Original scientific paper
Accepted 2. 11. 1998.

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How to Design Nice Tilings?

Dedicated to the memory of Professor Stanko Bilinski

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ABSTRACT

Motivated by famous nice tilings we classify all Γ–tilings $(\mathcal{T},\Gamma)$ in the planes of constant curvature with 5 barycentric triangle orbits under a discontinuous isometry group Γ. We describe the 13 infinite series of the resulting tilings by so-called D–diagrams and additional rotation parameters in our Table. Depending on the parameters, the tilings are realizable in the sphere ($\mathbb{S}^2$), in the Euclidean ($\mathbb{E}^2$) or hyperbolic ($\mathbb{H}^2$) plane. The starting examples are depicted in our figures. Summarizing two theorems are formulated in Section 3.

Key words: D–symbol, tiling in the plane

Kako projektirati lijepo popločavanje?

SAŽETAK

Motivirani lijepim popločavanjem klasificiramo sva Γ–popločavanja $(\mathcal{T},\Gamma)$ u ravninama konstantne zakrivljenosti s pet baricentričkih trokutastih orbita pod nekontinuiranom grupom izometrija Γ. U tabeli prikazujemo 13 beskonačnih serija dobivenih popločenja pomoću tzv. D–dijagrama i dodatnih parametara rotacije. Ovisno o parametrima popločenja se mogu realizirati u sfernoj ($\mathbb{S}^2$), euklidskoj ($\mathbb{E}^2$) ili hiperboličkoj ($\mathbb{H}^2$) ravnini. Početni primjeri prikazani su na slikama. Dva zaključna teorema izrečena su u odjeljku 4.

Ključne riječi: D–simbol, popločavanje ravnine

1 An Archimedian tiling and its generalization by D–symbols

We start with a tiling which seemingly was a favorite one of Professor Bilinski [1]. This tiling $(\mathcal{T},\Gamma)$ in Fig. 1 fils the Euclidean plane $\mathbb{E}^2$ with regular triangles and quadrates under a symmetry group Γ, acting transitively on the vertices of $\mathcal{T}$. Such an Archimedian tiling can be described by the symbol $(4,3,4,3,3)$ showing the cyclic order of the corresponding polygons about each vertex.

Now we introduce a concise symbol for $(\mathcal{T},\Gamma)$, called D–symbol (to honour of B. N. Delone (Delaunay), M. S. Delaney and A. W. M. Dress [7, 8, 11]), which reflects the combinatorics and periodicity of $\mathcal{T}$ at the same time.

We prepare the formal barycentric subdivision C of $\mathcal{T}$ with (labelled or coloured) sides

0 ······ , 1 ---- , 2 —— . (1.1)

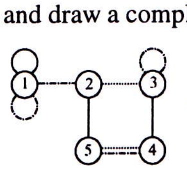
Each barycentric triangle has a 0–side opposite to its vertex (a 0–dimensional constituent of $\mathcal{T}$), a 1–side opposite to a (formal 1–dimensional) edge centre, a 2–side (polygon side) opposite to a (formal 2–dimensional) tile centre. We assume that this barycentric subdivision is invariant under the action of Γ. Thus, we obtain finitely many, now exactly 5, barycentric triangles (numbered by $1,\dots,5$ in Fig. 1) whose Γ–images induce the whole tiling $\mathcal{T}$. We can introduce adjacency operations

σ_0 ······ , σ_1 ---- , σ_2 ——

for the above Γ–orbits of barycentric triangles

$$\{1,2,3,4,5\} =: \mathcal{D} := \{D_1,D_2,D_3,D_4,D_5\} \qquad (1.2)$$

and draw a complete diagram



$$= (\Sigma_I, \mathcal{D}). \qquad (1.3)$$

This expresses exactly the above operations as involutive permutations of the set $\mathcal{D}$:

$$\sigma_0\,(1)(2,3)(4,5), \quad \sigma_1\,(1,2)(3)(4,5), \quad \sigma_2\,(1)(2,5)(3,4). \qquad (1.4)$$

Considering the barycentric triangles C_1, C_2, C_3, C_4, C_5 in Fig. 1, they form a fundamental domain

$$\mathcal{F} = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \qquad (1.5)$$

for the group Γ. The σ–operations describe the generators of Γ as follows:

$\sigma_0(1) = 1$ means that the triangle C_1 and $\sigma_0(C_1)$ lie in the same Γ–orbit D_1. We write $\sigma_0(C_1) = C_1^{m_0}$, the line reflection m_0, as a generator of Γ, is written into the exponent.

* Supported by the Hungarian National Science Grant (OTKA), Grant No. T020498/1996.

