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Sphere Covering by Rational Quadratic Beziér Patches

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ABSTRACT.
A covering of the sphere in the three-dimensional Euclidean space is constructed consisting of three-sided and two-sided regions. Each region is represented as a rational quadratic Beziér patch over a triangular or a rectangular domain, respectively.
Keywords: Beziér patches, rational spline-functions

Prekrivanje kugle racionalnim kvadratičnim Beziérovim plohama
SAŽETAK
Prekrivanje kugle u trodimenzionalnom euklidskom prostoru konstruirano je tako da se sastoji od trostranih i dvostranih područja. Svako područje predstavljeno je po dijelovima racionalnim kvadratičnim Beziérovim plohama nad trokutastom ili pravokutnom domenom.
Ključne riječi: Beziérove plohe, racionalne splajn funkcije

INTRODUCTION

In the practice of geometric modelling a recurrent request is to describe different objects by a given collection of spline-functions. The exact representation of the three-dimensional sphere in the Euclidean space by quadratic rational spline-functions has been the subject of several papers in the CAD-literature. Unfortunately, the first publication on constructing rational Beziér sphere patches [5] contains an error [2], and the paper on the exact representation of a spherical cap by a single rational Beziér patch [1] is not easily available. So we have no information about the type of Beziér patches in that representation.
Our subdivision technique of the sphere is based on the investigation of quadrics given in [4], especially on the following theorem:
The rational quadratic Beziér triangle is a quadric if and only if all three extended boundary curves meet in one point of the quadric.
From this theorem it follows that the sphere can't be subdivided into triangular regions represented by rational quadratic Beziér patches. Therefore, patches of other type are also necessary to fill the gaps between the triangular patches.
An obvious construction of triangular regions satisfying the condition of the theorem is the following. Four given points on the sphere determine a tetrahedron. The planes of the faces around a fixed vertex cut the sphere in three circular arcs bordering an appropriate triangular region (Fig. 1).

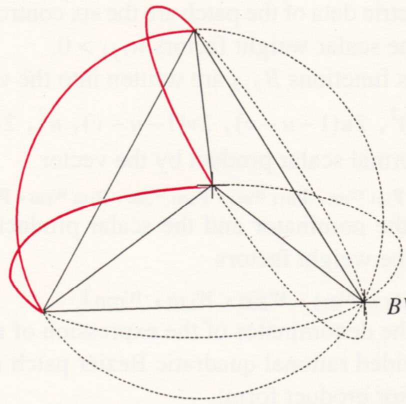

Fig.1: Appropriate triangular region of the sphere.

Four such triangular regions are determined by the tetrahedron. Between each two neighbouring triangles a two-sided gap arises. There are six such two-sided regions. Each of them will be covered by a degenerated four-sided Beziér patch (Fig.2).

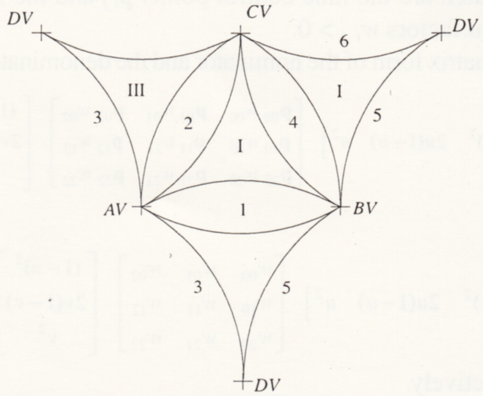

Fig.2: Subdivision of the sphere.

The computations of the geometric data of a two-sided patch have been carrried out by *Mathematica* [6], what is the new result of this paper.

MATHEMATICAL DESCRIPTION OF RATIONAL QUADRATIC PATCHES

The two-parametric vector equation of a rational quadratic Beziér patch over a triangular parameter domain has the form [3]:

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$$\mathbf{r}(u,v) = \sum_{\substack{i,j,k\ge 0\\ i+j+k=2}} \mathrm{p}_{ijk}\, w_{ijk}\, B_{ijk}(u,v,s) \;/\; \sum_{\substack{i,j,k\ge 0\\ i+j+k=2}} w_{ijk}\, B_{ijk}(u,v,s)$$

where u, v and s are the barycentric coordinates of a point in the triangular domain with respect to the vertices of the triangle, $u+v+s=1$ and $0 \le u,v,s \le 1$.

$$B_{ijk}(u,v,s) = \frac{2!}{i!\,j!\,k!}\, u^i v^j s^k$$

are the bivariate Bernstein basis functions of degree 2. The geometric data of the patch are the six control points $\mathbf{p}_{ijk}$ and the scalar weight factors $w_{ijk} > 0$.

