3,A_4\}$.

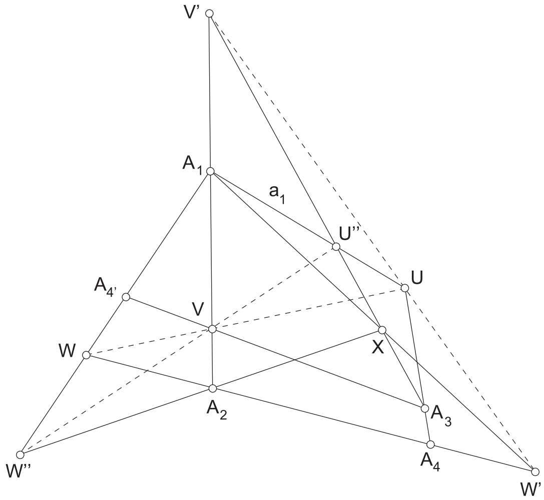

Figure 5

Theorem 3.2
$p(A_1a_1A_1,A_2,A_3,A_4) = p(A_1a_1A_1,A_{2'},A_{3'},A_{4'})$ for any different points $A_{2'},A_{3'},A_{4'} \in p(A_1a_1A_1,A_2,A_3,A_4)\setminus\{A_1\}$, i.e, an one-fold specialized Pascal set is uniquely determined by its flag $A_1a_1A_1$ and any three of its points, which are mutually different and different from A_1.

Proof. At least one of the points $A_{2'}$, $A_{3'}$, $A_{4'}$ is different from the points A_2, A_3. Let be e.g. $A_{4'} \neq A_2,A_3$. From $A_{4'} \in p(A_1a_1A_1,A_2,A_3,A_4)\setminus\{A_1,A_2,A_3\}$ by Theorem 3.1 $p(A_1a_1A_1,A_2,A_3,A_4) = p(A_1a_1A_1,A_{4'},A_2,A_3)$ follows. At least one of the points $A_{2'}$, $A_{3'}$ is different from the point A_2. Let be e.g. $A_{3'} \neq A_2$. From $A_{3'} \in p(A_1a_1A_1,A_{4'},A_2,A_3)\setminus\{A_1,A_{4'},A_2\}$ by Theorem 3.1 $p(A_1a_1A_1,A_{4'},A_2,A_3) = p(A_1a_1A_1,A_{3'},A_{4'},A_2)$ follows. Finally, from $A_{2'} \in p(A_1a_1A_1,A_{3'},A_{4'},A_2)\setminus\{A_1,A_{3'},A_{4'}\}$ by Theorem 3.1 $p(A_1a_1A_1,A_{3'},A_{4'},A_2) = p(A_1a_1A_1,A_{2'},A_{3'},A_{4'})$ follows.

Theorem 3.2 and the definition of one-fold specialized Pascal set p determined by the flag AaA imply that any three different points of p are non-collinear and that $X\,\mathrm{I}\,a$ iff $X = A$ for any point $X \in p$.

A line a_2 such that $P(A_1a_1A_1,A_2a_2A_2,A_3,A_4)$ holds is said to be a *tangent of the one-fold specialized Pascal set $p(A_1a_1A_1,A_2,A_3,A_4)$ at the point A_2*. According to Theorem 1.7 then a_2 is a tangent of $p(A_1a_1A_1,A_2,A_4,A_3)$ at the point A_2. The line a_1 is said to be the tangent of $p(A_1a_1A_1,A_2,A_3,A_4)$ at the point A_1.

Theorem 3.3
There is one and only one tangent of $p(A_1a_1A_1,A_2,A_3,A_4)$ at the point A_2.

Proof. Let $U = a_1 \cap A_2A_3$, $V = A_1A_2 \cap A_3A_4$. A line a_2 is a tangent of $p(A_1a_1A_1,A_2,A_3,A_4)$ at the point A_2 iff $P(A_1a_1A_1,A_2a_2A_2,A_3,A_4)$ holds, i.e. iff $U,V,W = a_2 \cap A_4A_1$ are collinear points, i.e. iff $a_2 = A_2W$, where $W = A_4A_1 \cap UV$.

Theorem 3.4
Let be $A_{4'} \in p(A_1a_1A_1,A_2,A_3,A_4)\setminus\{A_1,A_2,A_3\}$. A line a_2 is the tangent of $p(A_1a_1A_1,A_2,A_3,A_4)$ at the point A_2 iff a_2 is the tangent of $p(A_1a_1A_1,A_2,A_3,A_{4'})$ at the point A_2.

Proof. The statement is trivial if $A_{4'} = A_4$. Let further $A_{4'} \neq A_4$. By Theorem 1.5 $A_{4'} \in p(A_1a_1A_1,A_2,A_3,A_4)\setminus\{A_1,A_2,A_3\}$ implies $A_4 \in p(A_1a_1A_1,A_2,A_3,A_{4'})\setminus\{A_1,A_2,A_3\}$ and we have $P(A_1a_1A_1,A_4,A_2,A_{4'},A_3)$, i.e. $a_1 \cap A_2A_{4'} = U$, $A_1A_4 \cap A_{4'}A_3 = V$, $A_4A_2 \cap A_3A_1 = W$ are collinear points (Fig. 6). Let a_2 be the tangent of $p(A_1a_1A_1,A_2,A_3,A_4)$ at the point A_2, i.e. let $P(A_1a_1A_1,A_2a_2A_2,A_3,A_4)$ hold. Then by Theorem 1.6 we have $P(A_1a_1A_1,A_3,A_2a_2A_2,A_4)$, i.e. $a_1 \cap a_2 = U'$, $A_1A_3 \cap A_2A_4 = W$, $A_3A_2 \cap A_4A_1 = W'$ are collinear points. Consider the triangles with the vertices A_2, U', A_1 resp. A_3, W, V. The lines A_2A_3, $U'W$, A_1V pass through the point W' and Desargues theorem implies that $U'A_1 \cap WV = U$, $A_1A_2 \cap VA_3 = V''$, $A_2U' \cap A_3W = W''$ are collinear points. But, $U = a_1 \cap A_2A_{4'}$, $V'' = A_1A_2 \cap A_{4'}A_3$, $W'' = a_2 \cap A_3A_1$ and we have $P(A_1a_1A_1,A_2a_2A_2,A_{4'},A_3)$, i.e. a_2 is the tangent of $p(A_1a_1A_1,A_2,A_{4'},A_3) = p(A_1a_1A_1,A_2,A_3,A_{4'})$ at the point A_2. The proof of the converse follows by the substitution $A_4 \leftrightarrow A_{4'}$.

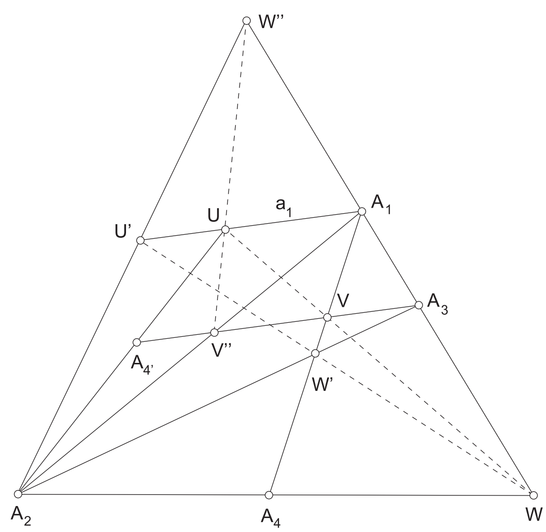

Figure 6

Theorem 3.5
Let $A_{3'}, A_{4'} \in p(A_1a_1A_1, A_2, A_3, A_4) \setminus \{A_1, A_2\}$ *be two different points. A line* a_2 *is the tangent of* $p(A_1a_1A_1, A_2, A_3, A_4)$ *at the point* A_2 *iff* a_2 *is the tangent of* $p(A_1a_1A_1, A_2, A_{3'}, A_{4'})$ *at the point* A_2.

Proof. By Theorem 3.2 $p(A_1a_1A_1, A_2, A_3, A_4) = p(A_1a_1A_1, A_2, A_{3'}, A_{4'})$. At least one of the points $A_{3'}$, $A_{4'}$ is different from the point A_3. Let be e.g. $A_{4'} \neq A_3$. From $A_{4'} \in p(A_1a_1A_1, A_2, A_3, A_4) \setminus \{A_1, A_2, A_3\}$ by Theorem 3.2 $p(A_1a_1A_1, A_2, A_3, A_4) = p(A_1a_1A_1, A_2, A_{4'}, A_3)$ follows and by Theorem 3.4 a_2 is the tangent of $p(A_1a_1A_1, A_2, A_3, A_4)$ at the point A_2 iff a_2 is the tangent of $p(A_1a_1A_1, A_2, A_{4'}, A_3)$ at the point A_2. From $A_{3'} \in p(A_1a_1A_1, A_2, A_{4'}, A_3) \setminus \{A_1, A_2, A_{4'}\}$ by Theorem 3.2 it follows that a_2 is the tangent of $p(A_1a_1A_1, A_2, A_{4'}, A_3)$ at the point A_2 iff a_2 is the tangent of $p(A_1a_1A_1, A_2, A_{4'}, A_{3'}) = p(A_1a_1A_1, A_2, A_{3'}, A_{4'})$ at the point A_2.