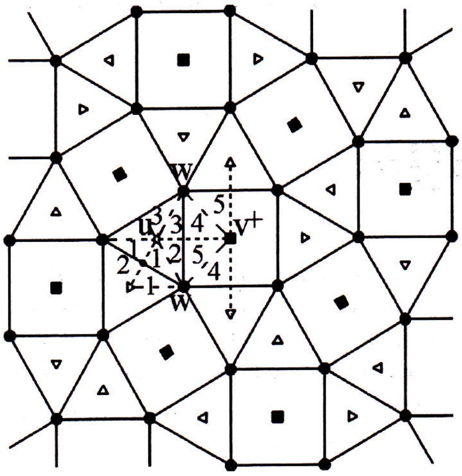


Fig. 1: $\Gamma_{5.12}(3\cdot 1, 4^{+}; 5\cdot 1)$

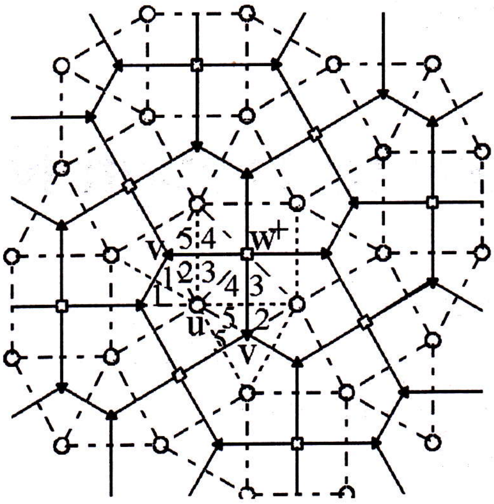


Fig. 2: $\Gamma_{5.13}(5\cdot 1; 3\cdot 1, 4^{+})$

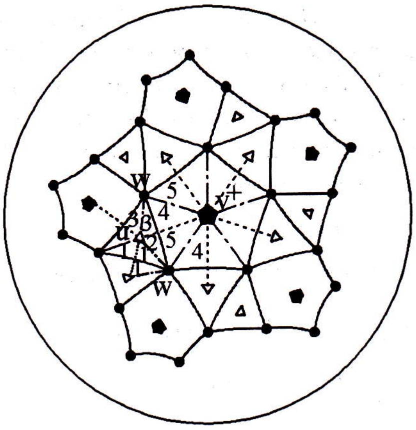


Fig. 3: $\Gamma_{5.12}(3\cdot 1, 5^{+}; 5\cdot 1)$

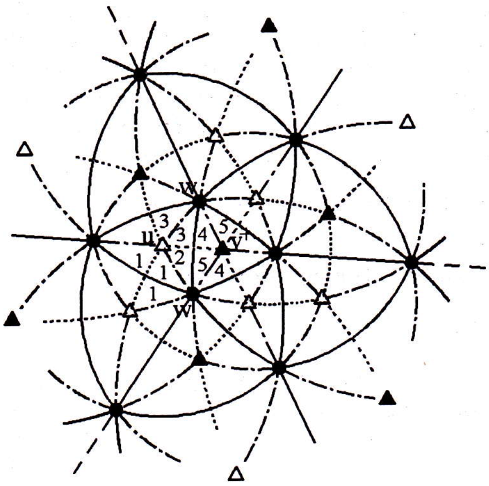


Fig. 4: $\Gamma_{5.12}(3\cdot 1, 3^{+}; 5\cdot 1)$

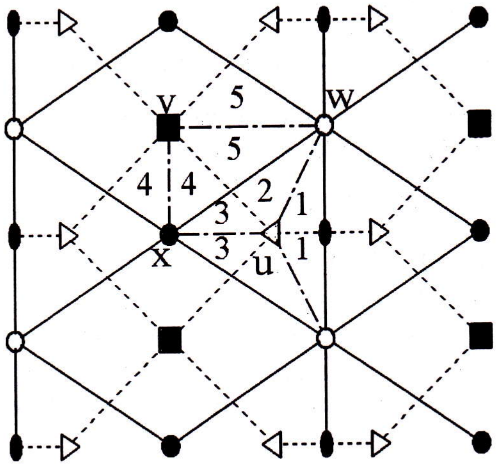


Fig. 5: $\Gamma_{5.11}(3\cdot 1, 2\cdot 2; 3\cdot 2, 2\cdot 2)$

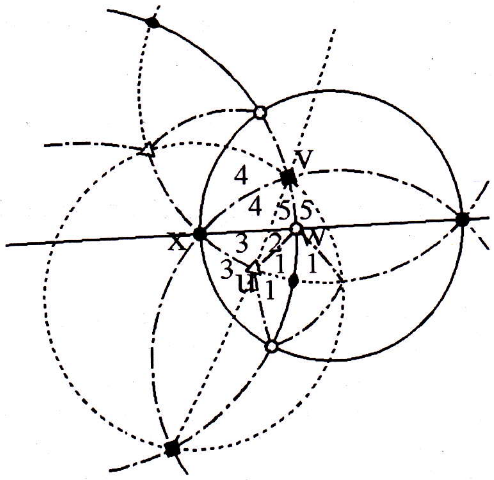


Fig. 6: $\Gamma_{5.11}(3\cdot 1, 2\cdot 2; 3\cdot 1, 2\cdot 2)$

Now $\sigma_1(3) = 3$ analogously means, that C_3 and $\sigma_1(C_3)$ belong to the same Γ–orbit D_3, $\sigma_1(C_3) = C_3^{m_1}$, m_1 is also a generator of Γ. Then $\sigma_2(1) = 1$ defines $\sigma_2(C_1) = C_1^{m_2}$, a new generating reflection m_2 for Γ.

Finally $\sigma_1 : (4,5)$ defines the generating rotation r by $\sigma_1(C_5) = \sigma_1(C_5) = C_4^r$ and its inverse r^{-1} by $\sigma_1(C_5) = \sigma_1(C_4) = C_5^{r^{-1}}$.

Notice almost the same situation in Figs. 3–4 as pictures of the hyperbolic plane $\mathbb{H}^2$ and the sphere $\mathbb{S}^2$, respectively, by their conforme models.

We see, these tilings have the same D–diagram, denoted by $(\Sigma_I, \mathcal{D})$, of 5 elements. Further, we say that the same adjacency structure

$$\Sigma_I := \{\sigma_i,\, i \in I = \{0,1,2\} \,:\, \sigma_i^2 := \sigma_i\sigma_i = 1\}, \text{ by (1.3)} \qquad (1.6)$$

as a 'free' Coxeter reflection group acts on $\mathcal{D}$, and induces first a group scheme

$$\Gamma := \left(m_0, m_1, m_2, r \,:\, 1 = m_0^2 = m_1^2 = m_2^2\right) \qquad (1.7)$$

and a surface (topological 2–space) is defined by $\overline{\mathcal{F}}$, the fundamental domain $\mathcal{F}$ in (1.5) is endowed by the side identification as (1.3) and (1.7) describe.

