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Closing Pipes by Extension of B–Spline Surfaces

Dedicated to the memory of Professor Stanko Bilinski

Zatvaranje cijevi proširenjem B–spline ploha

SAŽETAK

U radu je prikazan algoritam za zatvaranje cijevi reprezentiranih tenzorskim produktom B–spline ploha. Plohe u obliku cijevi proširuju se i zatvaraju pravokutnim dijelovima (trostranim). Geometrijski podaci dijela koji zatvara su zajednički singularni vrh novih rubnih dijelova i rubni uvjeti prvog reda na tom vrhu. Tada se točke proširenja kontrolne mreže računaju iz tih podataka. Korisnički unos algoritma je, osim cijevi, singularna točka zatvaranja. Rubni se uvjeti biraju automatski, a da bi se postigao glatki oblik dijela koji zatvara oni se djelomično računaju i iz uvjeta zaglađivanja. Što više, točka zatvaranja također se može izračunati iz uvjeta zaglađivanja što vodi do automatskog zatvaranja cijevi.

Ključne riječi: CAGD, B–spline plohe, zaglađivanje

Closing Pipes by Extension of B–Spline Surfaces

ABSTRACT

This paper presents an algorithm to close pipes represented as tensor product B–spline surfaces. The tube shaped surface will be extended and closed by degenerate rectangular (three–sided) patches. The geometric data of the closing part are the common singular vertex of the new bordering patches and first order boundary conditions at this vertex. Then the points of the extension of the control net are computed from these data. The user input of the algorithm, besides the pipe, is the single closing point. The boundary conditions are chosen automatically and, in order to achieve a fair shape of the closing part, are partly computed from a fairness condition. Moreover, also the closing point can be computed from the fairness condition, which leads to the automatic closing of the pipe.

Key words: CAGD, B–spline surface, fairing

1 Introduction

Pipes, that is tube shaped surfaces, occur frequently in design processes. Such surfaces are, for example, a handle, a bottle, a telephon receiver, etc. From the geometric viewpoint any rotational surface with full parallel circles and any swept surface with closed generator curves is a pipe. Since the modelling systems do not allow in general to specify a single point as a closed sectional curve, in order to get a closed end of a pipe a new surface has to be constructed and fitted to the pipe. It may also happen that the exact shape of the covering part at the end of a tube is not prescribed, it simply has to be smooth and fit correctly. For the solution of this modelling problem we present an algorithm developed for tensor product B–spline surfaces of degrees (3,2). The tube shaped surface will be extended and closed by degenerate rectangular (three–sided) patches. The user inputs of the algorithm are the closing point, which is the common singular vertex of the three–sided patches and, if required, the position of the tangent plane or the tangent direction of the longitudinal boundary curves at this point. The remaining data, which are necessary for the equations of the new patches will be computed from a fairness condition. We will also show an automatic closing of the pipe, where even the position of the closing point is computed from a fairness condition.

The tensor product B–spline surface is determined by $n \times m$ control points and by the B–spline basis functions of 3rd and 2nd degree over the periodic knot vectors $\{t\}_{-2}^{n+3}$ and $\{s\}_{-1}^{m+2}$, respectively. We assume that the (i,j)th patch of the tube shaped surface is given by a parametric vector equation in the following matrix form

$$\mathbf{r}_{i,j}(u,v) = [1\ u\ u^2\ u^3]\left[\mathbf{B}_i^{(3)}(t)\right][\mathbf{V}_{i,j}]\left[\mathbf{B}_j^{(2)}(s)\right]^{\mathrm{T}}[1\ v\ v^2]^{\mathrm{T}}, \tag{1}$$
$$(u,v) \in [0,1]\times[0,1], \quad i=1,\dots,n-3, \quad j=1,\dots,m-2,$$

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where

$$\left[\mathbf{B}_i^{(3)}(t)\right] = \begin{bmatrix} b_{11} & (1-b_{11}-b_{13}) & b_{13} & 0 \\ -3b_{11} & 3b_{11}-b_{23} & b_{23} & 0 \\ 3b_{11} & -(3b_{11}+b_{33}) & b_{33} & 0 \\ -b_{11} & b_{11}-b_{43}-b_{44} & b_{43} & b_{44} \end{bmatrix},$$

$$b_{11} = \frac{(t_{i+1}-t_i)^2}{(t_{i+1}-t_{i-1})(t_{i+1}-t_{i-2})},$$

$$b_{13} = \frac{(t_i-t_{i-1})^2}{(t_{i+2}-t_{i-1})(t_{i+1}-t_{i-1})},$$

$$b_{23} = \frac{3(t_{i+1}-t_i)(t_i-t_{i-1})}{(t_{i+2}-t_{i-1})(t_{i+1}-t_{i-1})},$$