Theorem 3.6
If a_2 *is the tangent of* $p = p(A_1a_1A_1, A_2, A_3, A_4)$ *at the point* A_2*, then* A_2 *is the unique point such that* $A_2 \in p$ *and* $A_2 \mathtt{I} a_2$.

Proof. We have $P(A_1a_1A_1, A_2a_2A_2, A_3, A_4)$ and therefore $A_i \mathtt{I} a_2$ iff $i = 2$. Suppose that there is a point $A_5 \in p \setminus \{A_1, A_2, A_3, A_4\}$ such that $A_5 \mathtt{I} a_2$. Owing to Theorem 3.2 we have $p = p(A_1a_1A_1, A_2, A_3, A_5)$ and by Theorem 3.5 a_2 is the tangent of $p(A_1a_1A_1, A_2, A_3, A_5)$ at the point A_2. Therefore we have $P(A_1a_1A_1, A_2a_2A_2, A_3, A_5)$ which contradicts with $A_5 \mathtt{I} a_2$.

If p is an one-fold specialized Pascal set and a_2 is a tangent of p at its point A_2, then we say that $A_2a_2A_2$ is a *flag* of p.

Theorem 3.7
If $A_2a_2A_2$ *is a flag of* $p(A_1a_1A_1, A_2, A_3, A_4)$*, then* $P(A_1a_1A_1, A_2, A_3, A_4) = P(A_2a_2A_2, A_1, A_3, A_4)$.

Proof. The line a_2 is the tangent of $p(A_1a_1A_1, A_2, A_3, A_4)$ at the point A_2 and so $P(A_1a_1A_1, A_2a_2A_2, A_3, A_4)$ holds, wherefrom by Theorem 1.7 $P(A_2a_2A_2, A_1a_1A_1, A_3, A_4)$ follows, i.e. a_1 is the tangent of $p(A_2a_2A_2, A_1, A_3, A_4)$ at the point A_1, and $a_1 \cap A_2A_3 = U$, $A_1A_2 \cap A_3A_4 = V$, $a_2 \cap A_4A_1 = W$ are collinear points (Fig. 7). Let $X \in p(A_1a_1A_1, A_2, A_3, A_4) \setminus \{A_1, A_2, A_3, A_4\}$, i.e. let $P(A_1a_1A_1, A_2, A_3, A_4, X)$ holds. We must prove $X \in p(A_2a_2A_2, A_1, A_3, A_4) \setminus \{A_1, A_2, A_3, A_4\}$, i.e. $P(A_2a_2A_2, A_1, A_3, A_4, X)$. According to Theorem 1.5 we have $P(A_1a_1A_1, A_2, X, A_3, A_4)$, i.e. $a_1 \cap XA_3 = U'$, $A_1A_2 \cap A_3A_4 = V$, $A_2X \cap A_4A_1 = W'$ are collinear points. The lines A_3X, VW' , UA_1 pass through the point U' and Desargues theorem implies the collinearity of the points $VU \cap W'A_1 = W$, $UA_3 \cap A_1X = V''$, $A_3V \cap XW' = W''$. But, we have $W = a_2 \cap A_4A_1$, $V'' = A_2A_3 \cap A_1X$, $W'' = A_3A_4 \cap XA_2$, i.e. $P(A_2a_2A_2, A_3, A_4, A_1, X)$, and Theorem 1.5 implies $P(A_2a_2A_2, A_1, A_3, A_4, X)$. On the same manner (by the substitutions $A_1 \leftrightarrow A_2$, $a_1 \leftrightarrow a_2$) it can be proved that $X \in p(A_2a_2A_2, A_1, A_3, A_4) \setminus \{A_1, A_2, A_3, A_4\}$ implies $X \in p(A_1a_1A_1, A_2, A_3, A_4) \setminus \{A_1, A_2, A_3, A_4\}$.

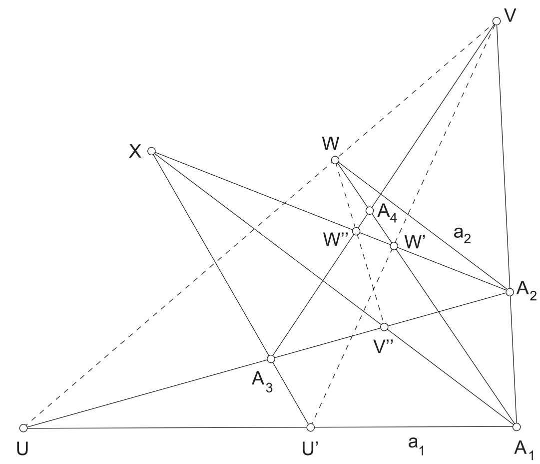

Figure 7

Theorem 3.8
Let $A_2a_2A_2$ *be a flag of* $p(A_1a_1A_1, A_2, A_3, A_4)$*. A line* a_3 *is the tangent of* $p(A_1a_1A_1, A_2, A_3, A_4)$ *at the point* A_3 *iff* a_3 *is the tangent of* $p(A_2a_2A_2, A_1, A_3, A_4)$ *at the point* A_3.

Proof. As in the proof of Theorem 3.7 we conclude that a_1 is the tangent of $p(A_2a_2A_2, A_1, A_3, A_4)$ at the point A_1. We have $P(A_1a_1A_1, A_2a_2A_2, A_3, A_4)$ i.e. by Theorem 1.6 $P(A_1a_1A_1, A_3, A_2a_2A_2, A_4)$, and $a_1 \cap a_2 = U$, $A_1A_3 \cap A_2A_4 = V$, $A_3A_2 \cap A_4A_1 = W$ are collinear points (Fig. 8). Let a_3 be the tangent of $p(A_1a_1A_1, A_2, A_3, A_4) = p(A_1a_1A_1, A_3, A_2, A_4)$ at the point A_3. Then $P(A_1a_1A_1, A_3a_3A_3, A_2, A_4)$ holds, i.e. $a_1 \cap A_3A_2 = U'$, $A_1A_3 \cap A_2A_4 = V$, $a_3 \cap A_4A_1 = W'$ are collinear points. The lines $W'V$, A_3A_2, A_1U pass through the point U' and Desargues theorem implies that $A_3A_1 \cap A_2U = U''$, $A_1W' \cap UV = W$, $W'A_3 \cap VA_2 = W''$ are collinear points. But, $U'' = a_2 \cap A_3A_1$, $W = A_2A_3 \cap A_1A_4$, $W'' = a_3 \cap A_4A_2$ and we have $P(A_2a_2A_2, A_3a_3A_3, A_1, A_4)$, i.e. a_3 is the tangent of $p(A_2a_2A_2, A_3, A_1, A_4) = p(A_2a_2A_2, A_1, A_3, A_4)$ at the point A_3. The proof of the converse follows by the substitutions $A_1 \leftrightarrow A_2$, $a_1 \leftrightarrow a_2$.

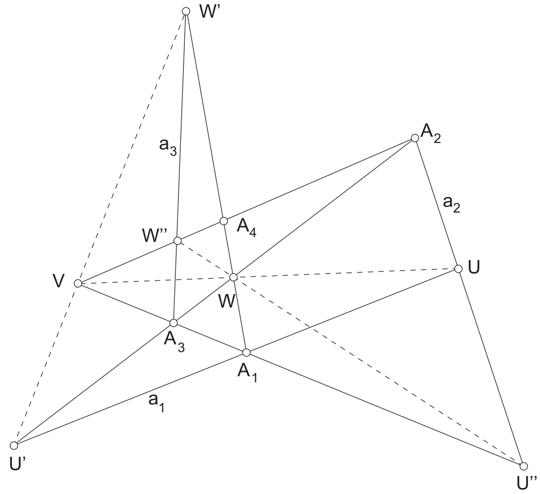

Figure 8

Theorem 3.9
If $A_{1'}$, $A_{2'}$, $A_{3'}$, $A_{4'} \in p = p(A_1a_1A_1,A_2,A_3,A_4)$ are four different points and if $a_{1'}$ is a tangent of p at the point $A_{1'}$ then $p = p(A_{1'}a_{1'}A_{1'},A_{2'},A_{3'},A_{4'})$, i.e. an one-fold specialized Pascal set is uniquely determined by anyone of its flags AaA and any three of its points which are mutually different and different from the point A.

Proof. If $A_{1'} = A_1$ then we use Theorem 3.2. Let be further $A_{1'} \neq A_1$. At most one of the points $A_{2'}$, $A_{3'}$, $A_{4'}$ is equal to A_1. Let be e.g. $A_1 \neq A_{2'}.A_{3'}$. Then Theorem 3.2 implies $p = p(A_1a_1A_1,A_{1'},A_{2'},A_{3'})$. By Theorem 3.5 $a_{1'}$ is the tangent of $p(A_1a_1A_1,A_{1'},A_{2'},A_{3'})$ at the point $A_{1'}$. Therefore, Theorem 3.7 implies $p(A_1a_1A_1,A_{1'},A_{2'},A_{3'}) = p(A_{1'}a_{1'}A_{1'},A_1,A_{2'},A_{3'})$. So we have $A_{4'} \in p(A_{1'}a_{1'}A_{1'},A_1,A_{2'},A_{3'})$ and finally Theorem 3.2 implies $p(A_{1'}a_{1'}A_{1'},A_1,A_{2'},A_{3'}) = p(A_{1'}a_{1'}A_{1'},A_{2'},A_{3'},A_{4'})$.