We have a symmeric matrix function with natural values

$$\begin{aligned} &r_{ij} : \mathcal{D} \to \mathbb{N}_{I\times I}, \quad \text{defined by (1.3)} \\ &r_{ij}(D) := \min\{r \,:\, (\sigma_j\sigma_i)^r \cdot D = D\} \end{aligned} \qquad (1.8)$$

which determines just the surface topology of $(\Sigma_I, \mathcal{D}) = \overline{\mathcal{F}}$.

Now, $r_{01}(D_1, D_2, D_3) = 3$, $r_{01}(D_4, D_5) = 1$ appear in our Table at $\Gamma_{5.12}(3u, v^+; 5w)$ as the coefficients of u and v^+, respectively. The coefficient of w is just

$$r_{12}(D_1, D_2, D_3, D_4, D_5) = 5. \qquad (1.9)$$

This indicates the vertex transitivity, since we have one $\langle\sigma_1, \sigma_2\rangle$–orbit. The coefficients

$$r_{02}(D_1) = 1, \quad r_{02}(D_2, D_3, D_4, D_5) = 2^+ \qquad (1.10)$$

will not appear in our Table, $r_{ii} = 1$ stands by convention $(i \in I)$.

Next we turn back to Fig. 1 by introducing the rotation parameters (orders, also a matrix function v_{ij} on $\mathcal{D}$):

$$\begin{aligned} &v_{01}(D_1, D_2, D_3) = u = 1, \\ &v_{01}(D_4, D_5) = v^+ = 4^+, \\ &v_{12}(D_1, D_2, D_3, D_4, D_5) = w = 1. \end{aligned} \qquad (1.11)$$

Here v^+ indicates that we have a cyclic rotation group as a stabilizer of the 2–centre at D_4, D_5. Else we have dihedral group with the corresponding rotation subgroup order.

Then as a last step we define the adjacency D–matrix by

$$\begin{aligned} &m_{ij} = r_{ij} \cdot v_{ij} \text{ with } \quad m_{02} = 2 \text{ for each } D \in \mathcal{D}, \\ &\text{where } m_{01} \geq 3 \text{ and } m_{12} \geq 3 \text{ are assumed for convention.} \end{aligned} \qquad (1.12)$$

This fixes some rotation orders $v_{02} := 2/r_{02}$, e. g.

$$v_{02}(D_1) = \tfrac{2}{1} = 2, \quad v_{02}(D_2, D_3, D_4, D_5) = 1^+. \qquad (1.13)$$

Moreover, **and this could be the starting point**, the matrix function, by the barycentric simplicies C in the full subdivision C:

$$\begin{aligned} &m_{ij} : \mathcal{D} \to \mathbb{N}_{I\times I}, \\ &m_{ij}(D) := \min\{m \,:\, (\sigma_j\sigma_i)^m \cdot C = C,\, C \in D\}, \end{aligned} \qquad (1.14)$$

will determine the combinatorial structure of the tiling $(\mathcal{T}, \Gamma)$, by fixing the defining relations for the generators of Γ, in addition to (1.7). By (1.11) we have

$$\begin{aligned} &(m_0 m_1)^u = 1, \quad r^v = 1, \quad (m_1 r m_2 r^{-1})^w = 1, \\ &\text{moreover, } (m_0 m_2)^2 = 1 \text{ by (1.13).} \end{aligned} \qquad (1.15)$$

With $(u, v; w) = (1, 4; 1)$ we obtain the Euclidean tiling $(\mathcal{T}, \Gamma)$ in Fig. 1, indeed, as the Table at $\Gamma_{5.12}(3u, v^+; 5w)$ contains. The group $\Gamma =$ **p4g** $= 4*2$ is a Euclidean plane crystallographic group, now.

In general we have vertex–transitive (Archimedian or uniform) tilings: $(v, 3u, v, 3u, 3u)$–polygons w–times around each vertex, with groups Γ by (1.7) and (1.15):

$$\Gamma = \Gamma_{5.12} = v*u, 2, w = (+, 0; [v]; \{(u, 2, w)\}). \qquad (1.16)$$

Here we indicated Conway's notation and Macbeath's signature, too [10].

Definition 1.1.

The D–diagram $(\Sigma_I, \mathcal{D})$ together with the matrix function $D \mapsto m_{ij}(D)$ will be called D–symbol.

We mention roughly the ***theorem*** of A. W. M. Dress:

A "good" D–symbol describes a tiling $(\mathcal{T}, \Gamma)$, up to an equivariant homeomorphism, uniquely in a space of constant curvature $\mathbb{S}^2$ $(>)$, $\mathbb{E}^2$ $(=)$, $\mathbb{H}^2$ (< 0) iff for the so–called curvature

$$K(\mathcal{D}, m_{ij}) := \sum_{D\in\mathcal{D}} \left(\frac{1}{m_{01}(D)} + \frac{1}{m_{12}(D)} - \frac{1}{2}\right) \gtreqless 0 \qquad (1.17)$$

holds.

Definition 1.2.