If the basis functions B_{ijk} are written into the vector

$$((1-u-v)^2,\ 2u(1-u-v),\ 2v(1-u-v),\ u^2,\ 2uv,\ v^2),$$

then the formal scalar product by the vector

$$(\mathbf{p}_{002}\, w_{002},\ \mathbf{p}_{101}\, w_{101},\ \mathbf{p}_{011}\, w_{011},\ \mathbf{p}_{200}\, w_{200},\ \mathbf{p}_{110}\, w_{110},\ \mathbf{p}_{020}\, w_{020})$$

stands in the nominator and the scalar product by the vector of the weight factors

$$(w_{002},\ w_{101},\ w_{011},\ w_{200},\ w_{110},\ w_{020})$$

stands in the denominator of the expression of $\mathbf{r}\,(u,v)$.

The four-sided rational quadratic Beziér patch is given by the tensor product form:

$$\mathbf{r}(u,v) = \sum_{i,j=0}^{2} \mathbf{p}_{ij}\, w_{ij}\, B_i(u) B_j(v) \;/\; \sum_{i,j=0}^{2} w_{ij}\, B_i(u) B_j(v),$$

where $0 \le u,v \le 1$ and

$$B_i(u) = \frac{2!}{i!(2-i)!}\, u^i (1-u)^{2-i}, \quad (i = 0,1,2)$$

is the quadratic Bernstein basis. The geometric data of the patch are the nine control points $\mathbf{p}_{ij}$ and the scalar weight factors $w_{ij} > 0$.

The matrix form of the nominator and the denominator are

$$\begin{bmatrix}(1-u)^2 & 2u(1-u) & u^2\end{bmatrix} \begin{bmatrix} \mathbf{p}_{00}\, w_{00} & \mathbf{p}_{01}\, w_{01} & \mathbf{p}_{02}\, w_{02} \\ \mathbf{p}_{10}\, w_{10} & \mathbf{p}_{11}\, w_{11} & \mathbf{p}_{12}\, w_{12} \\ \mathbf{p}_{20}\, w_{20} & \mathbf{p}_{21}\, w_{21} & \mathbf{p}_{22}\, w_{22} \end{bmatrix} \begin{bmatrix} (1-v)^2 \\ 2v(1-v) \\ v^2 \end{bmatrix}$$

and

$$\begin{bmatrix}(1-u)^2 & 2u(1-u) & u^2\end{bmatrix} \begin{bmatrix} w_{00} & w_{01} & w_{02} \\ w_{10} & w_{11} & w_{12} \\ w_{20} & w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} (1-v)^2 \\ 2v(1-v) \\ v^2 \end{bmatrix},$$

respectively.

SPHERICAL PATCHES

Each boundary curve of a triangular patch is a segment of the circumscribed circle of a regular triangle. Such a circular arc of 60° can be represented as a quadratic rational Beziér curve determined by the control points P_0, P_1 and P_2 and the corresponding weights 1, 1/2 and 1 (Fig.3).

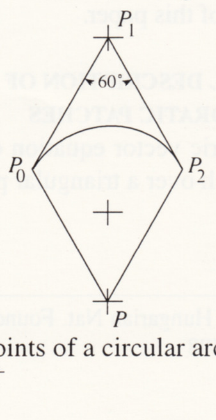

Fig.3: Control points of a circular arc.

Obviously, the point P_1 is the reflected lower vertex P of the triangle on the line of P_0 and P_2. The control points of the 4x3 boundary curves of the spherical patches can be computed from the vertices of the tetrahedron in a similar way by appropriate reflexions of the corresponding vertices. As the boundary curves of a Beziér patch are Beziér curves generated by the boundary control points of the triangular patch, all the geometric data of the triangular patches are determined by the four vertices *AV*, *BV*, *CV* and *DV* of the tetrahedron (Fig.4).