Theorem 3.10
Let $A_{1'}$, $A_{2'}$, $A_{3'} \in p = p(A_1a_1A_1,A_2,A_3,A_4)$ be different points such that $A_{1'}$, $A_{2'}$, $A_{3'} \neq A_4$ and let $a_{1'}$ be the tangent of p at the point $A_{1'}$. A line a_4 is the tangent of p at the point A_4 iff a_4 is the tangent of $p(A_{1'}a_{1'}A_{1'},A_{2'},A_{3'},A_4)$ at the point A_4, i.e. the tangent of an one-fold specialized Pascal set at anyone of its points is uniquely determined.

Proof. If $A_{1'} = A_1$, then we use Theorem 3.5. Let be further $A_{1'} \neq A_1$. At most one of the points $A_{2'}$, $A_{3'}$ is equal to A_1. Let be e.g. $A_1 \neq A_{2'}$. By Theorem 3.5 it follows that a_4 is the tangent of p at the point A_4 iff a_4 is the tangent of $p(A_1a_1A_1,A_{1'},A_{2'},A_4)$ at the point A_4. If we apply this fact to the point $A_{1'}$ instead of the point A_4, then it follows that $a_{1'}$ is the tangent of $p(A_1a_1A_1,A_{1'},A_{2'},A_4)$ at the point $A_{1'}$. Therefore, Theorem 3.8 implies that a_4 is the tangent of $p(A_1a_1A_1,A_{1'},A_{2'},A_4)$ at the point A_4 iff a_4 is the tangent of $p(A_{1'}a_{1'}A_{1'},A_1,A_{2'},A_4)$ at the point A_4. But, $A_{3'}$ $p(A_{1'}a_{1'}A_{1'},A_1,A_{2'},A_4)$ and Theorem 3.5 implies that a_4 is the tangent of $p(A_{1'}a_{1'}A_{1'},A_1,A_{2'},A_4)$ in the point A_4 iff a_4 is the tangent of $p(A_{1'}a_{1'}A_{1'},A_{2'},A_{3'},A_4)$ at the point A_4.

4 Two-fold specialized Pascal sets

Let A_1, A_2, A_3 be three non-collinear points and a_1, a_2 two lines such that $A_i \mathtt{I} a_j$ iff $i = j$. A *two-fold specialized Pascal set* determined by the flags $A_1a_1A_1$, $A_2a_2A_2$ and the point A_3 is the set of points $p(A_1a_1A_1,A_2a_2A_2,A_3) = \{A_1,A_2,A_3\} \cup \{X \mid P(A_1a_1A_1,A_2a_2A_2,A_3,X)\}$.

Theorem 4.1
$p(A_1a_1A_1,A_2a_2A_2,A_3) = p(A_1a_1A_1,A_2a_2A_2,A_{3'})$ for any point $A_{3'} \in p(A_1a_1A_1,A_2a_2A_2,A_3) \setminus \{A_1,A_2\}$.

Proof. If $A_{3'} = A_3$, the statement is trivial.
Let be further $A_{3'} \neq A_3$. As $A_{3'} \in P(A_1a_1A_1,A_2a_2A_2,A_3) \setminus \{A_1,A_2,A_3\}$, so we have $P(A_1a_1A_1,A_2a_2A_2,A_3,A_{3'})$, wherefrom by Theorem 1.7 $P(A_1a_1A_1,A_2a_2A_2,A_{3'},A_3)$ follows, i.e. $A_3 \in p(A_1a_1A_1,A_2a_2A_2,A_{3'}) \setminus \{A_1,A_2,A_{3'}\}$. Let now be $X \in p(A_1a_1A_1,A_2a_2A_2,A_3) \setminus \{A_1,A_2,A_{3'}\}$, i.e. let we have $P(A_1a_1A_1,A_2a_2A_2,A_3,X)$, and let $X \neq A_{3'}$. We must prove $X \in p(A_1a_1A_1,A_2a_2A_2,A_{3'}) \setminus \{A_1,A_2,A_{3'}\}$, i.e. $P(A_1a_1A_1,A_2a_2A_2,A_{3'},X)$. Therefore, because of Theorem 1.6, we must prove that $P(A_1a_1A_1,A_3,A_2a_2A_2,A_{3'}),P(A_1a_1A_1,A_3,A_2a_2A_2,X)$ and $A_{3'} \neq X$ imply $P(A_1a_1A_1,A_{3'},A_2a_2A_2,X)$. But, the first two hypotheses mean that $a_1 \cap a_2 = U$, $A_1A_3 \cap A_2A_{3'} = V$, $A_3A_2 \cap A_{3'}A_1 = W$ resp. $a_1 \cap a_2 = U$, $A_1A_3 \cap A_2X = V'$, $A_3A_2 \cap XA_1 = W'$ are collinear points (Fig. 9).

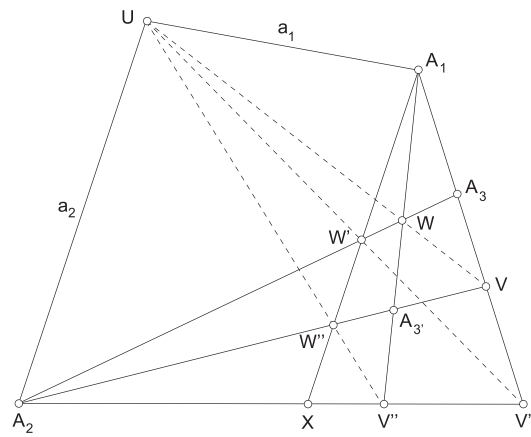

Figure 9

By Pappus theorem we have $P(V,W,A_1,W',V',A_2)$, i.e. $VW \cap W'V' = U$, $WA_1 \cap V'A_2 = V''$, $A_1W' \cap A_2V = W''$ are collinear points. But, $U = a_1 \cap a_2$, $V'' = A_1A_{3'} \cap A_2X$, $W'' = A_{3'}A_2 \cap XA_1$, and we have $P(A_1a_1A_1, A_{3'}, A_2a_2A_2, X)$. On the same manner (by the substitution $A_3 \leftrightarrow A_{3'}$) it can be proved that $X \in p(A_1a_1A_1, A_2a_2A_2, A_{3'}) \setminus \{A_1, A_2, A_{3'}\}$ and $X \neq A_3$ imply $X \in p(A_1a_1A_1, A_2a_2A_2, A_3) \setminus \{A_1, A_2, A_3\}$.

Theorem 4.1 and the definition of two-fold specialized Pascal set p determined by flags $A_1a_1A_1$ and $A_2a_2A_2$ imply that any point of $p \setminus \{A_1, A_2\}$ is not-incident with the lines a_1, a_2, A_1A_2.

A line a_3 such that $p(A_1a_1A_1, A_2a_2A_2, A_3a_3A_3)$ holds is said to be a *tangent of the two-fold specialized Pascal set* $p(A_1a_1A_1, A_2a_2A_2, A_3)$ *at the point* A_3. The lines a_1 and a_2 are said to be the tangents of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ at the points A_1 and A_2, respectively.

Theorem 4.2
There is one and only one tangent of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ *at the point* A_3.

Proof. Let $U = a_1 \cap A_2A_3$, $W = a_2 \cap A_3A_1$. A line a_3 is a tangent of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ at the point A_3 iff $P(A_1a_1A_1, A_2a_2A_2, A_3a_3A_3)$ holds, i.e. iff $A_3 \mathtt{I} a_3$ and iff $U, V = A_1A_2 \cap a_3, W$ are collinear points, i.e. iff $a_3 = A_3V$, where $V = A_1A_2 \cap UW$.

Theorem 4.3
If a_3 *is the tangent of* $p = p(A_1a_1A_1, A_2a_2A_2, A_3)$ *at the point* A_3*, then* A_3 *is the unique point such that* $A_3 \in p$ *and* $A_3 \mathtt{I} a_3$.

Proof. We have $P(A_1a_1A_1, A_2a_2A_2, A_3a_3A_3)$ and therefore $A_i \mathtt{I} a_3$ iff $i = 3$. The points $a_1 \cap A_2A_3 = U$, $A_1A_2 \cap a_3 = V$, $a_2 \cap A_3A_1 = W$ are collinear. Suppose that there is a point $A_4 \in p \setminus \{A_1, A_2, A_3\}$ such that $A_4 \mathtt{I} a_3$. Then we have $P(A_1a_1A_1, A_2a_2A_2, A_3, A_4)$, i.e. $a_1 \cap A_2A_3 = U$, $A_1A_2 \cap A_3A_4 = A_1A_2 \cap a_3 = V$, $a_2 \cap A_4A_1 = W'$ are collinear points. Therefore we have $W' \mathtt{I} UV$ and $W' = a_2 \cap UV = W$ i.e. finally $A_4 = a_3 \cap A_1W' = a_3 \cap A_1W = A_3$, contrary to the hypothesis.

If p is a two-fold specialized Pascal set and a_3 a tangent of p at its point A_3, then we say that $A_3a_3A_3$ is a *flag* of p.

Theorem 4.4
If $A_3a_3A_3$ *is a flag of* $p(A_1a_1A_1, A_2a_2A_2, A_3)$*, then* $p(A_1a_1A_1, A_2a_2A_2, A_3) = p(A_1a_1A_1, A_3a_3A_3, A_2)$.