Two tilings $(\mathcal{T}, \Gamma)$ and $(\mathcal{T}', \Gamma')$ are equivariantly homeomorphic if there is a bijection

$$\varphi : \mathcal{T} \to \mathcal{T}' \qquad (1.18)$$

preserving all incidences (thus all adjacencies) such that

$$\Gamma' = \varphi^{-1}\Gamma\varphi. \qquad (1.19)$$

In short, φ carries also the group action from $\mathcal{T}$ onto $\mathcal{T}'$.

The curvature to $\Gamma_{5.12}(3u, v^+; 5w)$ will be by (1.17)

$$K = \left(\frac{1}{3u}+\frac{1}{5w}-\frac{1}{2}\right)\cdot 3+\left(\frac{1}{v}+\frac{1}{5w}-\frac{1}{2}\right)\cdot 2 \\ = \frac{1}{u}+\frac{2}{v}+\frac{1}{w}=\frac{5}{2}. \tag{1.20}$$

This provides us spherical tiling for $(u,v;w) = \underline{(1,3;1)}$ in Fig. 4 by stereographic projection. The underline indicates, as it reads in the Table, that Γ is not maximal in that case. Namely, the combinatorial automorphism group of $\mathcal{T}$ is larger than the group $\Gamma = \mathbf{m\bar{3}} = 3*2$. The tiling $(\mathcal{T}, \Gamma = \mathrm{Aut}\,\mathcal{T})$ is just the icosahedron tiling on $\mathbb{S}^2$ that can be described by D–symbol of 1 element and matrix $m_{01} = 3$, $m_{12} = 5$, as $\Gamma(\bar{u} = 3u = v = 3, \bar{v} = 5w = 5)$ stands in the Table. This group is generated by 3 reflections on the sides of the spherical triangle with angles $\frac{\pi}{3}, \frac{\pi}{5}, \frac{\pi}{2}$. We excluded $v = 2$, serving digons on $\mathbb{S}^2$.

We see in Fig. 3 the minimal hyperbolic solution in $\mathbb{H}^2$: $(u,v;w) = (1,5;1)$ providing $K = -\frac{1}{10}$ negative curvature. But we have infinite hyperbolic series by choosing $1 < u$ or $1 < w$ with $3 \le v$, e. g. $(u,v;w) = (2,3;1)$ as Archimedian tiling $(3,6,3,6,6)$.

2 The face–transitive dual tiling (4, 3, 4, 3, 3) and its relatives. Self dual tilings

In Fig. 2 we have depicted the dual tiling $(\overline{\mathcal{T}}, \Gamma)$ to the former one in Fig. 1. We only remarks that the machinery by D–symbol is straightforward. We change the lines, adjacencies, σ_i–operations in the D–diagram (1.3) as

$$\begin{array}{lcl} 0 \cdots\cdots \sigma_0 & \longleftrightarrow & 2 \longrule \sigma_2, \\ 1 ---- \sigma_1 & \longleftrightarrow & 1 ---- \sigma_1 \end{array} \tag{2.1}$$

indicate. Then the matrix functions r_{ij}, v_{ij}, m_{ij}, introduced in the former section 1, also change the indices $0 \leftrightarrow 2$ and $1 \leftrightarrow 1$. This reads in our Table at $\Gamma_{5.13}(5u, 3v, w^+)$, where the shorter dual diagram of (1.3) can be seen with changing the nodes ③ $\leftrightarrow$ ⑤ for a technical reason later on.

For short we left the loops from the D–diagrams in our Table.

Fig. 2 directly depicts the dual tiling of Fig. 1. Since the former one was vertex transitive, the laer one is tile transitive by pentagons with vertex valences $(4,3,4,3,3)$ in the title of this section 2.

Fig. 3 and Fig. 4 depict the dual tilings by dotted $0 \cdots\cdots$ lines in $\mathbb{H}^2$ and $\mathbb{S}^2$, respectively. This convention will also be kept later on.

Self dual tilings with corresponding parameters can be seen in Fig. 6 to the group $\Gamma_{5.11}(3u = 3, 2v = 4; 3w = 3, 2x = 4)$.

For $(u,v;w,x) = (1,4;1,4)$ we would have self dual Euclidean tilings with concave 8–gons and triangles in a nice manner, left to draw by the Reader.

As the former analysis to D–diagram and group

$$\Gamma_{5.11}(3u, 2v; 3w, 2x) = *u,\, 2,\, w,\, v,\, x \tag{2.2}$$

in our Table show, the barycentric triangles form a fundamental domain

$$\overline{\mathcal{F}} = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \tag{2.3}$$

with reflections in its sides as the D–diagram $\mathcal{D}$ dictates.

So $\overline{\mathcal{F}}$ will be a disc with cornerpoints by (2.2) or angles

$$\frac{\pi}{u}, \frac{\pi}{2}, \frac{\pi}{w}, \frac{\pi}{v}, \frac{\pi}{x} \tag{2.4}$$

on the boundary. The corresponding reflections generate the group Γ and produce the tiling $(mtc\mathcal{T}, \Gamma)$ in $\mathbb{S}^2 (> 0)$, $\mathbb{E}^2 (= 0)$ or $\mathbb{H}^2 (< 0)$ by the curvature

$$K = \left(\frac{1}{3u}+\frac{1}{3w}-\frac{1}{2}\right)\cdot 2+\left(\frac{1}{3u}+\frac{1}{2x}-\frac{1}{2}\right) \\ +\left(\frac{1}{2v}+\frac{1}{2x}-\frac{1}{2}\right)+\left(\frac{1}{2v}+\frac{1}{3w}-\frac{1}{2}\right) \\ = \frac{1}{u}+\frac{1}{v}+\frac{1}{w}+\frac{1}{x}-\frac{5}{2} \gtreqless 0. \tag{2.5}$$

This is in conformity with the comparison of the angle sum in (2.4) and the angle sum 3π of a Euclidean pentagon.