$$b_{33} = \frac{3(t_{i+1}-t_i)^2}{(t_{i+2}-t_{i-1})(t_{i+1}-t_{i-1})},$$

$$b_{43} = -\left\{\frac{1}{3}b_{33}+b_{44}+\frac{(t_{i+1}-t_i)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i-1})}\right\},$$

$$b_{44} = \frac{(t_{i+1}-t_i)^2}{(t_{i+3}-t_i)(t_{i+2}-t_i)}$$

is the coefficient matrix [1] of the ith non–uniform cubic B–spline basis function determined by the given knot vector

$$t_{-2} \leq, \dots, \leq t_{n+3}.$$

The first parameter of the patch is

$$u = \frac{t-t_i}{t_{i+1}-t_i}, \quad t \in [t_i, t_{i+1}].$$

Denoting the elements of the matrix $\left[\mathbf{B}_j^{(2)}(s)\right]$ by b again,

$$\left[\mathbf{B}_j^{(2)}(s)\right] = \begin{bmatrix} b_{11} & b_{12} & 0 \\ -2b_{11} & 2b_{11} & 0 \\ b_{11} & b_{32} & b_{33} \end{bmatrix},$$

$$b_{11} = \frac{s_{j+1}-s_j}{s_{j+1}-s_{j-1}},$$

$$b_{12} = \frac{s_j-s_{j-1}}{s_{j+1}-s_{j-1}},$$

$$b_{32} = -(s_{j+1}-s_j)\left(\frac{1}{s_{j+1}-s_{j-1}}+\frac{1}{s_{j+2}-s_j}\right),$$

$$b_{33} = \frac{s_{j+1}-s_j}{s_{j+2}-s_j},$$

is the coefficient matrix of the jth quadratic B–spline basis function determined by the given knot vector

$$s_{-1} \leq \dots \leq s_{m+2}.$$

The second parameter of the patch is

$$v = \frac{s-s_j}{s_{j+1}-s_j}, \quad s \in [s_j, s_{j+1}].$$

In the expressions of the coefficient matrices we accept the convention $\frac{0}{0} = 0$.

The matrix

$$[\mathbf{V}_{i,j}] = \begin{bmatrix} \mathbf{V}_{i,j} & \mathbf{V}_{i,j+1} & \mathbf{V}_{i,j+2} \\ \mathbf{V}_{i+1,j} & \mathbf{V}_{i+1,j+1} & \mathbf{V}_{i+1,j+2} \\ \mathbf{V}_{i+2,j} & \mathbf{V}_{i+2,j+1} & \mathbf{V}_{i+2,j+2} \\ \mathbf{V}_{i+3,j} & \mathbf{V}_{i+3,j+1} & \mathbf{V}_{i+3,j+2} \end{bmatrix},$$

$$i = 1, \dots, n-3, \quad j = 1, \dots, m-2$$

is built from the control points $\mathbf{V}_{k,l}$ $(k = i, \dots, i+3,$ $l = j, \dots, j+2)$ of the (i,j)th patch.

If for the control points

$$\mathbf{V}_{i,m-1} = \mathbf{V}_{i,1}, \quad \mathbf{V}_{i,m} = \mathbf{V}_{i,2} \quad (i = 1, \dots, n)$$

hold, then the $(n-3)\times(m-2)$ patches form a tubular surface. The longitudinal u–parameter lines of the surface are cubic, the v–parameter lines in the cross directions are closed quadratic curves. These parameter lines are C^2 and C^1 continuous functions, respectively, if there are no coinciding knot values or control points [2].

By assumption, the knot vectors will be periodical in our representation, so that

$$t_{-2} < \dots < t_2 \leq \dots \leq t_{n-2} < \dots < t_{n+3}$$

and

$$s_{-1} < s_2 < \dots < s_{m+2}.$$

A usual choice of the knot values is the 'chord–length' parametrization, when the knot values are placed according to

$$t_{i+1} - t_i = \overline{\mathbf{V}_{i,*}, \mathbf{V}_{i+1,*}} \quad (i = 1, \dots, n-1)$$

and

$$s_{j+1} - s_j = \overline{\mathbf{V}_{*,j}, \mathbf{V}_{*,j+1}} \quad (j = 1, \dots, m-1).$$

Here the meaning of $*$ in the index is that the average of the corresponding distances or the distances in a 'typical' point sequence in the longitudinal and cross direction of the control net are considered. As the knot vector is longer than a longitudinal control point sequence, the remaining knot values at both ends are placed in equal distances. In the cross direction the equalities $s_m - s_{m-1} = s_2 - s_1$ and $s_{m+1} - s_m = s_3 - s_2$ are assumed.

