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Locus Curves in Triangle Families

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ABSTRACT

In this article, we observe a one-parameter triangle family, where two vertices are fixed and the third vertex lies on a given line. For this family of triangles, we observe the loci of centroids, orthocenters, circumcenters, incenters, excenters and some triangle elements associated to these triangle points.

Key words: family of triangles, centroid, orthocenter, circumcenter, incenter, excenter

MSC2010: 51M04, 51N20, 51M15

Lokus krivulje u familijama trokuta

SAŽETAK

U ovom članku proučava se jednaparametarska familija trokuta kojemu su dva vrha fiksna, a treći vrh leži na zadanom pravcu. Za takvu familiju trokuta promatraju se geometrijska mjesta težišta, ortocentara, središta opisanih kružnica, središta upisanih i pripisanih kružnica te nekih elemenata vezanih za te karakteristične točke trokuta.

Ključne riječi: familija trokuta, težište, ortocentar, središte opisane kružnice, središte upisane kružnice, središte pripisane kružnice

1 Introduction

The evergrowing field of triangle geometry can be systematically researched in [8] where we can find numerous points, lines, circles, curves, objects etc. associated to a triangle. If we observe a set of triangles somehow connected, i.e., a one-parameter family of triangles or a pencil of triangles, then the locus of a certain triangle center of those triangles lies on a curve. In the same way, we can observe what will be the locus of a certain line, or other object associated to triangles in a triangle family. Some results in this area, especially for the Euclidean plane can be found in [1, 2, 4, 9, 10, 12], while [5, 6, 7] deal with the situation in the isotropic plane. This paper contains a family of triangles whose basic elements are dual to the family of triangles in [4] but the resulting locus curves are different.

We will define the triangle family τ as it follows:
Let A and B be two different fixed points and let p be a fixed line. We study one-parameter family τ of triangles $\triangle ABC_i$ such that the point C_i lies on the line p.

$$\tau = \{\triangle ABC_i : C_i \in p\}, \quad i \in \mathbb{R} \cup \{\infty\}.$$

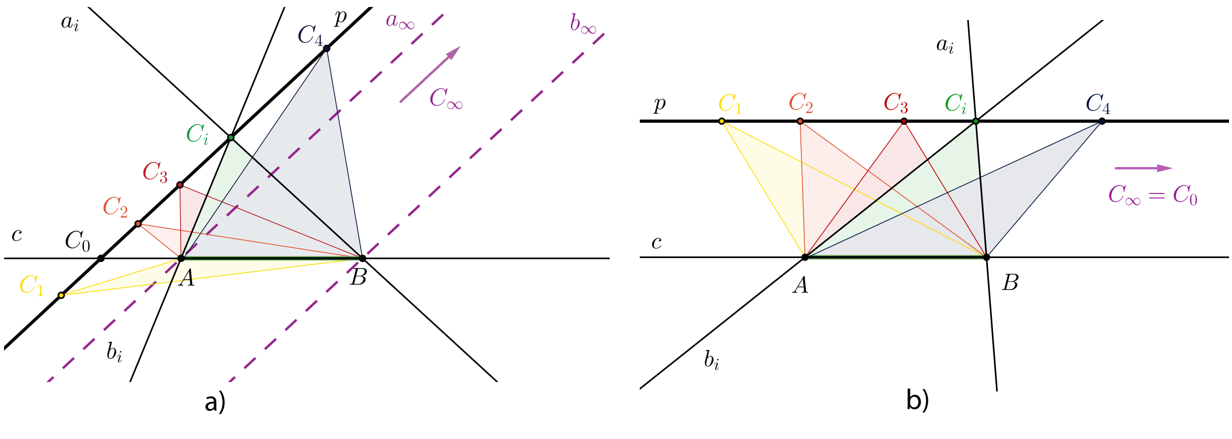

Figure 1: *The family of triangles $\triangle ABC_i$ where $C_i \in p$: a) p in general position, b) $p \parallel c$.*

We will use the following notation

$$
\begin{aligned}
a_i &= BC_i, \quad b_i = AC_i,\, i \in \mathbb{R} \cup \{\infty\}, \\
c &= AB, \quad C_0 = p \cap c,
\end{aligned}
\tag{1}
$$

and the ideal point on the line p will be denoted with C_∞ (see Fig. 1).