Proof. The line a_3 is the tangent of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ at the point A_3 and so $P(A_1a_1A_1, A_2a_2A_2, A_3a_3A_3)$, i.e. $P(A_1a_1A_1, A_3a_3A_3, A_2a_2A_2)$ holds, and a_2 is the tangent of $p(A_1a_1A_1, A_3a_3A_3, A_2)$ at the point A_2. Moreover, we have collinear points $a_1 \cap A_2A_3 = U$, $A_1A_2 \cap a_3 = V$, $a_2 \cap A_3A_1 = W$ (Fig. 10). Let $X \in p(A_1a_1A_1, A_2a_2A_2, A_3) \setminus \{A_1, A_2, A_3\}$, i.e. let $P(A_1a_1A_1, A_2a_2A_2, A_3, X)$ hold. Then $a_1 \cap A_2A_3 = U$, $A_1A_2 \cap A_3X = V'$, $a_2 \cap XA_1 = W'$ are collinear points. The lines WA_2, UV', A_1X pass through the point W' and by Desargues theorem $UA_1 \cap V'X = U''$, $A_1W \cap XA_2 = V''$, $WU \cap A_2V' = V$ are collinear points. But, $U'' = a_1 \cap A_3X$, $V'' = A_1A_3 \cap XA_2$, $V = a_3 \cap A_2A_1$ and we have $P(A_1a_1A_1, A_3a_3A_3, X, A_2)$, i.e. $P(A_1a_1A_1, A_3a_3A_3, A_2, X)$ because of Theorem 1.7. Hence $X \in p(A_1a_1A_1, A_3a_3A_3, A_2) \setminus \{A_1, A_2, A_3\}$. On the same manner (by the substitutions $A_2 \leftrightarrow A_3$, $a_2 \leftrightarrow a_3$) we can prove that $X \in p(A_1a_1A_1, A_3a_3A_3, A_2) \setminus \{A_1, A_2, A_3\}$ implies $X \in p(A_1a_1A_1, A_2a_2A_2, A_3) \setminus \{A_1, A_2, A_3\}$.

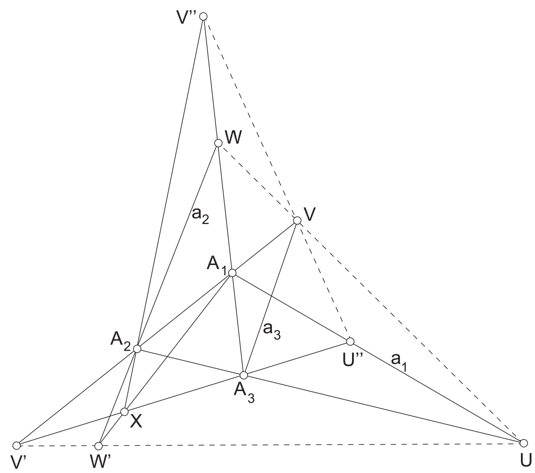

Figure 10

Theorem 4.5
Let $A_4 \in p(A_1a_1A_1, A_2a_2A_2, A_3) \setminus \{A_1, A_2, A_3\}$*. A line* a_3 *is the tangent of* $p(A_1a_1A_1, A_2a_2A_2, A_3)$ *at the point* A_3 *iff* a_3 *is the tangent of* $p(A_1a_1A_1, A_2, A_3, A_4)$ *at the point* A_3.

Proof. By Theorem 1.7 we have $P(A_1a_1A_1, A_2a_2A_2, A_4, A_3)$, i.e. $a_1 \cap A_2A_4 = U$, $A_1A_2 \cap A_4A_3 = V$, $a_2 \cap A_3A_1 = W$ are collinear points (Fig. 11). We must prove that $P(A_1a_1A_1, A_3a_3A_3, A_4, A_2)$ is equivalent to $P(A_1a_1A_1, A_2a_2A_2, A_3a_3A_3)$. If $a_1 \cap A_3A_4 = U'$, $A_1A_3 \cap A_4A_2 = V'$, $a_3 \cap A_2A_1 = W'$ are collinear points, then Pappus theorem implies $P(A_3, A_2, V, U, U', V')$, i.e. $A_3A_2 \cap UU' = U''$, $A_2V \cap U'V' = W'$, $VU \cap V'A_3 = W$ are collinear points. But, $U'' = a_1 \cap A_2A_3$, $W' = A_1A_2 \cap a_3$, $W = a_2 \cap A_3A_1$. Conversely, if $a_1 \cap A_2A_3 = U''$, $A_1A_2 \cap a_3 = W'$,

$a_2 \cap A_3A_1 = W$ are collinear points, then Pappus theorem implies $P(U'', U, A_2, V, A_3, W)$, i.e. $U''U \cap VA_3 = U'$, $UA_2 \cap A_3W = V'$, $A_2V \cap WU'' = W'$ are collinear points. But, $U' = a_1 \cap A_3A_4$, $V' = A_1A_3 \cap A_4A_2$, $W' = a_3 \cap A_2A_1$.

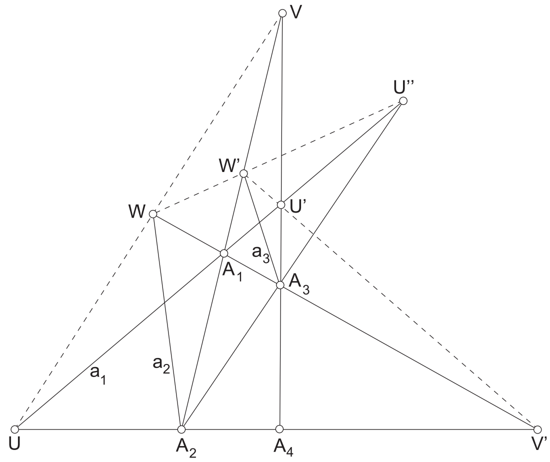

Figure 11

Theorem 4.6
Let $A_3a_3A_3$ be a flag and A_4 a point of $p(A_1a_1A_1, A_2a_2A_2, A_3)$. A line a_4 is a tangent of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ at the point A_4 iff a_4 is a tangent of $p(A_1a_1A_1, A_3a_3A_3, A_2)$ at the point A_4.

Proof. The statement is obvious if $A_4 \in \{A_1, A_2, A_3\}$. Let be further $A_4 \neq A_1, A_2, A_3$. We have $A_4 \in p(A_1a_1A_1, A_2a_2A_2, A_3) \setminus \{A_1, A_2, A_3\}$ and Theorem 1.7 implies $A_3 \in p(A_1a_1A_1, A_2a_2A_2, A_4) \setminus \{A_1, A_2, A_4\}$. Let us suppose that a_4 is a tangent of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ at the point A_4. Then, by the definition, a_4 is the tangent of $p(A_1a_1A_1, A_2a_2A_2, A_4)$ at the point A_4. Therefore, Theorem 4.5 implies (by the substitutions $A_3 \leftrightarrow A_4$, $a_3 \leftrightarrow a_4$) that a_4 is the tangent of $p(A_1a_1A_1, A_2, A_4, A_3) = p(A_1a_1A_1, A_2, A_3, A_4)$ at the point A_4. But $A_3a_3A_3$ is a flag of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ and Theorem 4.4 implies $p(A_1a_1A_1, A_2a_2A_2, A_3) = p(A_1a_1A_1, A_3a_3A_3, A_2)$. So we have $A_4 \in p(A_1a_1A_1, A_3a_3A_3, A_2)$ and by Theorem 1.7 we obtain $P(A_1a_1A_1, A_3a_3A_3, A_4, A_2)$, i.e. $A_2 \in p(A_1a_1A_1, A_3a_3A_3, A_4) \setminus \{A_1, A_3, A_4\}$. Moreover, a_4 is the tangent of $p(A_1a_1A_1, A_2, A_3, A_4) = p(A_1a_1A_1, A_3, A_4, A_2)$ at the point A_4 and Theorem 4.5 implies (by the substitutions $A_2 \to A_3$, $A_3 \to A_4$, $A_4 \to A_2$, $a_3 \to a_4$) that a_4 is the tangent of $p(A_1a_1A_1, A_3a_3A_3, A_4)$ at the point A_4. Then, by the definition, a_4 is a tangent of $p(A_1a_1A_1, A_3a_3A_3, A_2)$ at the point A_4. As $A_3a_3A_3$ is a flag of $p(A_1a_1A_1, A_2a_2A_2, A_3)$, so $A_2a_2A_2$ is a flag of $p(A_1a_1A_1, A_3a_3A_3, A_2)$. Moreover, $A_4 \in p(A_1a_1A_1, A_2a_2A_2, A_3)$ implies $A_4 \in p(A_1a_1A_1, A_3a_3A_3, A_2)$ because of Theorem 4.4. Now, if we suppose that a_4 is a tangent of $p(A_1a_1A_1, A_3a_3A_3, A_2)$ at the point A_4, then on the same way as in the first part of this proof (by the substitutions $A_2 \leftrightarrow A_3$, $a_2 \leftrightarrow a_3$) it follows that a_4 is a tangent of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ at the point A_4.