2 Definition of the closing part

For defining a closing part at the starting borderline $i = 1, j = 1, \dots, m-2$ we specify the closing point $\mathbf{P}$ and generate two additional rows of patches. For this purpose we extend the control net by $2 \times m$ control points $\mathbf{V}_{1,j}$, $\mathbf{V}_{2,j}$ $(j = 1, \dots, m)$, and reindex the former control points to $\mathbf{V}_{i+2,j}$ $(i = 1, \dots, n,\ j = 1, \dots, m)$. The closing part of the pipe will be a C^2–continuous extension of the original surface in the (longitudinal) u–direction, if there are no coinciding control points or knot values along the connection line. The new control points will be determined from the point **P** and geometric criteria prescribed at the new end of the pipe. The chosen geometric conditions ensure that the boundary patches meet at the given point **P**, and their longitudinal boundary lines end up there with prescribed tangent vectors. We shall show that the control points of the extension are uniquely determined by suitable conditions.

Theorem.

Six control vertices $\mathbf{w}_{i,j}$ $(i=1,\dots,2,\ j=1,\dots,3)$ *of the control net* $\mathbf{w}_{i,j}$ $(i=1,\dots,4,\ j=1,\dots,3)$ *determining a tensor product B–spline patch of degrees (3,2) over the rectangular domain* $(u,v)\in[0,1]\times[0,1]$ *are uniquely determined by the following boundary data: the endpoints* $(u=0)$ *of the cubic boundary curves* $v=0$ *and* $v=1$*, the tangent vectors of the same curves at the given endpoints, the cross directional tangent vector of the boundary curve* $u=0$ *at the corner point* $(u,v)=(0,0)$ *and the twist vector at the same corner point.*

Proof. Denote a single bordering patch $\mathbf{r}_{1,j}$ $(j=1,\dots,m-2)$ by $\mathbf{R}(u,v)$ and its control points by $\mathbf{w}_{ij}$ $(i=1,\dots,4,\ j=1,\dots,3)$. In order to make the description of the formulae simpler, we assume uniform parametrization, i. e. $t_i=i$ and $s_j=j$. In this case the coefficient matrices of the periodic B–spline functions are constant, and the parametric equation of a patch has the following form

$$\mathbf{R}(u,v)=\left[1\ u\ u^2\ u^3\right]\left[\mathbf{B}^{(3)}\right][\mathbf{W}]\left[\mathbf{B}^{(2)}\right]^{\mathrm{T}}\left[1\ v\ v^2\right]^{\mathrm{T}},\qquad (u,v)\in[0,1]\times[0,1], \tag{2}$$

where

$$\mathbf{B}_i^{(3)}(t)=\mathbf{B}^{(3)}=\frac{1}{6}\begin{bmatrix}1&4&1&0\\-3&0&3&0\\3&-6&3&0\\-1&3&-3&1\end{bmatrix},$$

$$\mathbf{B}_j^{(2)}(s)=\mathbf{B}^{(2)}=\frac{1}{2}\begin{bmatrix}1&1&0\\-2&2&0\\1&-2&1\end{bmatrix},$$

$$\mathbf{W}=\begin{bmatrix}\mathbf{w}_{11}&\mathbf{w}_{12}&\mathbf{w}_{13}\\\mathbf{w}_{21}&\mathbf{w}_{22}&\mathbf{w}_{23}\\\mathbf{w}_{31}&\mathbf{w}_{32}&\mathbf{w}_{33}\\\mathbf{w}_{41}&\mathbf{w}_{42}&\mathbf{w}_{43}\end{bmatrix}.$$

In this equation the six control vertices $\mathbf{w}_{1,j}$, $\mathbf{w}_{2,j}$ $(j=1,\dots,3)$ of the extension are unknown.

The usual technique to force a curve or a surface through a given point is to specify the point as a multiple control point with the multiplicity $d+1$ (d= degree) or to raise the multiplicity of the corresponding knot value accordingly. In this case the curve or the surface and also the derivatives are uniquely determined and no freedom is left for the shape control. In our representation the knot vectors are periodic and the multiplicity of the control vertices equals one, which enables us to prescribe additional boundary conditions besides the interpolation point **P**.

The assumptions that the boundary lines $\mathbf{R}(u,0)$ and $\mathbf{R}(u,1)$ end at the closing point **P** with the tangents $\mathbf{T}_0$ and $\mathbf{T}_1$, respectively, are expressed by the equations

$$\mathbf{P}=\mathbf{R}(0,0), \tag{3}$$

$$\mathbf{P}=\mathbf{R}(0,1), \tag{4}$$

$$\mathbf{T}_0=\frac{\partial}{\partial u}\mathbf{R}(u,v)\big|_{u=0,v=0}, \tag{5}$$

$$\mathbf{T}_1=\frac{\partial}{\partial u}\mathbf{R}(u,v)\big|_{u=0,v=1}. \tag{6}$$

The assumption that the boundary line $u=0$ shrinks to the point **P** implies that the patch degenerates into a triangular one, therefore we require that the vectors

$$\mathbf{T}_v=\frac{\partial}{\partial v}\mathbf{R}(u,v)\big|_{u=0,v=0}, \tag{7}$$

$$\mathbf{T}_{uv}=\frac{\partial^2}{\partial u\partial v}\mathbf{R}(u,v)\big|_{u=0,v=0} \tag{8}$$

will be set to zero.