From the view point of projective geometry, in this structure we view a line p as a range of points (C_i), $i \in \mathbb{R} \cup \{\infty\}$, the points A and B as vertices of pencils of lines a_i and b_i, $i \in \mathbb{R} \cup \{\infty\}$, which contain the triangle sides $\overline{BC_i}$ and $\overline{AC_i}$ of the triangles $\triangle ABC_i$ in the family τ. The pencils will be denoted with (A) and (B).

Fig. 1 shows two different families of triangles:

- (case a) when the line p is in an arbitrary position to the line c,
- (case b) when the line p and c are parallel.

In every family, we have two special triangles that are degenerate which occur when $i = \{0, \infty\}$:

- (case 0) $\triangle ABC_0$, $C_0 = p \cap c$, i.e., when vertices of the triangle are collinear and the triangle degenerates to a line segment,
- (case ∞) $\triangle ABC_\infty$, where C_∞ is the ideal point of the line p, i.e., when a triangle has a vertex at infinity.

If the line p is parallel to the line c (case b) then the degenerate triangles coincide, i.e., $C_0 = C_\infty$ (see Fig. 1b).

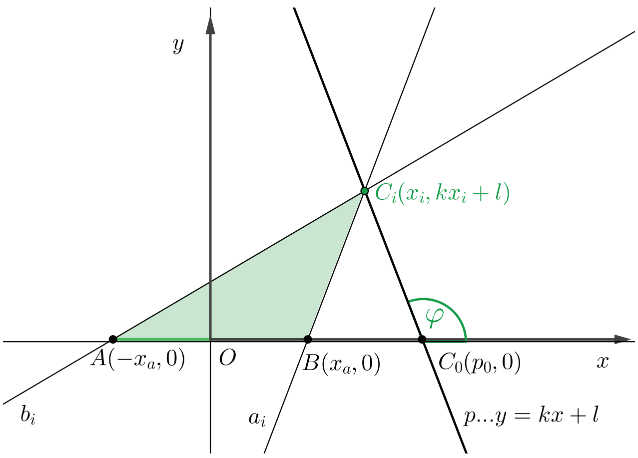

Figure 2: *The family of triangles τ positioned in the coordinate system.*

For all analytic treatment, we will put the coordinate system such that the x-axis is the line c and the points A and B are symmetric regarding to the origin of the coordinate system (see Fig. 2), i.e., we will put the family of triangles τ in the coordinate system as follows:

$$
\begin{aligned}
&A = (-x_a, 0), \quad B = (x_a, 0), \\
&c \ldots y = 0, \quad p \ldots y = kx + l, \\
&C_i = (x_i, kx_i + l), \quad C_0 = (p_0, 0), \quad p_0 = -\frac{l}{k} \\
&a_i \ldots y = \frac{kx_i + l}{x_i - x_a}(x - x_a), \quad b_i \ldots y = \frac{kx_i + l}{x_i + x_a}(x + x_a).
\end{aligned}
\tag{2}
$$

2 The locus of centroids

Lemma 1 *The midpoints M_{a_i} and M_{b_i} of the corresponding triangle sides $\overline{BC_i}$ and $\overline{AC_i}$ of the triangle family τ lie on lines parallel to the line p.*

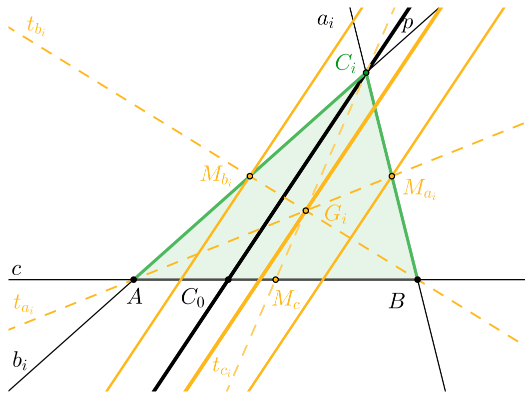

Figure 3: *The locus of midpoints and centroids are parallel lines.*

Lemma 1 is an immediate consequence of the Intercept Theorem.