Theorem 4.7
If $A_{1'}$, $A_{2'}$, $A_{3'} \in p(A_1a_1A_1, A_2a_2A_2, A_3)$ are different points and $a_{1'}$, $a_{2'}$ are two tangents of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ at the points $A_{1'}$, $A_{2'}$, respectively, then $p(A_1a_1A_1, A_2a_2A_2, A_3) = p(A_{1'}a_{1'}A_{1'}, A_{2'}a_{2'}A_{2'}, A_{3'})$ i.e. a two-fold specialized Pascal set is uniquely determined by any two of its flags AaA, BaB and anyone of its points different from A, B.

Proof. At least one of the points $A_{1'}$, $A_{2'}$ is different from A_1. Let be e.g. $A_{2'} \neq A_1$. At first let be $A_{2'} \neq A_2$. Then $A_{2'} \in p(A_1a_1A_1, A_2a_2A_2, A_3) \setminus \{A_1, A_2\}$ implies by Theorem 4.1 $p(A_1a_1A_1, A_2a_2A_2, A_3) = p(A_1a_1A_1, A_2a_2A_2, A_{2'})$. As $a_{2'}$ is a tangent of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ at the point $A_{2'}$ so $a_{2'}$ is the tangent of $p(A_1a_1A_1, A_2a_2A_2, A_{2'})$ at this point. Therefore, Theorem 4.4 implies $p(A_1a_1A_1, A_2a_2A_2, A_{2'}) = p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_2)$. At least one of the points A_1, A_2 is different from $A_{3'}$. Let be e.g. $A_1 \neq A_{3'}$. Then $A_{3'} \in p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_2) \setminus \{A_1, A_{2'}\}$ implies $p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_2) = p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_{3'})$. Therefore, if we have $A_{2'} \neq A_2$, then $p(A_1a_1A_1, A_2a_2A_2, A_3) = p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_{3'})$ holds. As $a_{1'}$ is a tangent of $p(A_1a_1A_1, A_2a_2A_2, A_3)$ at the point $A_{1'}$, then $a_{1'}$ is a tangent of $p(A_1a_1A_1, A_2a_2A_2, A_{2'})$ at this point. By Theorem 4.6 $a_{1'}$ is a tangent of $p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_2)$ at the point $A_{1'}$, i.e. a tangent of $p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_{3'})$ at this point. If we have $A_{2'} = A_2$ and then necessarily $A_{2'} \neq A_{3'}$ then obviously $p(A_1a_1A_1, A_2a_2A_2, A_3) = p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_3)$ and we conclude again that $p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_3) = p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_{3'})$ and that $a_{1'}$ is a tangent of $p(A_1a_1A_1, A_{2'}a_{2'}A_{2'}, A_{3'})$ at the point $A_{1'}$. Therefore, in every case we have $p(A_1a_1A_1, A_2a_2A_2, A_3) = p(A_{2'}a_{2'}A_{2'}, A_1a_1A_1, A_{3'})$ and so $A_{1'} \in p(A_{2'}a_{2'}A_{2'}, A_1a_1A_1, A_{3'})$. Moreover, $a_{1'}$ is a tangent of $p(A_{2'}a_{2'}A_{2'}, A_1a_1A_1, A_{3'})$ at the point $A_{1'}$. Now, let $A_{1'} \neq A_1$ at first. From $A_{1'} \in p(A_{2'}a_{2'}A_{2'}, A_1a_1A_1, A_{3'}) \setminus \{A_{2'}, A_1\}$ we obtain $p(A_{2'}a_{2'}A_{2'}, A_1a_1A_1, A_{3'}) = p(A_{2'}a_{2'}A_{2'}, A_1a_1A_1, A_{1'})$ by Theorem 4.1. As $a_{1'}$ is a tangent of $p(A_{2'}a_{2'}A_{2'}, A_1a_1A_1, A_{3'})$ at the point $A_{1'}$ so $a_{1'}$ is the tangent of $p(A_{2'}a_{2'}A_{2'}, A_1a_1A_1, A_{1'})$ at the same point $A_{1'}$. Therefore, Theorem 4.4 implies $p(A_{2'}a_{2'}A_{2'}, A_1a_1A_1, A_{1'}) = p(A_{2'}a_{2'}A_{2'}, A_{1'}a_{1'}A_{1'}, A_1)$. From $A_{3'} \in p(A_1a_1A_1, A_2a_2A_2, A_3) \setminus \{A_{1'}, A_{2'}\} = p(A_{1'}a_{1'}A_{1'}, A_{2'}a_{2'}A_{2'}, A_1) \setminus \{A_{1'}, A_{2'}\}$ we obtain finally

$p(A_{1'}a_{1'}A_{1'},A_{2'}a_{2'}A_{2'},A_1) = p(A_{1'}a_{1'}A_{1'},A_{2'}a_{2'}A_{2'},A_{3'})$ by Theorem 4.1. If we have $A_{1'} = A_1$, then $p(A_1a_1a_1,A_{2'}a_{2'}A_{2'},A_{3'}) = p(A_{1'}a_{1'}A_{1'},A_{2'}a_{2'}A_{2'},A_{3'})$ obviously holds.

Theorem 4.8
Let $A_{1'}a_{1'}A_{1'}$, $A_{2'}a_{2'}A_{2'}$ be two different flags of $p(A_1a_1A_1,A_2a_2A_2,A_3)$ and let $A_{1'},A_{2'} \neq A_3$. A line a_3 is the tangent of $p(A_1a_1A_1,A_2a_2A_2,A_3)$ at the point A_3 iff a_3 is a tangent of $p(A_{1'}a_{1'}A_{1'},A_{2'}a_{2'}A_{2'},A_{3'})$ at this point, i.e. the tangent of a two-fold specialized Pascal set in anyone of its points is uniquely determined.

Proof. At least one of the points $A_{1'}$, $A_{2'}$ is different from A_1. Let be e.g. $A_{2'} \neq A_1$. At first, let $A_{2'} \neq A_2$. According to proof of Theorem 4.7 we have $p(A_1a_1A_1,A_2a_2A_2,A_3) = p(A_1a_1A_1,A_2a_2A_2,A_{2'}) = p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_2)$. Then $A_3 \in p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_2) \setminus \{A_1,A_{2'}\}$ and Theorem 4.1 implies $p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_2) = p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_3)$. Moreover, $A_{2'}a_{2'}A_{2'}$ is a flag of $p(A_1a_1A_1,A_2a_2A_2,A_3)$ and therefore a flag of $p(A_1a_1A_1,A_2a_2A_2,A_{2'})$. So, Theorem 4.6 implies that a_3 is a tangent of $p(A_1a_1A_1,A_2a_2A_2,A_{2'})$ at the point A_3 iff a_3 is a tangent of $p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_2)$ at this point. Moreover, we conclude that a_3 is the tangent of $p(A_1a_1A_1,A_2a_2A_2,A_3)$ at the point A_3 iff a_3 is a tangent of $p(A_1a_1A_1,A_2a_2A_2,A_{2'})$ at this point and that a_3 is a tangent of $p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_2)$ at the point A_3 iff a_3 is the tangent of $p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_3)$ at this point. Therefore, it follows finally in the case $A_{2'} \neq A_2$ that a_3 is the tangent of $p(A_1a_1A_1,A_2a_2A_2,A_3)$ at the point A_3 iff a_3 is the tangent of $p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_3)$ at this point. In the case $A_{2'} = A_2$ this statement is trivial. Therefore, we have the conclusion: if $A_{2'}a_{2'}A_{2'}$ is a flag of $p(A_1a_1A_1,A_2a_2A_2,A_3)$ and $A_{2'} \neq A_1,A_3$ then $p(A_1a_1A_1,A_2a_2A_2,A_3) = p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_3)$ and a_3 is the tangent of $p(A_1a_1A_1,A_2a_2A_2,A_3)$ at the point A_3 iff a_3 is the tangent of $p(A_1a_1A_1,A_{2'}a_{2'}A_{2'},A_3)$, i.e. of $p(A_{2'}a_{2'}A_{2'},A_1a_1A_1,A_3)$ at this point. So, we have now a flag $A_{1'}a_{1'}A_{1'}$ of $p(A_1a_1A_1,A_2a_2A_2,A_3) = p(A_{2'}a_{2'}A_{2'},A_1a_1A_1,A_3)$ and $A_{1'} \neq A_{2'},A_3$ and on the same manner (by the substitutions $A_1 \to A_{2'}$, $A_2 \to A_1$, $A_{2'} \to A_{1'}$, $a_1 \to a_{2'}$, $a_2 \to a_1$, $a_{2'} \to a_{1'}$) we conclude that $p(A_{2'}a_{2'}A_{2'},A_1a_1A_1,A_3) = p(A_{2'}a_{2'}A_{2'},A_{1'}a_{1'}A_{1'},A_3)$ and that a_3 is the tangent of $p(A_{2'}a_{2'}A_{2'},A_1a_1A_1,A_3)$ at the point A_3 iff a_3 is the tangent of $p(A_{2'}a_{2'}A_{2'},A_{1'}a_{1'}A_{1'},A_3)$, i.e. of $p(A_{1'}a_{1'}A_{1'},A_{2'}a_{2'}A_{2'},A_3)$ at the point A_3.

5 Pascal sets

Now, we shall investigate the mutual relationships between different types of Pascal sets.