Fig. 6 shows a self dual spherical tiling from the infinite series $(u,v;w,x) = (1,2;1,x)$. The self dual tiling with the group

$$\Gamma_{5.11}(3\cdot 1, 2\cdot 3, 3\cdot 1, 2\cdot 3) = \mathbf{\bar{4}3m} = *2,3,3 \tag{2.6}$$

realizes on $\mathbb{S}^2$ with concave hexagons and triangles or by a polyhedron with 4 hexagons and 12 triangles as faces. The Reader would make this polyhedron.

Fig. 5 shows us again a Euclidean tiling from the series $\Gamma_{5.11}$ with the same D–diagram, however, it is not self dual because of the parameters $(u,v;w,x) = (1,2;2,2)$. Then we have

$$\Gamma = \mathbf{pmm} = *2,2,2,2 \tag{2.7}$$

a plane chrystallographic group with a rectangle disc as fundamental domain. We have depicted the continuous triangles and rhombs, moreover the dotted hexagons and rhombs.

All tilings to $\Gamma_{5.11}$ is $(2,2,2)$–transitive, i. e. we have 2–2–2 Γ–orbits of tiles, edges and vertices, respectively.

This transitivity property is indicated at the diagrams in our Table for every series.

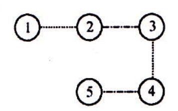
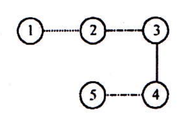
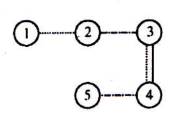
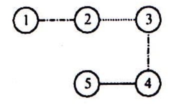
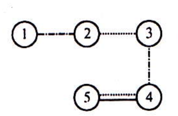
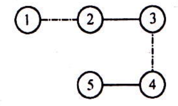
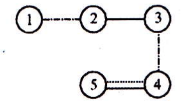
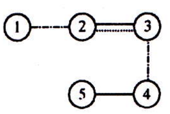
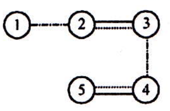
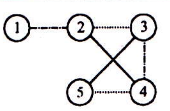
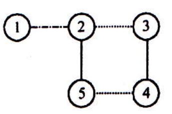
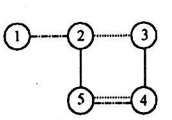
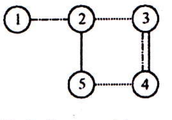
3 The classification of D–symbols and their tiling series with $|\mathcal{D}| = 5$

The D–symbol method provides us a systematic tool to classify tilings in the plane, moreover in the space, and in d–dimensional space, in general.

This program was proposed in [11] and solved up to $d = 3$, $|\mathcal{D}| = 3$, i. e. up to 3 barycentric simplices in the fundamental domain. In the plane ($d = 2$) the tilings have been classified up to $|\mathcal{D}| = 4$ for another publication just by the program of [11] illustrated in this paper. However, we face to great difficulties in dimension $d = 3$ as [11] indicates. E. g. 8 homogeneous 3–spaces (Thurston-geometries) may occur, if the metric realization is still possible. Splittings on 2–surfaces (orbifolds) occur already in [11] where different geometries are realizable in the different pieces. Behind these stands the Thurston–conjecture to be decided.

In [11] we showed that for any cardinality $|\mathcal{D}|$ fixed there exists finitely many D–diagrams $(\Sigma_I, \mathcal{D})$ in each dimension d, where the index set $I = \{0, 1, \ldots, d\}$ occurs for the adja-