From the view point of projective geometry, the range of points C_i is in perspectivity with ranges of points M_{a_i} and M_{b_i} where the centers of perspectivity are the points A and B

$$(C_i) \overset{=}{\barwedge} (M_{a_i}), \quad (C_i) \overset{=}{\barwedge} (M_{b_i}).$$

Theorem 1 *The triangle centroids G_i of the triangle family τ lie on a line parallel to the line p.*

Proof. The points A and B are fixed, hence the midpoint M_c of the triangle side $\overline{AB}$ is fixed. Since the centroid divides the segment $\overline{M_c C_i}$ in ratio 1:2, the locus of all points G_i is a line parallel to the line p. $\square$

From (2) the equations of the triangle medians are

$$
\begin{aligned}
t_{a_i} \ldots y &= \frac{kx_i + l}{x_i + 3x_a}(x + x_a), \\
t_{b_i} \ldots y &= \frac{kx_i + l}{x_i - 3x_a}(x - x_a), \\
t_{c_i} \ldots y &= \frac{kx_i + l}{x_i}(x - x_i).
\end{aligned}
\tag{3}
$$

Hence, the coordinates of the centroids G_i are

$$G_i = \left(\frac{x_i}{3}, \frac{kx_i + l}{3}\right), \tag{4}$$

and satisfy the following equation of a line

$$\mathcal{G}_i \ldots y = kx + \frac{l}{3} \tag{5}$$

From the view point of projective geometry, the correspondence between the pencils (A) and (B) is established so that for every triangle $\triangle ABC_i$ of the family τ the median t_{a_i} corresponds to the median t_{b_i}. This is a 1-1 correspondence, but in the degenerate triangle $\triangle ABC_0$ the medians coincide and the centroid G_0 can be interpreted as line c. Therefore, we have a projectivity between the pencils (A) and (B) for which the product degenerates to two lines, the one with the locus of all centroids of the triangle family τ and the line c of the degenerate case 0.

3 The locus of orthocenters

Theorem 2 *The orthocenters H_i of the triangle family τ lie on a conic, which is a hyperbola or parabola or degenerates to two lines.*

Proof. From (2) we can calculate the equations of the triangle altitudes

$$v_{a_i} \ldots y = \frac{x_a - x_i}{kx_i + l}(x + x_a),$$
$$v_{b_i} \ldots y = -\frac{x_a + x_i}{kx_i + l}(x - x_a), \tag{6}$$
$$v_{c_i} \ldots x = x_i,$$

and they are lines of the pencils (A), (B) and a pencil of parallel lines orthogonal to the line c, respectively. The coordinates of the orthocenters H_i are

$$H_i = \left(x_i, \frac{x_a^2 - x_i^2}{kx_i + l}\right), \tag{7}$$

and satisfy the following equation

$$\mathcal{H}_i \ldots x^2 + kxy + ly - x_a^2 = 0 \tag{8}$$

which is an equation of a conic. A conic is degenerate if the coefficient matrix $(c_{ij}), i, j \in \{0, 1, 2\}$ of its homogeneous equation is singular, i.e, the determinant of (c_{ij}) equals zero.
In our case, this yields

$$\begin{vmatrix} 1 & \frac{k}{2} & 0 \\ \frac{k}{2} & 0 & \frac{l}{2} \\ 0 & \frac{l}{2} & -x_a^2 \end{vmatrix} = -\frac{l^2}{4} + \frac{k^2}{4}x_a^2 = 0, \tag{9}$$

$$p_0 = -\frac{l}{k} \Longrightarrow x_a = \pm p_0.$$

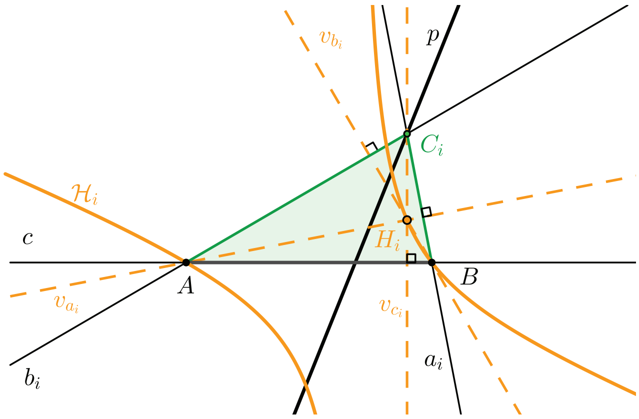

Figure 4: *The locus of orthocenters is a hyperbola when p is in general position.*