In the equations (3)–(8) the vectors on the left hand sides are prescribed and the expressions on the right hand sides are linear in the control vertices. These expressions are easy to compute from equation (2). The unknown control vertices can be determined from the system of vector equations (3)–(8), and we get the following solution:

$$\begin{aligned}\mathbf{w}_{11}&=-2\mathbf{T}_0+\mathbf{T}_{uv}+\mathbf{w}_{31},\\\mathbf{w}_{12}&=-2\mathbf{T}_0-\mathbf{T}_{uv}+\mathbf{w}_{32},\\\mathbf{w}_{13}&=-4\mathbf{T}_1+2\mathbf{T}_0+\mathbf{T}_{uv}+\mathbf{w}_{33},\\\mathbf{w}_{21}&=\tfrac{1}{4}\left(6\mathbf{P}+2\mathbf{T}_0-\mathbf{T}_{uv}-3\mathbf{T}_v-2\mathbf{w}_{31}\right),\\\mathbf{w}_{22}&=\tfrac{1}{4}\left(6\mathbf{P}+2\mathbf{T}_0+\mathbf{T}_{uv}+3\mathbf{T}_v-2\mathbf{w}_{32}\right),\\\mathbf{w}_{23}&=\tfrac{1}{4}\left(6\mathbf{P}+4\mathbf{T}_1-2\mathbf{T}_0-\mathbf{T}_{uv}-3\mathbf{T}_v-2\mathbf{w}_{33}\right).\end{aligned} \tag{9}$$

These control points are uniquely determined by the prescribed boundary conditions and are called phantom points or pseudo vertices [4]. *QED*

The first extension of the method of control points to boundary control of surfaces is given in [5].

While generating the closing part of the pipe the control net of the boundary patches slide around in the (cross–sectional) v–direction for $j=1,\dots,m-2$. Although the phantom points $\mathbf{w}_{i1}$ and $\mathbf{w}_{i2}$ overwrite the phantom points $\mathbf{w}_{i2}$ and $\mathbf{w}_{i3}$ $(i=1,2)$ of the preceeding neighbouring patch, we shall show through the examples below that starting with coplanar tangent vectors $\mathbf{T}_{0,j}$ and $\mathbf{T}_{1,j}$, $(j=1,\dots,m-2)$, all the tangent vectors of the longitudinal u–parameter lines of the degenerate closing patches will lie in this plane, which is the tangent plane at the closing point **P**. Moreover, the boundary line $u=0$ shrinks to the point **P** with the tangent vector $\mathbf{T}_v\equiv\mathbf{0}$.