Table: Two–dimensional D–symbols with $|\mathcal{D}| = 5$

 (3, 3, 1)–transitive	$\Gamma_{5.1}(5u; v, 2w, 2x) = *u, v, w, x, 2$ $= (+, 0; []; \{(u, v, w, x, 2)\})$; $1 \le u, 3 \le v, 2 \le w, x$ max. iff $v = 2w = 2x$ does not hold, else $\Gamma_1(\bar{u} = 5u; \bar{v} = v = 2w = 2x)$ is its super group. $(u; v, w, x)$ — $\mathbb{H}^2$: $(1;3,2,2), \ldots$
 (2, 3, 2)	$\Gamma_{5.2}(3u, 2v; w, 4x) = *u, v, 2, x, w$ $= (+, 0; []; \{(u, v, 2, w, x)\})$; $1 \le u, 2 \le v, 3 \le w, 1 \le x$ max. iff $3u \ne 2v$ or $w \ne 4x$, else $\Gamma_1(\bar{u} = 3u = 2v; \bar{v} = w = 4x)$ is its super group. $(u; v, w, x)$ — $\mathbb{S}^2$: $(1,2;w,1), (1,3;3,1),$ $(1,3;4,1), (1,3;5,1), (1,4;3,1), (1,5;3,1)$ — $\mathbb{E}^2$: $(1,3;6,1), (1,4;4,1), (1,6;3,1)$ — $\mathbb{H}^2$: else
 (2, 3, 1)	$\Gamma_{5.3}(5u; v, 4w) = 2 * u, v, w, 2$ $= (+, 0; [2]; \{(u, v, w, 2)\})$; $1 \le u, 3 \le v, 1 \le w$ max. iff $v \ne 4w$, else $\Gamma_1(\bar{u} = 5u; \bar{v} = v = 4w)$ is its super group. $(u; v, w,)$ — $\mathbb{H}^2$: $(1;3,1), \ldots$
 (2, 3, 2)	$\Gamma_{5.4}$ is dual to $\Gamma_{5.2}$ by $1 \leftrightarrow 5, 2 \leftrightarrow 4, 3 \leftrightarrow 3, 4 \leftrightarrow 2, 5 \leftrightarrow 1$ and $--- \leftrightarrow ---, \cdots\cdot \leftrightarrow$ ——
 (2, 3, 1)	$\Gamma_{5.5}(5u; 2v, 3w) = 2 * u, 2, v, w$ $= (+, 0; [2]; \{(u, 2, v, w)\})$; $1 \le u, 2 \le v, 1 \le w$ max. iff $2v \ne 3w$, else $\Gamma_1(\bar{u} = 5u; \bar{v} = 2v = 3w)$ is its super group. $(u; v, w)$ — $\mathbb{E}^2$: $(1;2,1)$ — $\mathbb{H}^2$: else
 (1, 3, 3)	$\Gamma_{5.6}$ is dual to $\Gamma_{5.1}$ by $1 \leftrightarrow 5, 2 \leftrightarrow 4, 3 \leftrightarrow 3, 4 \leftrightarrow 2, 5 \leftrightarrow 1$ and $--- \leftrightarrow ---, \cdots\cdot \leftrightarrow$ ——
 (1, 3, 2)	$\Gamma_{5.7}$ is dual to $\Gamma_{5.5}$ by $1 \leftrightarrow 1, 2 \leftrightarrow 2, 3 \leftrightarrow 3, 4 \leftrightarrow 4, 5 \leftrightarrow 5$ and $--- \leftrightarrow ---, \cdots\cdot \leftrightarrow$ ——
 (1, 3, 2)	$\Gamma_{5.8}$ is dual to $\Gamma_{5.3}$ by $1 \leftrightarrow 5, 2 \leftrightarrow 4, 3 \leftrightarrow 3, 4 \leftrightarrow 2, 5 \leftrightarrow 1$ and $--- \leftrightarrow ---, \cdots\cdot \leftrightarrow$ ——
 (1, 3, 1) Self dual (Sd)	$\Gamma_{5.9}(5u; 5v) = 2, 2 * u, 2, v$ $= (+, 0; [2, 2]; \{(u, 2, v)\})$; $1 \le u, 1 \le v$ non maximal: $\Gamma_1(\bar{u} = 5u; \bar{v} = 5v)$ is its super group. $(u; v)$ — $\mathbb{H}^2$: $\underline{(1;1)}, \ldots$
 (1, 2, 1) Self dual: $3 \leftrightarrow 4$	$\Gamma_{5.10}(5u; 5v) = *u, 2, v, \times$ $= (-, 1; []; \{(u, 2, v)\})$; $1 \le u, 1 \le v$ non maximal: $\Gamma_1(\bar{u} = 5u; \bar{v} = 5v)$ is its super group. $(u; v)$ — $\mathbb{H}^2$: $\underline{(1;1)}, \ldots$
 (2, 2, 2) Self dual: $3 \leftrightarrow 5$	$\Gamma_{5.11}(3u, 2v; 3w, 2x) = *u, 2, w, v, x$ $(+, 0; []; \{(u, 2, w, v, x)\})$; $1 \le u, 2 \le v, 1 \le w, 2 \le x$ max. iff $3u \ne 2v$ or $3w \ne 2x$, else $\Gamma_1(\bar{u} = 3u = 2v; \bar{v} = 3w = 2x)$ is its super group. $(u, v; w, x)$ — $\mathbb{S}^2$: $(1,2;1,x), (1,v;1,2),$ $(1,3;1,3), (1,3;1,4), (1,3;1,5), (1,4;1,3),$ $(1,5;1,3)$ — $\mathbb{E}^2$: $(1,2;2,2), (1,3;1,6),$ $(1,4;1,4), (1,6;1,3), (2,2;1,2)$ — $\mathbb{H}^2$: else
 (1, 2, 2)	$\Gamma_{5.12}(3u, v+; 5w) = v * u, 2, w$ $= (+, 0; [v]; \{(u, 2, w)\})$; $1 \le u, 3 \le v, 1 \le w$ max. iff $3u \ne v$, else $\Gamma_1(\bar{u} = 3u = v; \bar{v} = 5w)$ is its super group. $(u, v; w)$ — $\mathbb{S}^2$: $(1,3;1)$ — $\mathbb{E}^2$: $(1,4;1)$ — $\mathbb{H}^2$: else, e. g. $(1,5;1)$
 (2, 2, 1)–transitive	$\Gamma_{5.13}$ is dual to $\Gamma_{5.12}$ by $3 \leftrightarrow 5$; $\cdots\cdot \leftrightarrow$ —— $\Gamma_{5.13}(5u; 3v, w+) = w * u, 2, v$ $= (+, 0; [w]; \{(u, 2, v)\})$; $1 \le u, 1 \le v, 3 \le w$ max. iff $3v \ne w$, else $\Gamma_1(\bar{u} = 5u; \bar{v} = 3v = w)$ is its super group. $(u; v, w)$ — $\mathbb{S}^2$: $(1;1,3)$ — $\mathbb{E}^2$: $(1;1,4)$ — $\mathbb{H}^2$: else, e. g. $(1;1,5)$

cencies, colours, matrix functions, etc. The geometric reason, namely that for the adjacency matrix

$$m_{ij} : \mathcal{D} \to \mathbb{N}_{I\times I},$$
$$m_{ij} = 2 \text{ stands for } |i-j| > 1,\ i,j \in I \tag{3.1}$$

considerably reduces the number of cases, and we gave an ordering and listing algorithm for *D*–diagrams and for the *D*–symbols, in general. This is of very large complexity, but for $d = 2$, in the plane, the realizability in $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$ is no question more. Althugh we need computer as in [7, 8, 10], e. g., the algorithms are elaborated (D. H. Huson, O. Delgado Friedrichs are cited in our references).