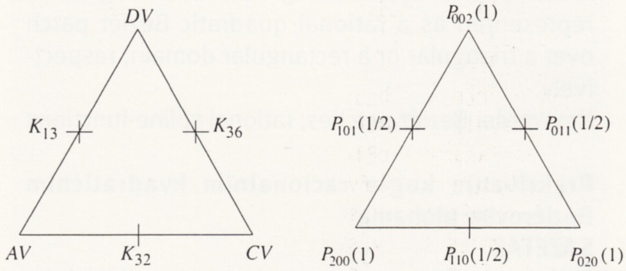

Fig.4: Control points and weights of a triangular patch.

The coordinates of the vertices of the tetrahedron and the control points are listed in Fig.5 in the input form of *Mathematica*.

```
AV={-1,1,-1}; BV={1,-1,-1}; CV={1,1,1}; DV={-1,-1,1};
K11={1,1,-3}; K21={0,0,-3}; K31={-1,-1,-3};
K12={1,3,-1}; K22={0,3,0}; K32={-1,3,1};
K13={-3,1,1}; K23={-3,0,0}; K33={-3,-1,-1};
K14={3,1,-1}; K24={3,0,0}; K34={3,-1,1};
K15={1,-3,1}; K25={0,-3,0}; K35={-1,-3,-1};
K16={1,-1,3}; K26={0,0,3}; K36={-1,1,3};
```

Fig.5. The coordinates of the control points.

The control points $K\,I\,J$, (I = 1,2,3 and J = 1...6) are numbered according to the two-sided patches, where the second index is the patchnumber. The circles building the patch boundaries on the sphere are drawn by *Mathematica* in Fig.6.

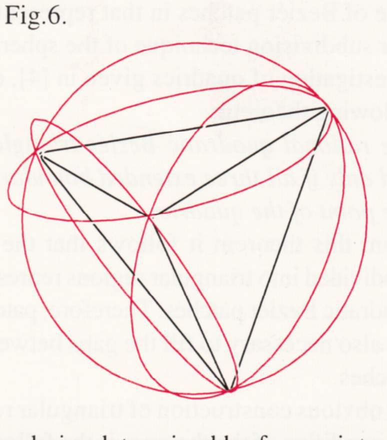

Fig.6: Patch boundaries determined by four points of the sphere.

The usual way of representing a triangular patch by parameter lines is drawing three sets of surface curves according to constant barycentric coordinates in the triangular parameter domain: $u = const$, $v = const$ and $s = 1 - u - v = const$. Unfortunately, *Mathematica* requires constant limits for the parameter values in the command ParametricPlot3D. Therefore, the triangular domain bordered by the lines $u = 0$, $v = 1$ and $u + v = 1$ has to be transformed into a rectangular domain for example by the parameter transformation $u = t - st$ and $v = st$, $(0 \le s,t \le 1)$. In our notation *bfig*3 [*u*_,*v*_], [*k*] ($k = 1,..,4$) is the two-parametric vector equation of a triangular Beziér patch (Fig.7).

```
Array[bfig3[u_,v_],4];
fu[s_,t_]:=t-s t;
fv[s_,t_]:=s t;
A[RGBColor[r_,g_,b_]]:=CMYKColor[0,r,r,1-r];
A[GrayLevel[x_]]:=GrayLevel[x];
triangs=ParametricPlot3D
   [{bfig3[fu[s,t],fv[s,t]][1],
   bfig3[fu[s,t],fv[s,t]][2],
   bfig3[fu[s,t],fv[s,t]][3],
   bfig3[fu[s,t],fv[s,t]][4]}
//Evaluate,{s,0,1},{t,0,1},PlotPoints->{12,12},
Boxed->False,Axes->None,
ColorOutput->A,Background->GrayLevel[0.8],
ViewPoint->{1,-3,.5}];
```

Fig. 7: Drawing command for the triangular patches.

The four patches drawn by *Mathematica* are shown in Fig. 8 together with the control net of one patch.