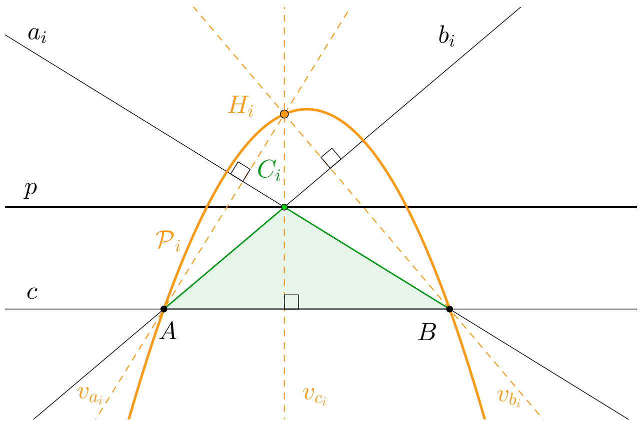

Figure 5: *The locus of orthocenters is a parabola when $p \parallel c$.*

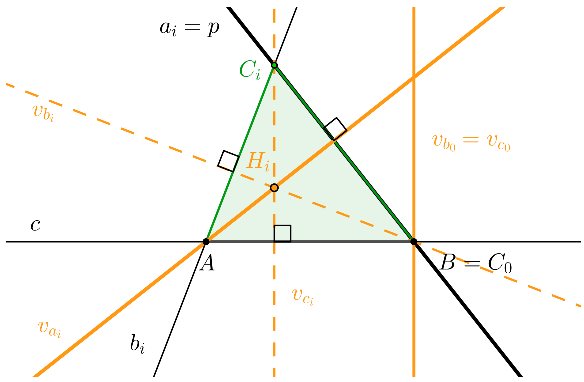

Figure 6: *The locus of orthocenters are two lines when $C_0 = A$.*

From this and (2) follows that the conic degenerates to two lines if the line p intersects c at point A or B, i.e., $C_0 = A$ or $C_0 = B$ (see Fig. 6).

The matrix H associated to the quadratic form of the conic (8) is

$$H = \begin{pmatrix} 1 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} \tag{10}$$

and its determinant is

$$\det H = -\frac{k^2}{4}, \tag{11}$$

wherefrom we can read off the affine type of the conic $\mathcal{H}_i$ ([3], p.20).
The determinant of the conic can never be positive, so $\mathcal{H}_i$ cannot be an ellipse. For $k = 0$, we have $\det H = 0$ and the line p is parallel to the line c and the conic is a parabola (see Fig. 5). From (8), we can derive that the equation of the parabola is

$$\mathcal{P}_i \dots y = \frac{-x^2 + x_a^2}{l}.$$

For $k \neq 0$, the determinant is always negative, and therefore, $\mathcal{H}_i$ is a hyperbola. □

From the view point of projective geometry, for every line in the pencil (A) there is a unique triangle $\triangle ABC_i$ where that line is the altitude v_{a_i} which corresponds to one line from the pencil (B) which is the altitude v_{b_i}. This is an 1-1 correspondence between these two projective pencils, hence the locus of orthocenters is a conic. For the degenerate triangle $\triangle ABC_0$, the altitudes are orthogonal to the line c, hence they are parallel and the orthocenter H_0 is at infinity. For the degenerate triangle $\triangle ABC_\infty$ the altitudes are orthogonal to the line p, thus the orthocenter H_∞ is also an ideal point. Therefore the conic has two different real points at infinity, hence is a hyperbola where from the degenerate triangles, we can conclude the directions of the asymptotes.
In the case $p \parallel c$, $C_0 = C_\infty$ and $H_0 = H_\infty$, the conic is a parabola. In this case, we can also conclude, that the infinite point of the conic is the infinite point of the line orthogonal to the line AB so that the axis of the parabola will also be orthogonal to the line AB.
In the case $p \cap c = \{A, B\}$, the altitudes v_{a_i} (or v_{b_i}) coincide if $i \neq 0$. For $i = 0$, the altitudes of the degenerate triangle $\triangle ABC_0$ are parallel lines orthogonal to the line c whereby altitudes v_{a_0} and v_{c_0} (or v_{b_0} and v_{c_0}) coincide.