Our Table lists all *D*–diagrams of elements $|\mathcal{D}| = 5$ in our lexicographic order by the index set $I = \{0, 1, 2\}$. We mention the steps to this ordering:

i) Let a *D*–diagram $(\Sigma_I, \mathcal{D}(D_1))$, with a fixed starting element D_1, be given. Assume, that $D_1, \dots, D_r$, $r < |\mathcal{D}| =: n$ have already been numbered. Consider $\sigma_0(D_r)$, $\sigma_1(D_r)$. The first of them, not listed yet, will be D_{r+1} if it exists.

ii) Else we take $\sigma_2(D_r), \dots, \sigma_2(D_1); \dots; \sigma_d(D_r), \dots, \sigma_d(D_1)$. The first new one will be D_{r+1}.

iii) Then we proceed with $r \mapsto r+1$ as above, still we end at D_n, $n = |\mathcal{D}|$.

iv) The distance of two elements D_x, D_y can be obtained:

We chose $D_x = D_{1'}$ for starting element and proceed as above. If we get $D_y = D_{k'}$ then the distance is $D_x D_y = k - 1$.

v) Let two *D*–diagrams $\mathcal{D}(D_1)$ and $\mathcal{D}'(D_{1'})$ be given, each with distinguished starting elements as above. We define $\mathcal{D} < \mathcal{D}'$ by the following preferences a–d:

a) $|I| < |I'|$ (dimension);

b) $|\mathcal{D} < |\mathcal{D}'|$ (cardinality);

c) Consider equally numbered elements and their σ–images. In reverse preference on $I = I'$ we consider distances: $D_1\sigma_d(D_1) < D_{1'}\sigma_d(D_{1'})$; if $=$ holds then $D_2\sigma_d(D_2) < D_{2'}\sigma_d(D_{2'})$; …; if $=$ holds then $D_1\sigma_{d-1}(D_1) < D_1\sigma_{d-1}(D_{1'})$; …

d) If $=$ stands in each place then the *D*–diagrams are isomorphic. Then come the matrix function m_{ij} and m'_{ij} by increasing preferences in their 01, 12, $(d-1)d$ entries for the equal elements.

You see that these algorithms can be implemented onto computer; although we can proceed now by hands. We formulate our results by the Table and figures.

Theorem 3.1.

There are exactly 13 *isomorphism classes of D–diagrams with* $|\mathcal{D}| = 5$ *elements in dimension* 2, *numbered by group schemes* $\Gamma_{5.1}$–$\Gamma_{5.13}$ *and the free rotation orders as parameters.*

Among them we have 3 *self dual classes:* 2 *non–maximal and* 1 *possibly maximal series* $\Gamma_{5.11}$ *for the dually equal parameters.*

We have 5 *dual pairs of series for the dually equal parameters for which the corresponding groups are conjugate by the duality mapping.*

The last concept is just the same as in (1.18–19) but φ changes dotted and continuous lines, preserves broken lines (see (1.1)) in the barycentric subdivisions.

The corresponding tilings are realizable in a plane of constant curvature, we listed in our Table and read

Theorem 3.2.

There are exactly 15 *Euclidean tilings* $(\mathcal{T}, \Gamma)$ *with* 5 *barycentric triangle orbits under* Γ, *up to equivariant homeomorphism: One self dual tiling in* $\Gamma_{5.11}$ *and seven dual pairs:* 3 *pairs in* $\Gamma_{5.2}$–$\Gamma_{5.4}$; 1 *pairs in* $\Gamma_{5.5}$–$\Gamma_{5.7}$; 2 *pairs in* $\Gamma_{5.11}$; 1 *pairs in* $\Gamma_{5.12}$–$\Gamma_{5.13}$.

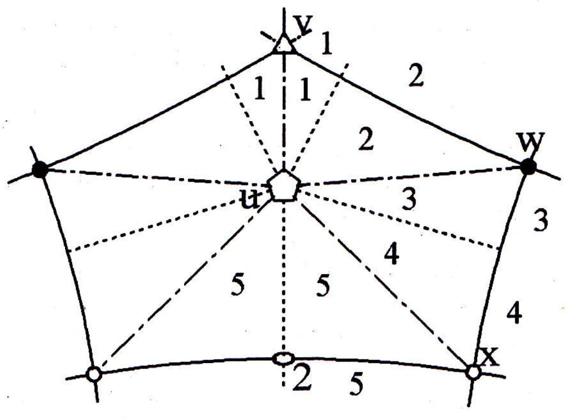
In Fig. 7 we have depicted the minimal representants of the other series. One dual pair to $\Gamma_{5.5}$–$\Gamma_{5.7}$ is Euclidean. One minimal spherical dual pair belongs to $\Gamma_{5.2}$–$\Gamma_{5.4}$. The other minimal representants are hyperbolic. The last ones are non–maximal. Combinatorally both are the regular hyperbolic tilings with pentagons of angles $2\pi/5$.