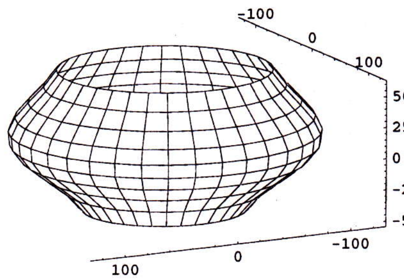

Fig. 1: Pipe

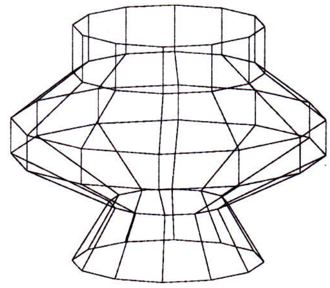

Fig. 2: Control net of the pipe

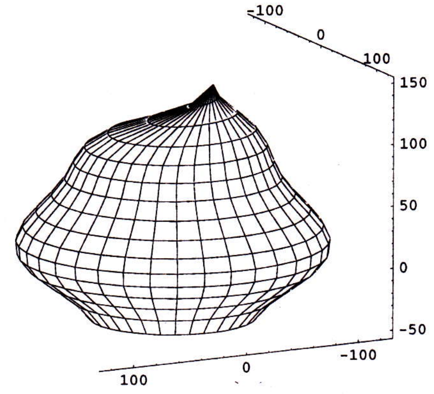

Fig. 3: Closing with fairing

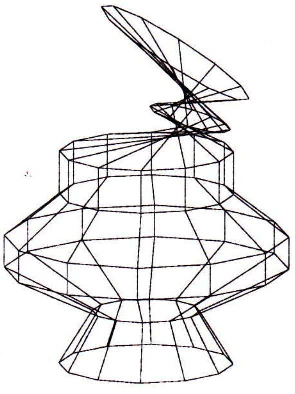

Fig. 4: Control net of the surface in Fig. 3

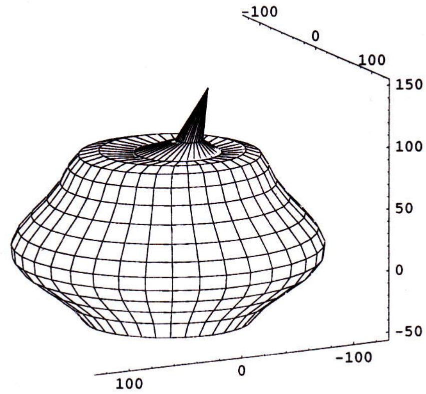

Fig. 5: Closing with a fixed common tangent

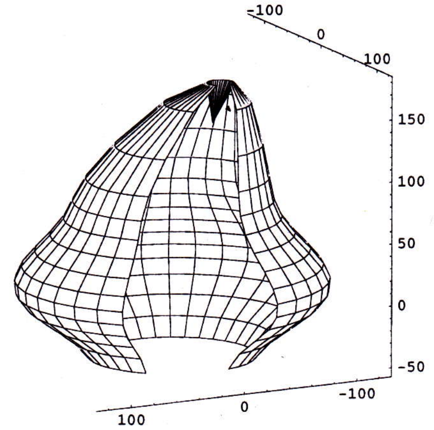

Fig. 6: Closing with reversed tangent

3 Examples

In the following examples the boundary data in (3)–(6) are user inputs, the derivatives in (7) and (8) are chosen to be zero. The closing parts of a given pipe are generated by the phantom points (9) obtained as the solution of the system of vector equations (3)–(8). Three surfaces are illustrated in Figures (3)–(6), which are generated in this way from the tube shown in Figure 1 determined by the control net shown in Figure 2. The tube is composed of 3×12 patches, each represented by 3×3 facets determined by 4×4 parameter lines. At the upper end of the tube a row of 12 rectangular and a row of 12 degenerate patches are computed from the same boundary data, the closing point **P**, the vector $\mathbf{T}_u = \mathbf{T}_0 = \mathbf{T}_1$ determining the tangent direction of the longitudinal parameter lines at **P** and the null vector for the initial value of $\mathbf{T}_v$ and $\mathbf{T}_{uv}$. In order to generate a non symmetric solution, the point **P** does not lie on the rotational axis of the pipe and the given direction is not parallel to the axis. The three closings differ in the magnitude and orientation of $\mathbf{T}_u$. In the first solution (Fig. 3) the magnitude of $\mathbf{T}_u$ is computed from a fairness condition for each bordering patch separately (see later). In the second solution (Fig. 5) the vector $\mathbf{T}_u$ is twice as long as the average in the first one. In the third example (Fig. 6) the prescribed tangent vector $\mathbf{T}_u$ is oriented in the reversed direction and has the same length as in the second example. Here two stripes of patches are not drawn in order to make the inwards turned peak **P** visible. The angular effect in the projections of the surfaces is due to the small number of the parameter lines.

The three examples illustrate the shaping effect of the orientation and magnitude of the boundary data $\mathbf{T}_u = \mathbf{T}_0 = \mathbf{T}_1$. The length of a tangent vector depends on the parametrization of the surface, which is hidden for the user. As it may cause non desired shaping effects, should be determined by the algorithm and not by the user. A well proved method for the solution of this problem is to compute the vector magnitudes from a fairness condition [6].

4 Fairing

We consider the magnitudes of the tangent vectors $\mathbf{T}_0$ and $\mathbf{T}_1$ of the longitudinal parameter lines at the closing point **P** as scalar parameters, and write $\lambda\mathbf{T}_0$ and $\mu\mathbf{T}_1$ in the equations (9) instead of $\mathbf{T}_0$ and $\mathbf{T}_1$, respectively. Then we compute the values of λ and μ from a fairness condition in order to achieve a satisfactorily smooth shape. This condition is that λ and μ have their values where an appropriate fairness functional is minimal.

A frequently used fairness functional is the area integral

$$A(\lambda,\mu) = \int_0^1\int_0^1 \left[\left(\frac{\partial^2}{\partial u^2}\mathbf{R}(u,v)\right)^2 + \left(\frac{\partial^2}{\partial v^2}\mathbf{R}(u,v)\right)^2\right] du\,dv, \qquad (10)$$

which approximates the energy function of a thin elastic plate [3]. This fairness functional has the advantage to be quadratic in the variables λ and μ, what keeps the optimization process simple.