Theorem 5.1
a) If $A_1a_1A_1$ is a flag of $p(A_1,A_2,A_3,A_4,A_5)$, then $p(A_1,A_2,A_3,A_4,A_5) = p(A_1a_1A_1,A_2,A_3,A_4)$.

b) If $A_5 \in p(A_1a_1A_1,A_2,A_3,A_4) \setminus \{A_1,A_2,A_3,A_4\}$, then $p(A_1a_1A_1,A_2,A_3,A_4) = p(A_1,A_2,A_3,A_4,A_5)$.

Proof. The hypothesis of a) resp. b) is that $P(A_1a_1A_1,A_2,A_3,A_4,A_5)$ holds, wherefrom by Theorem 1.4 $P(A_1a_1A_1,A_2,A_3,A_5,A_4)$ follows, i.e, the points $a_1 \cap A_3A_5 = U'$, $A_1A_2 \cap A_5A_4 = U$, $A_2A_3 \cap A_4A_1 = W'$ are collinear. We must show that $X \in p(A_1,A_2,A_3,A_4,A_5)$ iff $X \in p(A_1a_1A_1,A_2,A_3,A_4)$. This is obvious if $X \in \{A_1,A_2,A_3,A_4,A_5\}$. Let be further $X \neq A_1,A_2,A_3,A_4,A_5$. We must show that $P(A_1,A_2,A_3,A_4,A_5,X)$ implies $P(A_1a_1A_1,A_2,A_3,A_4,X)$ and conversely that $P(A_1a_1A_1,A_2,A_3,A_4,X)$ implies $P(A_1,A_2,A_3,A_4,A_5,X)$. The first statement was proved in fact in the proof of Theorem 2.3 (instead of $A_{5'}$ it must be taken X). Let us prove the second statement. $P(A_1a_1A_1,A_2,A_3,A_4,X)$ implies by Theorem 1.5 $P(A_1a_1A_1,A_3,A_2,X,A_4)$, i.e. $a_1 \cap A_2X = U''$, $A_1A_3 \cap XA_4 = W$, $A_3A_2 \cap A_4A_1 = W'$ are collinear points (Fig. 4) with X instead of $A_{5'}$). By Pappus theorem we have $P(A_1,A_2,U'',W',U',A_3)$, i.e. $A_1A_2 \cap W'U' = U$, $A_2U'' \cap U'A_3 = V$, $U''W' \cap A_3A_1 = W$ are collinear points. But, $U = A_1A_2 \cap A_4A_5$, $V = A_2X \cap A_5A_3$, $W = XA_4 \cap A_3A_1$ and so $P(A_1,A_2,X,A_4,A_5,A_3)$ holds and Theorem 1.1 implies $P(A_1,A_2,A_3,A_4,A_5,X)$.

Theorem 5.2
If A_1, A_2, A_3, A_4 are four different points of an ordinary Pascal set p and a_1 the tangent of p at the point A_1, then p is equal to the one-fold specialized Pascal set $p(A_1a_1A_1,A_2,A_3,A_4)$. Conversely, if A_1, A_2, A_3, A_4, A_5 are five different points of an one-fold specialized Pascal set p, then p is equal to the ordinary Pascal set $p(A_1,A_2,A_3,A_4,A_5)$.

Proof. Let p be an ordinary Pascal set, A_1, A_2, A_3, $A_4 \in p$ four different points and a_1 the tangent of p at the point A_1. There is a point $A_5 \in p \setminus \{A_1,A_2,A_3,A_4\}$ and by Theorem 2.1 we have $p = p(A_1,A_2,A_3,A_4,A_5)$. By Theorem 2.4 a_1 is the tangent of $p(A_1,A_2,A_3,A_4,A_5)$ at the point A_1. So Theorem 5.1 implies $p(A_1,A_2,A_3,A_4,A_5) = p(A_1a_1A_1,A_2,A_3,A_4)$. Conversely, let p be an one-fold specialized Pascal set and $A_1,A_2,A_3,A_4,A_5 \in p$ five different points. By Theorem 3.10 there is the tangent

a_1 of p at the point A_1 and according to Theorem 3.9 we have $p = p(A_1a_1A_1, A_2, A_3, A_4)$. As we have $A_5 \in p(A_1a_1A_1, A_2, A_3, A_4) \setminus \{A_1, A_2, A_3, A_4\}$, so Theorem 5.1 implies $p(A_1a_1A_1, A_2, A_3, A_4) = p(A_1, A_2, A_3, A_4, A_5)$.

Theorem 5.3

a) Let $A_1a_1A_1$ be a flag of $p(A_1, A_2, A_3, A_4, A_5)$.

b) Let $A_5 \in p(A_1a_1A_1, A_2, A_3, A_4) \setminus \{A_1, A_2, A_3, A_4\}$.

In both cases a line a_2 is the tangent of $p(A_1, A_2, A_3, A_4, A_5)$ at the point A_2 iff it is tangent of $p(A_1a_1A_1, A_2, A_3, A_4)$ at the same point.

Proof. The hypothesis of a) resp. b) is that $P(A_1a_1A_1, A_2, A_3, A_4, A_5)$ holds, wherefrom by Theorem 1.5 $p(A_1a_1A_1, A_2, A_5, A_3, A_4)$ follows, i.e. $a_1 \cap A_5A_3 = U$, $A_1A_2 \cap A_3A_4 = V$, $A_2A_5 \cap A_4A_1 = W$ are collinear points (Fig. 12). We must show that $P(A_2a_2A_2, A_1, A_3, A_4, A_5)$ holds iff $P(A_1a_1A_1, A_2a_2A_2, A_3, A_4)$. The hypothesis $P(A_2a_2A_2, A_1, A_3, A_4, A_5)$ implies by Theorem 1.5 $P(A_2a_2A_2, A_4, A_1, A_3, A_5)$, i.e. $a_2 \cap A_1A_3 = W'$, $A_2A_4 \cap A_3A_5 = V'$, $A_4A_1 \cap A_5A_2 = W$ are collinear points. Using the Pappus theorem we have $P(A_1, U, W', V', A_4, A_3)$, i.e. $A_1U \cap V'A_4 = U''$, $UW \cap A_4A_3 = V$, $WV' \cap A_3A_1 = W'$ are collinear points. But, $U'' = a_1 \cap A_2A_4$, $V = A_1A_2 \cap A_4A_3$, $W' = a_2 \cap A_3A_1$ and so $P(A_1a_1A_1, A_2a_2A_2, A_4, A_3)$ holds, wherefrom by Theorem 1.7 $P(A_1a_1A_1, A_2a_2A_2, A_3, A_4)$ follows. Conversely, let $P(A_1a_1A_1, A_2a_2A_2, A_4, A_3)$, i.e. $P(A_1a_1A_1, A_2a_2A_2, A_3, A_4)$ holds. Then U'', V, W' are collinear points. The Pappus theorem implies now $P(A_1, A_3, U, V, U'', A_4)$, i.e. $A_1A_3 \cap VU'' = W'$, $A_3U \cap U''A_4 = V'$, $UV \cap A_4A_1 = W$ are collinear points. But, $W' = a_2 \cap A_1A_3$, $V' = A_2A_4 \cap A_3A_5$, $W = A_4A_1 \cap A_5A_2$ and so $P(A_2a_2A_2, A_4, A_1, A_3, A_5)$, $P(A_2a_2A_2, A_1, A_3, A_4, A_5)$ holds.

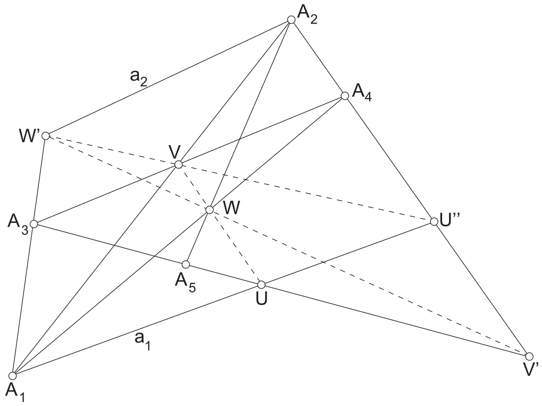

Figure 12

Theorem 5.4

Let p_1 be an ordinary Pascal set and p_2 an one-fold specialized Pascal set such that $p_1 = p_2$ and let $A_2 \in p_1 = p_2$. A line a_2 is the tangent of p_1 at the point A_2 iff it is the tangent of p_2 at this point.

Proof. Let A_1, A_3, A_4, $A_5 \in p_1 \setminus \{A_2\} = p_2 \setminus \{A_2\}$ be four different points and let a_1 be the tangent of p_2 at the point A_1. Then by Theorem 2.1 we have $p_1 = p(A_1, A_2, A_3, A_4, A_5)$ and by Theorem 3.9 $p_2 = p(A_1a_1A_1, A_2, A_3, A_4)$ holds. Moreover, by Theorems 2.4 and 3.10 it follows that p_1 and $p(A_1, A_2, A_3, A_4, A_5)$ resp. p_2 and $p(A_1a_1A_1, A_2, A_3, A_4)$ have the same tangent at the point A_2. As $A_5 \in p_2 \setminus \{A_1, A_2, A_3, A_4\} = p(A_1a_1A_1, A_2, A_3, A_4) \setminus \{A_1, A_2, A_3, A_4\}$, so by Theorem 5.3 b) it follows that a_2 is the tangent of $p(A_1, A_2, A_3, A_4, A_5)$ at the point A_2 iff a_2 is the tangent of $p(A_1a_1A_1, A_2, A_3, A_4)$ at the same point.