Lemma 2 *The intersection N_{a_i} and N_{b_i} of the triangle altitudes v_{a_i} and v_{b_i} with the triangle sides $\overline{BC_i}$ and $\overline{AC_i}$, respectively, of the triangle family τ lie on a circle whose diameter is the line segment $\overline{AB}$.*

The Lemma 2 is an immediate consequence of the Thales's theorem.
The equation of the circle k is

$$k \dots x^2 + y^2 = x_a^2$$

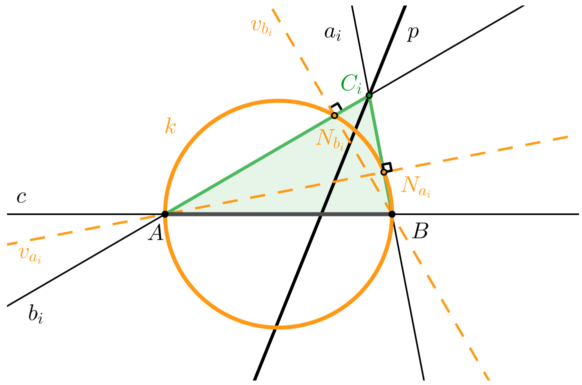

Figure 7: *The locus of $v_{a_i} \cap \overline{AC_i}$ and $v_{b_i} \cap \overline{BC_i}$ is a circle.*

4 The locus of circumcenters

Theorem 3 *The circumcenters O_i for the triangle family τ lie on a line.*

Proof. The circumcenter of a triangle is the intersection of the bisectors of the triangle sides and since all the triangles $\triangle ABC_i$ of the triangle family τ share the same side $\overline{AB}$ the bisector of that side for every triangle is always the same line s_c. Therefore the circumcenters O_i for the triangle family τ lie on it. □

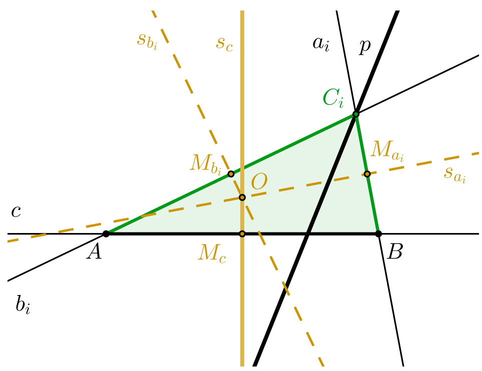

Figure 8: *The locus of circumcenters is the bisector of s_c.*

Theorem 4 *The bisectors s_{a_i} and s_{b_i} of the triangle sides $\overline{BC_i}$ and $\overline{AC_i}$, respectively, in the triangle family τ envelope parabolas. The line p is the common directrix of the parabolas and the points B and A are the foci of the parabolas, respectively.*

Proof. Let P_i be the intersection point of the bisector s_{a_i} and the line p, and S_{a_i} the intersection point of the orthogonal line from the points C_i to the line p and the bisector s_{a_i} (see Fig. 9). It follows that

$$\triangle P_i S_{a_i} C_i \cong \triangle P_i S_{a_i} B$$

because two sides of triangles and the angle between them are congruent. This is valid for every triangle $\triangle ABC_i$ in the triangle family τ, thus from

$$|\overline{C_iS_{a_i}}| = |\overline{BS_{a_i}}|, \quad \angle C_iS_{a_i}P_i = \angle P_iS_{a_i}B$$

we can conclude that the bisector s_{a_i} will be the tangent line with the contact point S_{a_i} of a parabola whose focus is the point B and directrix p. We can argue, analogously, for the other envelope corresponding to the point A. □