As the phantom points of the closing patch $\mathbf{R}(u,v) \equiv \mathbf{r}_{1,j}(u,v)$ $(j = 1,\dots,m-2)$ influence also the neighbouring patch $\mathbf{r}_{2,j}(u,v)$, we summarize the area integral for these two patches in each step, and consider the following fairness functional:

$$F_j(\lambda,\mu) = \sum_{i=1,2}\int_0^1\int_0^1 \left[\left(\frac{\partial^2}{\partial u^2}\mathbf{r}_{i,j}(u,v)\right)^2 + \left(\frac{\partial^2}{\partial v^2}\mathbf{r}_{i,j}(u,v)\right)^2\right] du\,dv \qquad (11)$$

for the jth stripe while moving around the borderline of the pipe. If the functional F_j has a local minimum, then there

$$\frac{\partial F_j}{\partial \lambda} = 0, \qquad \frac{\partial F_j}{\partial \mu} = 0, \qquad (12)$$

which is a system of linear equations for λ and μ. It can be verified numerically that the solution of this system of equations is a local minimum of F_j.

Then we substitute λ and μ computed in this way into the equations (9) in order to compute the control net of the closing patch $\mathbf{r}_{1,j}(u,v)$.

We can observe in the following examples that the calculated surfaces have a round, smooth shape and satisfy the prescribed boundary conditions.

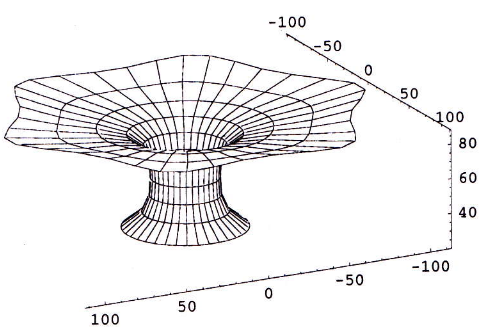

Fig. 7: Pipe

The given pipe is composed of 3×12 patches and is rotational symmetric in 6th order (Fig. 7). The closing point $\mathbf{P}$ is given on the rotational axis and the vectors $\mathbf{T}_0$ and $\mathbf{T}_1$ are orthogonal to the axis at $\mathbf{P}$. These vectors are generated for each closing patch by projecting the longitudinal tangent vectors of the given pipe at the corresponding points of the borderline onto the plane orthogonal to the rotational axis. The tangent plane of the closing part is prescribed by the 12 vectors $\mathbf{T}_0$ at the closing point in this way. Then we determine the magnitudes of these vectors by solving the system of the equations (12), and compute the control points of the extension by (9). The patches of the generated closing part (Fig. 8) have a collapsed edge at $\mathbf{P}$ with identically zero tangent vector $\mathbf{T}_v$. The tangent vectors of the longitudinal u–parameter lines at $\mathbf{P}$ are all coplanar lying in the prescribed tangent plane. The smooth round shape of the closing part can be seen in the front view (Fig. 9).

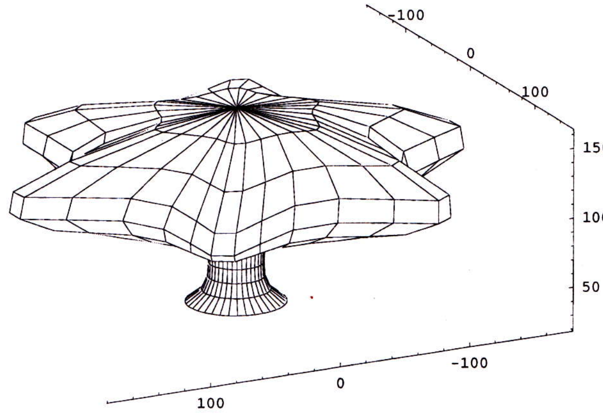

Fig. 8: Closing with fairing

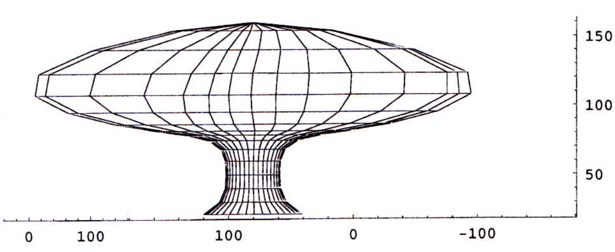

Fig. 9: Front view

This example, the solution in Figure 3 and several other generated surfaces show that the closing algorithm with the chosen fairing condition generates well shaped smooth surfaces. Other surface generation techniques have some disadvantages. The effect of multiple control points is shown in the next example (Fig. 10). This closing is generated without fairing with null vectors $\mathbf{T}_0$ at the closing point $\mathbf{P}$. The assumption $\lambda = \mu = 0$ implies that $\mathbf{w}_{1,j} = \mathbf{w}_{2,j}$ $(j = 1,\dots,m-2)$, and the surface has a slightly peaked shape (Fig. 11).