Theorem 5.5

a) If $A_2a_2A_2$ is a flag of $p(A_1a_1A_1, A_2, A_3, A_4)$, then $p(A_1a_1A_1, A_2, A_3, A_4) = p(A_1a_1A_1, A_2a_2A_2, A_3)$.

b) If $A_4 \in p(A_1a_1A_1, A_2a_2A_2, A_3) \setminus \{A_1, A_2, A_3\}$, then $p(A_1a_1A_1, A_2a_2A_2, A_3) = p(A_1a_1A_1, A_2, A_3, A_4)$.

Proof. The hypothesis of a) resp. b) implies $P(A_1a_1A_1, A_2a_2A_2, A_4, A_3)$ by Theorem 1.7, i.e. $a_1 \cap A_2A_4 = U$, $A_1A_2 \cap A_4A_3 = V$, $a_2 \cap A_3A_1 = W$ are oollinear points (Fig. 13).

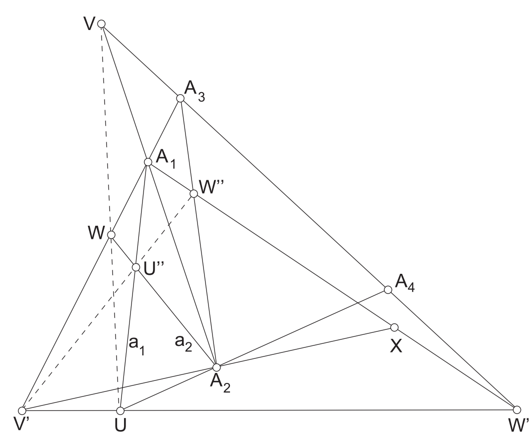

Figure 13

We must prove that $X \in p(A_1a_1A_1, A_2, A_3, A_4)$ iff $X \in p(A_1a_1A_1, A_2a_2A_2, A_3)$. The statement is obvious if $X \in \{A_1, A_2, A_3, A_4\}$. Let be now $X \neq A_1, A_2, A_3, A_4$. Because of Theorem 1.5 and 1.6 we must show that $P(A_1a_1A_1, A_3, A_4, A_2, X)$ holds iff $P(A_1a_1A_1, A_3, A_2a_2A_2, X)$ holds. If we have

$P(A_1a_1A_1,A_3,A_4,A_2,X)$, then $a_1 \cap A_4A_2 = U$, $A_1A_3 \cap A_2X = V'$, $A_3A_4 \cap XA_1 = W'$ are collinear points. As the lines $W'A_3$, A_1A_2, UW pass through the point V, so Desargues theorem implies that $A_1U \cap A_2W = U''$, $UW' \cap WA_3 = V'$, $W'A_1 \cap A_3A_2 = W''$ are collinear points. But, $U'' = a_1 \cap a_2$, $V' = A_1A_3 \cap A_2X$, $W'' = A_3A_2 \cap XA_1$ and so $P(A_1a_1A_1,A_3,A_2a_2A_2,X)$ holds. Conversely, if $P(A_1a_1A_1,A_3,A_2a_2A_2,X)$ holds, then U'', V', W'' are collinear points. As the lines $W''A_3$, A_1V, $U''W$ pass through the point A_2 so by Desargues theorem $A_1U'' \cap VW = U$, $U''W'' \cap WA_3 = V'$, $W''A_1 \cap A_3V = W'$ are collinear points. But, $U = a_1 \cap A_4A_2$, $V' = A_1A_3 \cap A_2X$, $W' = A_3A_4 \cap XA_1$ and we have $P(A_1a_1A_1,A_3,A_4,A_2,X)$.

Theorem 5.6
If A_1, A_2, A_3 are three different points of an one-fold specialized Pascal set p and a_1, a_2 are the tangents of p at the points A_1, A_2, respectively, then p is equal to the two-fold specialized Pascal set $p(A_1a_1A_1,A_2a_2A_2,A_3)$. Conversely, if A_1, A_2, A_3, A_4 are four different points of a two-fold specialized Pascal set p and a_1 the tangent of p at the point A_1, then p is equal to the one-fold specialized Pascal set $p(A_1a_1A_1,A_2,A_3,A_4)$.

Proof. Let p be an one-fold specialized Pascal set, A_1, A_2, $A_3 \in p$ three different points and a_1, a_2 the tangents of p at the points A_1, A_2, respectively. There is a point $A_4 \in p \setminus \{A_1,A_2,A_3\}$ and by Theorem 3.9 we have $p = p(A_1a_1A_1,A_2,A_3,A_4)$. According to Theorem 3.10 a_4 is the tangent of $p(A_1a_1A_1,A_2,A_3,A_4)$ at the point A_2. Therefore, Theorem 5.5 implies $p(A_1a_1A_1,A_2,A_3,A_4) = p(A_1a_1A_1,A_2a_2A_2,A_3)$. Conversely, let p be a two-fold specialized Pascal set, $A_1,A_2,A_3,A_4 \in p$ four different points and a_1 the tangent of p at the point A_1. According to Theorem 4.8 there is the tangent a_2 of p at the point A_2 and because of Theorem 4.7 we have the equality $p = p(A_1a_1A_1,A_2a_2A_2,A_3)$. As $A_4 \in p(A_1a_1A_1,A_2a_2A_2,A_3) \setminus \{A_1,A_2,A_3\}$, so Theorem 5.5 implies $p(A_1a_1A_1,A_2a_2A_2,A_3) = p(A_1a_1A_1,A_2,A_3,A_4)$.

Theorem 5.7
Let $A_2a_2A_2$ be a flag of $p(A_1a_1A_1,A_2,A_3,A_4)$. A line a_3 is the tangent of $p(A_1a_1A_1,A_2,A_3,A_4)$ at the point A_3 iff it is the tangent of $p(A_1a_1A_1,A_2a_2A_2,A_3)$ at the same point.

Proof. The hypothesis $P(A_1a_1A_1,A_2a_2A_2,A_3,A_4)$ is the same as the hypothesis of Theorem 4.5 and so the proof is the same as the proof of Theorem 4.5.

Theorem 5.8
Let p_1 be an one-fold specialized Pascal set and p_2 a two-fold specialized Pascal set such that $p_1 = p_2$ and let $A_3 \in p_1 = p_2$. A line a_3 is the tangent of p_1 at the point A_3 iff it is the tangent of p_2 at this point.

Proof. Let $A_1,A_2,A_4 \in p_1 \setminus \{A_3\} = p_2 \setminus \{A_3\}$ be three different points and let a_1, a_2 be the tangents of p_2 at the points A_1, A_2, respectively. Then by Theorem 3.9 we have $p_1 = p(A_1a_1A_1,A_2,A_3A_4)$ and by Theorem 4.7 $p_2 = p(A_1a_1A_1,A_2a_2A_2,A_3)$. Moreover, by Theorem 3.10 resp. Theorem 4.8 it follows that p_1 and $p(A_1a_1A_1,A_2,A_3,A_4)$ resp. p_2 and $p(A_1a_1A_1,A_2a_2A_2,A_3)$ have the same tangent at the point A_3. As $A_4 \in p_2 \setminus \{A_1,A_2,A_3\} = p(A_1a_1A_1,A_2a_2A_2,A_3) \setminus \{A_1,A_2,A_3\}$ so by Theorem 4.5 it follows that a_3 is the tangent of $p(A_1a_1A_1,A_2,A_3,A_4)$ at the point A_3 iff it is the tangent of $p(A_1a_1A_1,A_2a_2A_2,A_3)$ at this point.

Any ordinary Pascal set, any one-fold specialized Pascal set and any two-fold specialized Pascal set are said to be a *Pascal set*. Because of Theorems 5.1 and 5.5 any Pascal set is simultaneously an ordinary Pascal set, an one-fold specialized Pascal set, and a two-fold specialized Pascal set.

6 Pascal-Brianchon sets

A *simple 6-line* $a_1a_2a_3a_4a_5a_6$ is a set of six lines a_1, a_2, a_3, a_4, a_5, a_6 taken in this cyclic order in which any two consecutive lines and any other line are non-concurrent. We say that this 6-line is a *Brianchonian 6-line* and we write $B(a_1,a_2,a_3,a_4,a_5,a_6)$ if the lines $(a_1 \cap a_2)(a_4 \cap a_5)$, $(a_2 \cap a_3)(a_5 \cap a_6)$, $(a_3 \cap a_4)(a_6 \cap a_1)$ are concurrent.

The Pappus theorem can be stated now in the dual form:

If a_1, a_3, a_5 resp. a_2, a_4, a_6 are concurrent lines then $B(a_1,a_2,a_3,a_4,a_5,a_6)$.

Now, we shall dualize the whole above-mentioned theory. E.g. a *two-fold specialized simple 6-line* $a_1A_1a_1a_2A_2a_2a_3a_4$ *of type 1* is a set of four lines a_1, a_2, a_3, a_4 taken in this cyclic order in which any three lines are non-concurrent, and of two points A_1, A_2 such that $A_i \mathtt{I} a_j$ iff $i = j$. We say that this 6-line is a *Brianchonian two-fold specialized 6-line of type 1* and we write $B(a_1A_1a_1,a_2A_2a_2,a_3,a_4)$ if the lines $A_1(a_2 \cap a_3)$, $(a_1 \cap a_2)(a_3 \cap a_4)$, $A_2(a_4 \cap a_1)$ are concurrent. A *two-fold specialized Brianchon set* determined by two flags $a_1A_1a_1$, $a_2A_2a_2$ and a line a_3 such that $A_i \mathtt{I} a_j$ iff $i = j$ is the set of lines $b(a_1A_1a_1,a_2A_2a_2,a_3) = \{a_1,a_2,a_3\} \cup \{x \mid B(a_1A_1a_1,a_2A_2a_2,a_3,x)\}$.