4 Closing remarks and memories

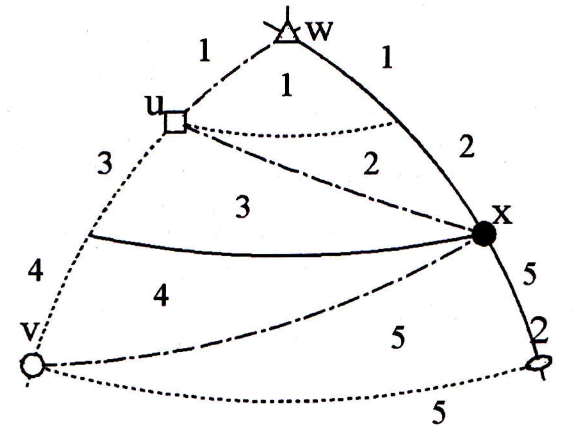
Professor Stanko Bilinski renewed the topic of combinatorial tilings in his pioneering papers [1, 2]. The hyperbolic plane $\mathbb{H}^2$ is very rich with possibilities. Those surfaces — the orientable ones of genus equal or bigger than 2 — whose universal cover is $\mathbb{H}^2$ were involved into his researches.

The theory of *D*–symbols, initiated and systematically developed by Andreas Dress and his school in Bielefeld, received an important influence from Professor Bilinski at an Oberwolfach seminary in 1984. The lecture held in his kind place and published in [3] gave the task to look for the quasi–regular polyhedra of genus 2 by *D*–symbols (that time Delaney–symbols), thus by computer. This was completely solved in [7, 8].

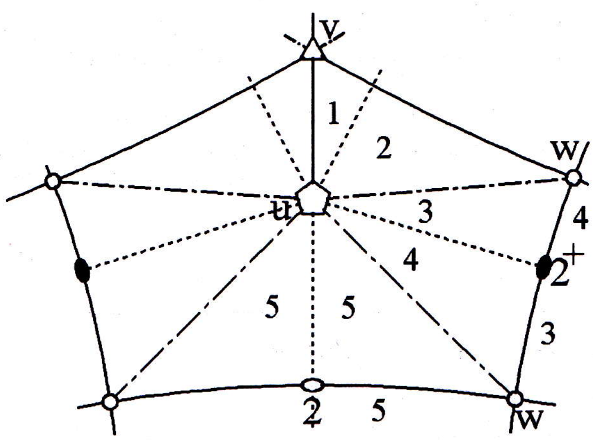
The second author met him first in that Oberwolfach conference (organized by A. W. M. Dress and Jörg Wills), we all could enjoy his kind anecdotes. We think he was paternal friend (väterlicher Freund) of many geometricians all over the world, in particular in the German and Slavic cultural



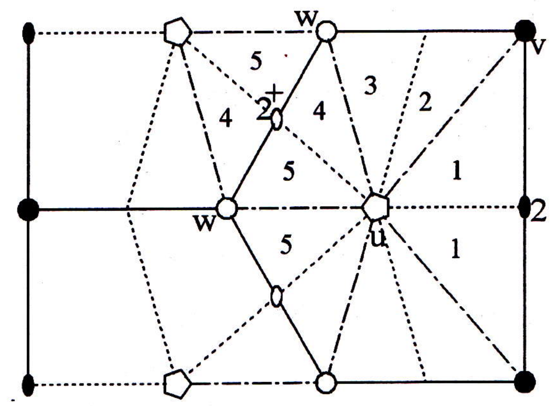
$\Gamma_{5.1}(5\cdot 1; 3, 2\cdot 2, 2\cdot 2)$



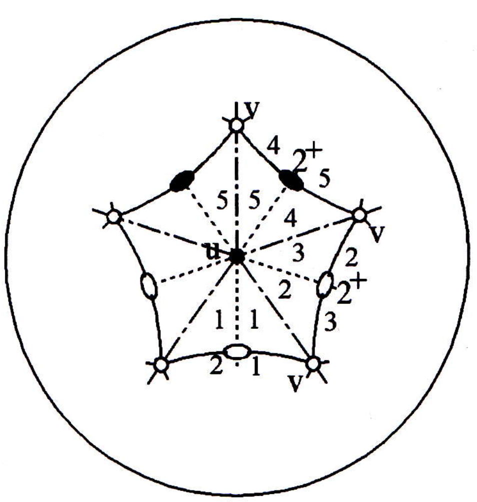
$\Gamma_{5.2}(3\cdot 1, 2\cdot 2; 3, 4\cdot 1)$



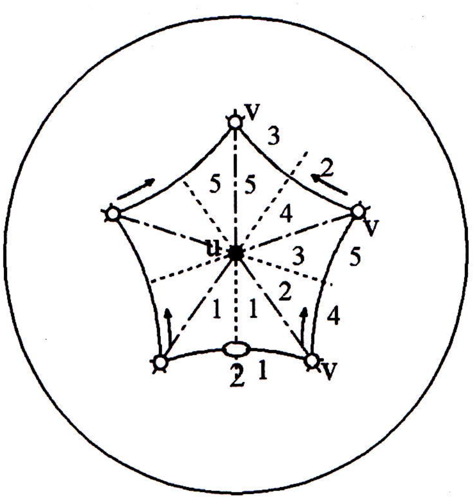
$\Gamma_{5.3}(5\cdot 1; 3, 4\cdot 1)$



$\Gamma_{5.5}(5\cdot 1; 2\cdot 2, 3\cdot 1)$



$\Gamma_{5.9}(5\cdot 1; 5\cdot 1)$



$\Gamma_{5.10}(5\cdot 1; 5\cdot 1)$

Fig. 7

territory. He was member of the Croatian and the Austrian Academy of Sciences [3–6]. He formed a strong geometry school in the University of Zagreb with many PhD students in various fields of geometry. His works [1–5] implicitely influenced our paper [9] where his favorite Archimedian tilings were discussed on the base of plane (NEC) crystallographic groups and their fundamental domains, found finally by a computer program COMCLASS [10] (e-mail adress: `comclass@matf.bg.ac.yu`).

We thanks the Editors of *KoG* for the invitation to this paper to honour of Professor Stanko Bilinski.

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