The fairing process presented above leads to automatic generation of the closing part if the closing point $\mathbf{P}$ is supposed to be moving, and its coordinates are considered as variables of the fairness functional. In the example shown in Figure 12 the lower end of the same pipe (Fig. 7) is closed in this way. The automatic closing of the first pipe (Fig. 1) is shown in Figure 13. In these examples the tangent plane at the closing point is orthogonal to the rotational axis of the

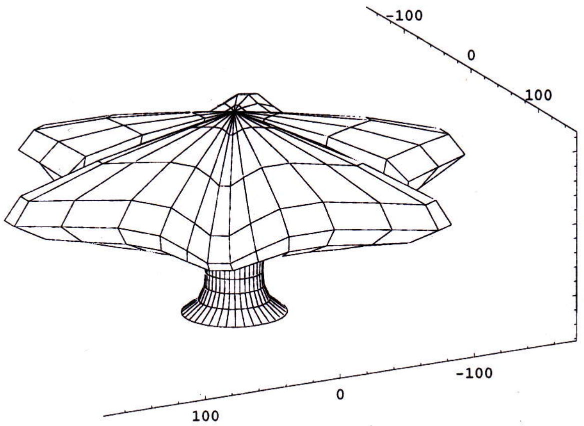

Fig. 10: Closing with null tangent vectors

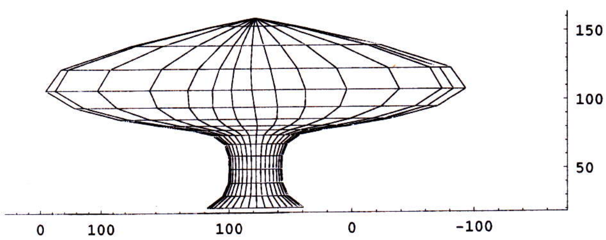

Fig. 11: Front view

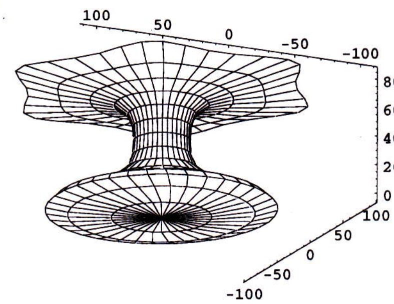

Fig. 12: Automatic closing of the pipe in Fig. 7

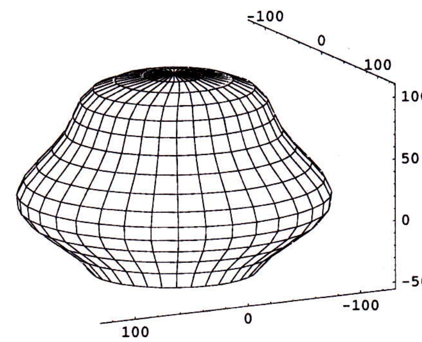

Fig. 13: Automatic closing of the pipe in Fig. 1

pipe. The prescribed tangent directions are then constructed in radial directions in this plane. The closing point is supposed to be moving along the axis. Its distance h from a fixed point of the axis is considered as a variable of the fairness functional. Then the minimization of $F(\lambda,\mu,h)$ gives a solution for the position of the closing point and for the unknown control points as well. In this case no user inputs are required.

Remark. The fairness functional

$$\int_0^1\int_0^1\left[\left(\frac{\partial^2}{\partial u^2}\mathbf{R}(u,v)\right)^2+2\left(\frac{\partial^2}{\partial u\partial v}\mathbf{R}(u,v)\right)^2+\left(\frac{\partial^2}{\partial v^2}\mathbf{R}(u,v)\right)^2\right]dudv \qquad (13)$$

has been also used in the fairness condition, and its effect has been compared with that of the functional in (10). Though there are differences in the numerical solutions, no differences can be observed on the generated surfaces. Therefore, those examples are not illustrated.

5 About the curvature entities

Consider a three–sided patch $\mathbf{R}(u,v)=x(u,v)\,\mathbf{i}+y(u,v)\,\mathbf{j}+z(u,v)\,\mathbf{k}$ of the generated closing part presented in a local coordinate system, the origin of which is in the singular vertex $\mathbf{P}(u=0, v\in[0,1])$ and the x,y axes lie in the tangent plane at this vertex (Fig. 14). Though the normal vector of this tangent plane is the null vector due to $\mathbf{T}_v=\frac{\partial}{\partial v}\mathbf{R}(u,v)\equiv\mathbf{0}$ $(u=0, v\in[0,1])$, the tangent plane of the surface considered as a point set does exist. The u–parameter lines of the patch end up at the singular vertex with coplanar tangent vectors, no pair of which are parallel. Under these conditions the technique of the so–called height function can be applied for the computation of the Gauß curvature at the singular point [8]. The height function $z=h(x,y)$ is defined in the neighbourhood of the singular point over the tangent plane. It is a single–valued, C^2–smooth uniquely defined function, and provides a local second order approximation of the degenerate surface at the singular point. The exact representation of the height function is not necessary for the computation of the Gauß curvature, only its second derivatives in the singular vertex.