A tangent of a Pascal set at one of its points has for the dual the notion of a *point of contact* of a Brianchon set with one of its lines.

Now, we can prove:

Theorem 6.1
The set of tangents of a Pascal set is a Brianchon set. Conversely, the set of points of contact of a Brianchon set is a Pascal set.

Proof. It suffices to prove only the first statement. Let p be the given Pascal set and $A_1a_1A_1$, $A_2a_2A_2$, $A_3a_3A_3$ three different flags of p. Let AaA be any flag of p. We shall prove that a is a line of the Brianchon set $B(a_1A_1a_1,a_2A_2a_2,a_3)$. The statement is trivial if $A\in\{A_1,A_2,A_3\}$, i.e. $a\in\{a_1,a_2,a_3\}$. Let be now $A\neq A_1,A_2,A_3$, i.e. $a\neq a_1,a_2,a_3$. We must show that $B(a_1A_1a_1,a_2A_2a_2,a_3,a)$ holds. By the hypothesis we have $P(A_1a_1A_1,A_2,A_3a_3a_3,A)$, $P(A_2a_2A_2,A_1,AaA,A_3)$, $P(A_1a_1A_1,A_2,AaA,A_3)$ and $P(A_2a_2A_2,A_1,A_3a_3A_3,A)$, i.e. the triples of points $a_1\cap a_3=V''$, $A_1A_2\cap A_3A=W$, $A_2A_3\cap AA_1=U$; $a_2\cap a=V'$, $A_2A_1\cap AA_3=W$, $A_1A\cap A_3A_2=U$; $a_1\cap a=U'$, $A_1A_2\cap AA_3=W$, $A_2A\cap A_3A_1=V$ and $a_2\cap a_3=U''$, $A_2A_1\cap A_3A=W$, $A_1A_3\cap AA_2=V$ are collinear (Fig. 14). Therefore, we have V'', $V'\mathtt{I}UW$, and U', $U''\mathtt{I}VW$, i.e. V', V'', W resp. U', U'', W are collinear points. As the lines A_1A_2, $(a_1\cap a_3)(a_2\cap a)=V''V'$, $(a_3\cap a_2)(a\cap a_1)=U''U'$ pass through the point W, so $B(a_1A_1a_1,a_3,a_2A_2a_2,a)$ holds, wherefrom by the dual of Theorem 1.6 $B(a_1A_1a_1,a_2A_2a_2,a_3,a)$ follows.

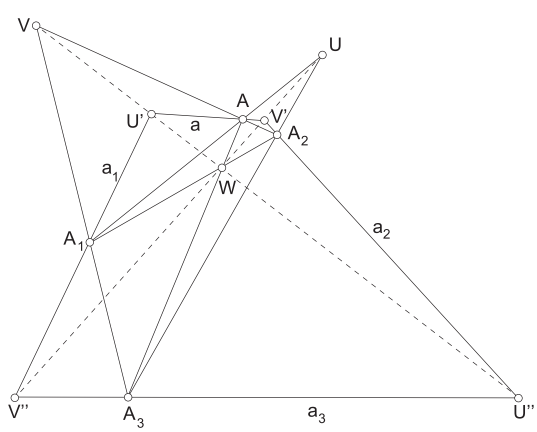

Figure 14

If p is a Pascal set and b a Brianchon set such that b is the set of tangents of p, i.e. p is the set of points of contact of b, then the ordered pair (p,b) is said to be a *Pascal-Brianchon set*. If $A\in p$ is a point and $a\in b$ a line such that AaA is a flag of p, i.e. aAa is a flag of b, then we say that (A,a) is a *flag* of (p,b).

According to Theorems 2.1, 3.9, 4.7 and their duals the following theorem follows:

Theorem 6.2
A Pascal-Brianchon set is uniquely determined by:

a) any five different of its points;

b) anyone of its flags (A,a) and any three of its points which are mutual different and different from A;

c) any two different of its flags (A_1,a_1), (A_2,a_2) and anyone of its points different from A_1 and A_2;

d) any two different of its flags (A_1,a_1), (A_2,a_2) and anyone of its lines different from a_1 and a_2;

e) anyone of its flags (A,a) and any three of its lines which are mutual different and different from a;

f) any five different of its lines.

Theorems 2.5, 3.6 and 4.3 and their duals imply:

Theorem 6.3
If (A,a) is a flag of a Pascal-Brianchon set (p,b) and $A_1\in p$, $a_1\in b$, then $A_1\mathtt{I}a$ implies $A_1=A$ and $A\mathtt{I}a_1$ implies $a_1=a$.

Let us prove the following theorem.

Theorem 6.4
Let (A_1,a_1) be a flag of a Pascal-Brianchon set (p,b). If b_1 is any line such that $A_1\mathtt{I}b_1$ and $b_1\neq a_1$, then there is one and only one point X such that $X\mathtt{I}b_1$ and $X\in p\setminus\{A_1\}$. Dually, if B_1 is any point such that $B\mathtt{I}a_1$ and $B_1\neq A_1$, then there is one and only one line x such that $B_1\mathtt{I}x$ and $x\in b\setminus\{a_1\}$.

Proof. It suffices to prove the statement for an ordinary Pascal set p, any flag $A_1a_1A_1$ of p and any line b_1 such that $A_1\mathtt{I}b_1$ and $b_1\neq a_1$. At first let us prove the existence of the required point X. Let $A_2,A_3,A_4,A_5\in p\setminus\{A_1\}$ be four different points. The statement of theorem is obvious if $A_i\mathtt{I}b_1$, for any $i\in\{2,3,4,5\}$. Let be further A_2, A_3, A_4, A_5 non-incident with b_1. Put $A_1A_2\cap A_4A_5=U$, $A_3A_4\cap b_1=W$, $A_2A_3\cap UW=V$, $b_1\cap A_5V=X$. If it were $X=A_1$, then would be $P(A_1b_1A_1,A_2,A_3,A_4,A_5)$ because of the collinearity of the points $b_1\cap A_3A_4=W$,

$A_1A_2 \cap A_4A_5 = U$, $A_2A_3 \cap A_5A_1 = V$. But, then $A_1b_1A_1$ would be a flag of p, which is in contradiction with $b_1 \neq a_1$. Therefore, we have $X \neq A_1$. The points $A_1A_2 \cap A_4A_5 = U$, $A_2A_3 \cap A_5X = V$, $A_3A_4 \cap XA_1 = W$ are collinear and we have $P(A_1, A_2, A_3, A_4, A_5, X)$, i.e. $X \in p$. Let now X' be a point such that $X' \mathtt{I} b_1$ and $X' \in p \setminus \{A_1\}$. Because of non-collinearity of any three different points of p it follows necessarily $X' = X$.

Theorem 6.4 implies that any Pascal or Brianchon set contains $n+1$ points resp. lines, where n is the order (finite or infinite) of the projective plane.

In virtue of Theorem 6.4 we can define two new notions.

Let (A, a) be a flag of a Pascal-Brianchon set (p, b). If c is any line such that $A\mathtt{I}c$ and $c \neq a$, then the point X such that $X\mathtt{I}c$ and $X \in p \setminus \{A\}$ is said to be *the second intersection* of the line c with the Pascal set p. If $c = a$, then we say that A is the second intersection of the line c with p. If C is any point, such that $C\mathtt{I}a$ and $C \neq A$, then the line x such that $C\mathtt{I}x$ and $x \in b \setminus \{a\}$ is said to be *the second tangent* from the point C onto the Brianchon set b. If $C = A$, then we say that a is the second tangent from the point C onto b.

We shall say that the simple 6-points $A_1A_2A_3A_4A_5A_6$, $A_1a_1A_1A_2A_3A_4A_5$, $A_1a_1A_1A_2a_2A_2A_3A_4$, $A_1a_1A_1A_2A_3a_3A_3A_4$, or $A_1a_1A_1A_2a_2A_2A_3a_3A_3$ are *inscribed* resp. that the simple 6-lines $a_1a_2a_3a_4a_5a_6$, $a_1A_1a_1a_2a_3a_4a_5$, $a_1A_1a_1a_2A_2a_2a_3a_4$, $a_1A_1a_1a_2a_3A_3a_3a_4$ or $a_1A_1a_1a_2A_2a_2a_3A_3a_3$ are *circumscribed* to a Pascal-Brianchon set (p, b) if $A_i \in p$ and $a_i \in b$ for $i = 1, 2, 3, 4, 5, 6$.

Now, the definitions of various types of Pascal and Brianchon sets and of tangents of Pascal sets or of points of contact of Brianchon sets imply:

Theorem 6.5 *(generalized Pascal theorem)*
A simple 6-point is a Pascalian 6-point iff it is inscribed to a Pascal-Brianchon set.

Theorem 6.6 *(generalized Brianchon theorem)*
A simple 6-line is a Brianchonian 6-line iff it is circumscribed to a Pascal-Brianchon set.

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