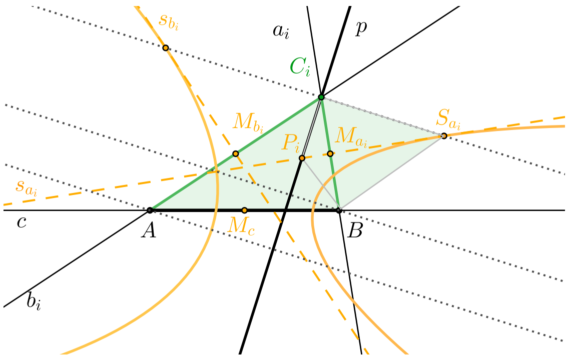

Figure 9: *The locus of bisectors of s_{a_i} and s_{b_i} envelop parabolas.*

From here, we can conclude that the axis of parabolas will be orthogonal to the line p and passing through vertices A or B, respectively. This property of a tangent to a parabola can be found in ([3], p. 31-32), from where we can also conclude that the locus of midpoints M_{a_i} and M_{b_i} of the triangle sides of the family τ from Lemma 1 are the tangent lines to parabolas at the vertex of the parabola.

Since the bisectors of a triangle side do not depend on the opposite vertex, we can choose the line p to be vertical

$$p \dots x = l.$$

Then the equations of the parabolas are

$$y^2 = 2(l \pm x_a)x.$$

5 The locus of incenters and excenters

Theorem 5 *The incenters I_i and excenters I_{a_i}, I_{b_i}, I_{c_i} of the triangle family τ lie on a cubic or degenerate cubic which can be the union of a hyperbola and a straight line, or even the union of three lines.*

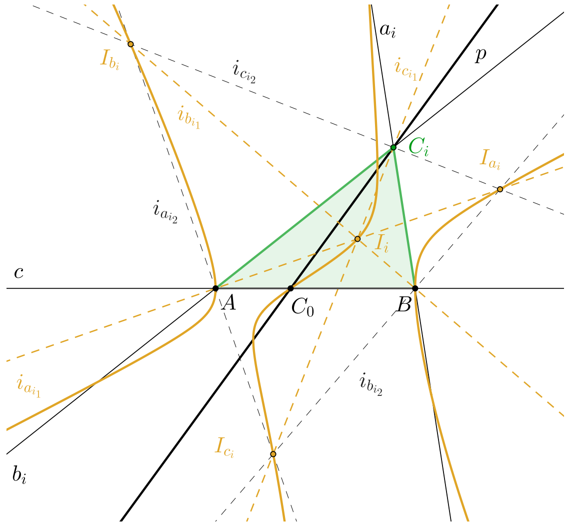

Figure 10: *The locus of incenters and excenters when p intersects $\overline{AB}$ in an interior point.*

For the triangle $\triangle ABC_i$, the angle bisectors at the vertex A will be denoted with $i_{a_{i_1}}$ (interior bisector) and $i_{a_{i_2}}$ (exterior bisector). Analogously, we denote the angle bisectors $i_{b_{i_1}}$, $i_{b_{i_2}}$ at vertex B and angle bisectors $i_{c_{i_1}}$, $i_{c_{i_2}}$ at vertex C_i (see Fig. 10).

Proof. From the view point of projective geometry, every line of the pencil (A) corresponds to two lines of the pencil (B), i.e., to every line of the pencil (A) which is an angle bisector at vertex A correspond two angle bisectors (interior and exterior) at vertex B, and vice versa. Three out of these four points of intersection are different points, i.e., one of them is counted twice, therefore the result of these two projective pencils is a curve of degree three.

We can also view it in this way: for every triangle $\triangle ABC_i$, the angle bisectors $i_{a_{i_1}}$, $i_{a_{i_2}}$, $i_{b_{i_1}}$, $i_{b_{i_2}}$ determine a quadrilateral whose vertices and two diagonal points are points A, B, I_i, I_{a_i}, I_{b_i}, and I_{c_i}. Diagonals of the quadrilateral are the line c and the angle bisectors $i_{c_{i_1}}$, $i_{c_{i_2}}$.