By assumption, $h(0,0)=0$ and $h'_x(0,0)=h'_y(0,0)=0$, therefore the Gauß curvature at the origin is

$$K=h''_{xx}(0,0)\,h''_{yy}(0,0)-[h''_{xy}(0,0)]^2, \qquad (14)$$

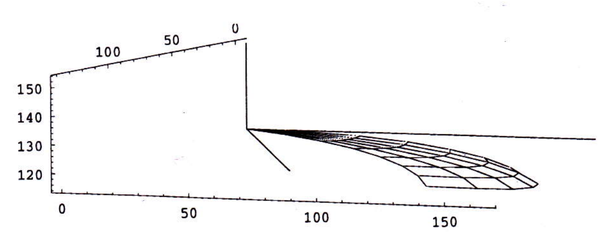

Fig. 14: Closing patch and the local coordinate system

and the normal curvature in a direction given by the unit vector (x_α, y_α) in the tangent plane is

$$\kappa_\alpha = h''_{xx}(0,0)x_\alpha^2 + 2h''_{xy}(0,0)x_\alpha y_\alpha + h''_{yy}(0,0)y_\alpha^2. \qquad (15)$$

Now we substitute the normal curvatures κ_α of the approximating surface $z = h(x,y)$ by the curvature at the point $(0,0)$ of the isoparametric surface curve $v = const$ of the surface $\mathbf{R}(u,v)$, which has the tangent direction (x_α, y_α). Choosing three u–parameter lines with pairwise linearly independent tangent directions, three linear equations can be written in this way for the three unknown values $h''_{xx}(0,0)$, $h''_{xy}(0,0)$ and $h''_{yy}(0,0)$. Solving the system of equations (15) we get the Gauß curvature of the surface $z = h(x,y)$. As stated in [8] and verified by our calculations, this Gauß curvature does not depend (with relatively large of 1 percent error) on the three chosen surface curves $v = const$ of the surface $\mathbf{R}(u,v)$, if a height function exists. The existence of the height function is ensured by the conditions

$$(\partial_u \psi(0,0) - \partial_u \psi(0,1)) \cdot \partial_u \partial_v \psi(0,v) \neq 0$$

and

$$\partial_u \det(\psi'(0,v)) \neq 0 \quad \text{for all} \quad v \in [0,1],$$

where $\psi(u,v) = (x(u,v), y(u,v))$ and $\det\psi'(u,v))$ is the determinant of the Jacobian of $\psi(u,v)$ [8]. These conditions can be checked easily for polynomial spline functions.

In our example (Fig. 8) the Monge representation of the surface does not exist, because the determinant of the Jacobian matrix of $(x(u,v), y(u,v))$ in the singular point is zero, but the conditions of the existence of the height function hold. Consequently, the Gauß curvature computed by the above described method can be defined as the Gauß curvature of the surface $\mathbf{R}(u,v)$ in the singular vertex. The result of the computation for the patches of the closing part in Figure 8 was zero. For other surfaces generated without fairing, where the tangent vectors $\mathbf{T}_u$ specified in radial directions in the tangent plane at the closing point are shorter then those computed from the fairness condition, the Gauß curvature was positive.

For the surface in Figure 10 the conditions of the existence of a height function do not hold, consequently, the Gauß curvature at the closing point does not exist.

6 Conclusions

The presented algorithm developed for a special extension of tube shaped surfaces works on tensor product B–spline surfaces of (3,2) degrees. The extension for closing the pipe at one end is composed from degenerate rectangular patches. Their control nets have been created by using the method of phantom (pseudo) vertices. These control points are computed from prescribed boundary conditions, therefore cannot be used for interactive shape control in the usual way. A fairness condition has been applied to avoid unconvenient user inputs, for example, specifying the magnitudes of tangent vectors. Also, an automatic closing of a pipe has been shown based on the minimization of a fairness functional. The given method can be extended without significant changes to rational B–spline surfaces with the restriction that the weights of the control points of the newly generated patches are fixed and the weights of the phantom points equal one.

The computations have been carried out with the help of Mathematica [7], the control nets have been generated by the modelling system of the author implemented on a 16 MB PC.

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