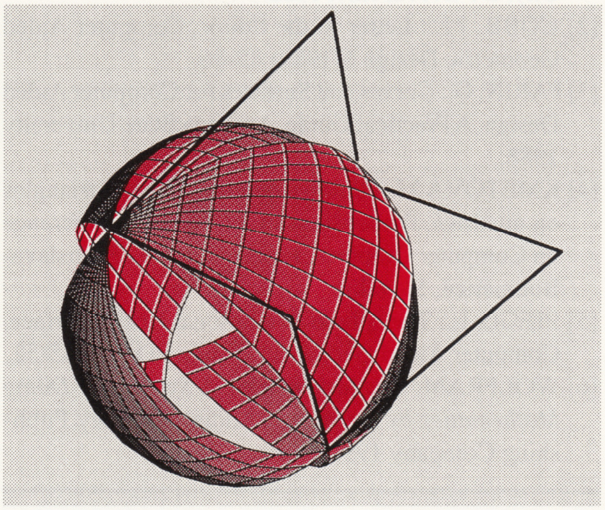

Fig. 8: Triangular patches drawn by *Mathematica*.

The two-sided patches can be considered as degenerated four-sided patches. The geometric data of such a patch are not determined by the two opposite boundary curves coinciding with the boundaries of the neighbouring triangular patches. In Fig.9 the degenerated control net of the 2nd patch and the corresponding Beziér control net over a rectangular parameter domain are shown. The unknown data are the geometric data of the middle longitudinal parameter curve, namely, the control point P_{11} and its weight w, moreover the weights of the points P_{10} and P_{12} coinciding with the vertex CV and AV, respectively. For symmetry reasons these weights are equal, denoted by c.

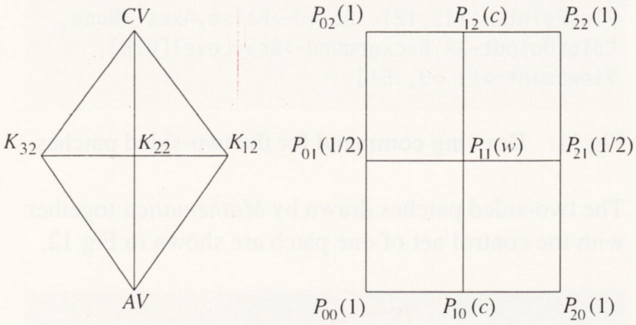

Fig.9: Geometric data of a degenerated two-sided patch.

These unknown data in the vector equation of the rectangular Beziér patch can be determined from the condition that any point of the generated patch must lie on the sphere. After substituting the coordinates of an arbitrary point of the patch into the equation of the sphere (with center point in the origin and radius=3) and reordering the equation, the coefficients of the terms $u^i v^j$ ($i = 0..4$, $v = 0..4$) can be collected with *Mathematica*. The condition that each coefficient equals to zero leads to a system of equations for the unknowns. However, the number of equations is greater than the number of unknowns, and the equations are not linear, the solution

$$P_{11} = (0,3,0), \quad w = \sqrt{3}/3, \quad c = \sqrt{3}/2$$

can be found interactively by *Mathematica*. After that it can be verified easily that an arbitrary point of the generated patch by these data lies on the sphere indeed. The control net of a two-sided patch is shown in Fig.10.

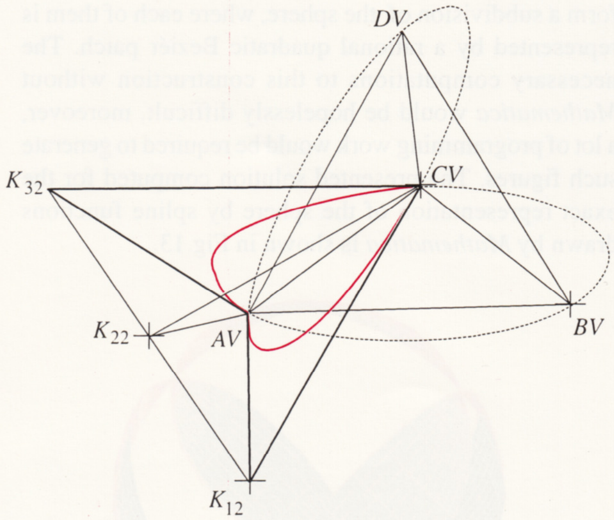

Fig.10: The control net of a two-sided patch.