If the line p is the bisector s_c of the line segment $\overline{AB}$, then the family τ is a family of isosceles triangles (see Fig. 11). The intersection of bisectors $i_{a_{i_1}}$, $i_{b_{i_1}}$ and $i_{a_{i_2}}$, $i_{b_{i_2}}$, i.e., the incenters I_i and the excenters I_{c_i} lie on the bisector s_c. To any line from the (A) which is an interior angle bisector $i_{a_{i_1}}$ there exists a corresponding line from the pencil (B) which is an exterior angle bisector $i_{b_{i_2}}$, and vice versa. The intersection points are excenters I_{a_i} and I_{b_i}. Hence, from the latter we have a 1-1 correspondence and the set of intersection points is a conic. For the degenerate triangle $\triangle ABC_\infty$, the angle bisector $i_{a_{\infty_1}}$ is parallel to the angle bisector $i_{b_{\infty_2}}$, and vice versa, hence the conic is a hyperbola. Note that these two points of the locus curve of incenters and excenters are at infinity. This is also true in the case of an arbitrary line p (see Fig. 13). The third diagonal point of the aforementioned quadrilateral is the point $C_0 = M_c$.

If the line p is incident with the point A (or B), then the sides $\overline{AC_i}$ (or $\overline{BC_i}$) of triangles in the family τ lie on the line p. Therefore, for every triangle $\triangle ABC_i$ the angle bisectors at the vertex A (or B) are always the same two lines i_{a_1} and i_{a_2} (or i_{b_1}, i_{b_2}) which is the part of the degenerate cubic. The third line of the degenerate cubic is the exterior angle bisector $i_{b_{0_2}}$ at the vertex B for the degenerate triangle ABC_0, respectively $i_{a_{0_2}}$ if the line p is incident with the point B (see Fig. 12). $\square$

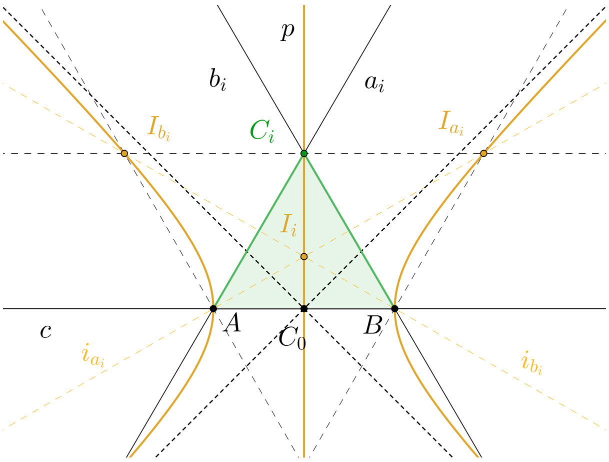

Figure 11: *The locus of incenters and excenters if* $p = s_c$.

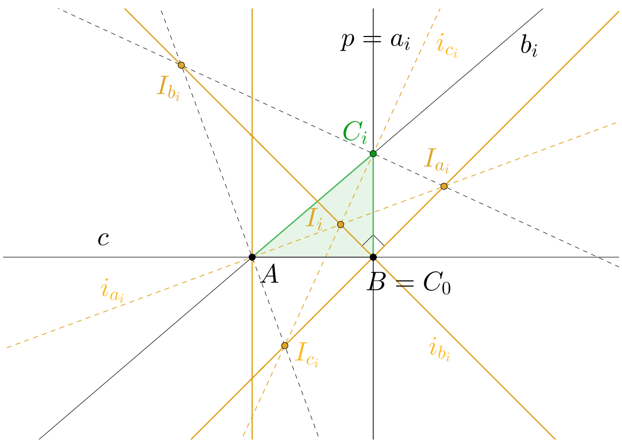

Figure 12: *The locus of incenters and excenters consists of three lines if* $C_0 = B$.