The drawing command for the six patches is written in Fig.11, where bfig2 [u_,v_] [k], (k = 1,..,6) denotes the two-parametric vector equation of a rational quadratic Beziér patch over a rectangular domain.

```
Array[bfig2[u_,v_],6];
monds=
ParametricPlot3D[{bfig2[u,v][1],bfig2[u,v][2],
   bfig2[u,v][3],bfig2[u,v][4],
   bfig2[u,v][5],bfig2[u,v][6]}
//Evaluate,{u,0,1},{v,0,1},
PlotPoints->{12,12}, Boxed->False,Axes->None,
ColorOutput->A,Background->GrayLevel[0.8],
ViewPoint->{1,-3,.5}]
```

Fig.11: Drawing command for the two-sided patches.

The two-sided patches drawn by *Mathematica* together with the control net of one patch are shown in Fig.12.

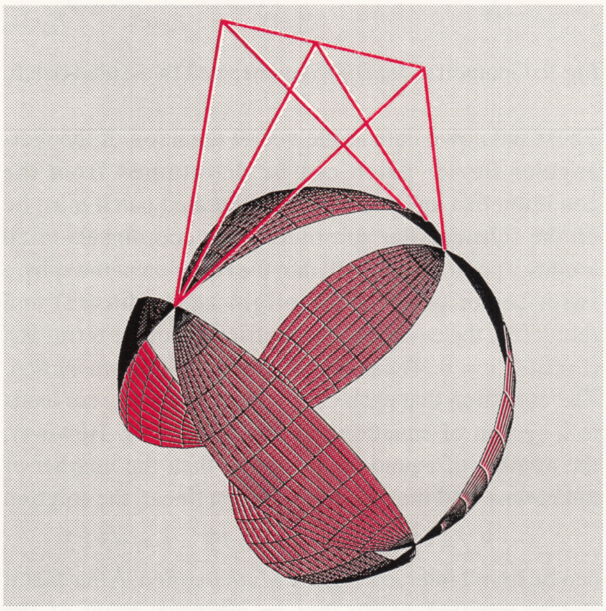

Fig.12: Two-sided patches drawn by *Mathematica*.

Finally, the four triangular and six two-sided patches form a subdivision of the sphere, where each of them is represented by a rational quadratic Beziér patch. The necessary computations to this construction without *Mathematica* would be hopelessly difficult, moreover, a lot of programming work would be required to generate such figures. The presented solution computed for the exact representation of the sphere by spline functions drawn by *Mathematica* is shown in Fig.13.

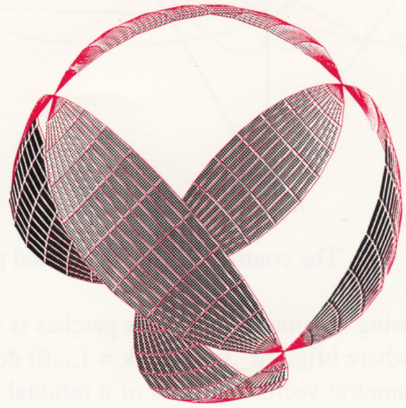

Fig. 13 a

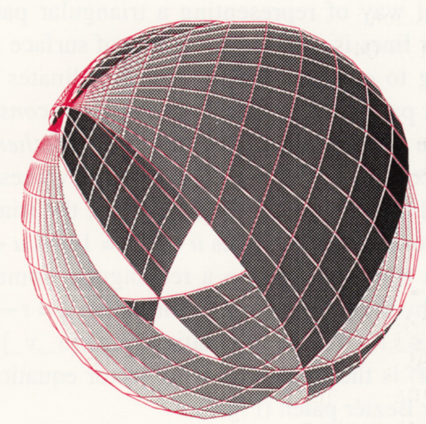

Fig. 13 b

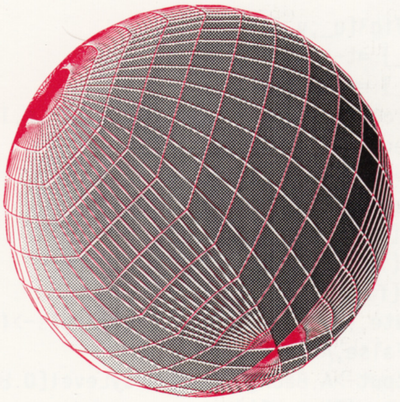

Fig. 13 c

Fig.13: The covering of the sphere.

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