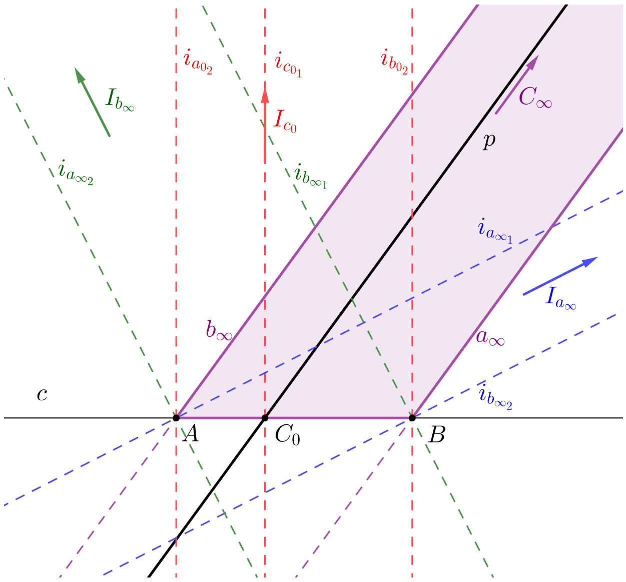

Figure 13: *The directions of the asymptotes of the cubic.*

We can distinguish these cases for the initial elements:

- (case a) The line p is in general position relative to the line AB when the intersection point $C_0 = p \cap AB$ lies between the points A and B.
- (case b) The line p is in general position to the line AB when the intersection point $C_0 = p \cap AB$ lies outside the segment $\overline{AB}$.
- (case c) The line p is parallel to AB.
- (case d) The line p is the bisector of segment $\overline{AB}$.
- (case e) The line p passes through either A or B.

For the first three cases, the locus curve is a cubic. For the last two, according to Theorem 5, the curve degenerates.
In (case a), for the degenerate triangle $\triangle ABC_0$, the bisectors $i_{a_{0_2}}$, $i_{b_{0_2}}$, and $i_{c_{0_1}}$ are parallel lines which are orthogonal to the line AB. Therefore one asymptote of the locus curve of the incenters and the excenters is orthogonal to the line AB. The other two directions of the asymptotes we can deduce from the triangle $\triangle ABC_\infty$ where two of the triangle excenters are at infinity whereas the angle bisectors are parallel as stated in the proof of Theorem 5 (see Fig. 12).
In (case b), for the degenerate triangle $\triangle ABC_0$, the bisectors $i_{a_{0_2}}$, $i_{b_{0_1}}$, and $i_{c_{0_2}}$ are parallel but the way to find the direction of the asymptotes is the same as in (case a).
We will now derive the equation of the curve.
The general equation of a cubic is

$$\begin{aligned} P(x,y) =& Ax^3 + Bx^2y + Cxy^2 + Dy^3 \\ & + Ex^2 + Fxy + Gy^2 + Hx + Ky + L = 0 \end{aligned} \qquad (12)$$

where we will denote the cubic homogeneous part as

$$P_3(x,y) = Ax^3 + Bx^2y + Cxy^2 + Dy^3 \tag{13}$$

and the quadratic part as

$$P_2(x,y) = Ex^2 + Fxy + Gy^2. \tag{14}$$

Let $k_1, -\frac{1}{k_1}$ be the slopes of the angle bisectors between the lines p and c, $k_1 = \tan\frac{\varphi}{2}$, and $p_0 = -\frac{l}{k}$ (see Fig. 2) hence the asymptotes of the cubic are

$$x = p_0,\ y = k_1x + l_1,\ y = -\frac{1}{k_1}x + l_2. \tag{15}$$

Now one can use results from [11] and [13] regarding asymptotes of algebraic curves and their relations to the equation of the curve. In [11] we find relations between linear factors of the highest degree homogeneous part of the equation and equations of asymptotes as follows in our case of degree 3 polynomial P:

- a linear factor $(ax+by)$ is a simple factor of $P_3(x,y)$ if $P_3(x,y) = (ax+by)Q_3(x,y)$ where $Q_3(b,-a) \neq 0$, with
$$P_3(x,y) = (ax+by)Q_3(x,y)$$
and to this simple factor is associated the single asymptote to $P(x,y) = 0$ given by
$$(ax+by)Q_3(b,-a) + P_2(b,-a) = 0. \tag{16}$$

From (15) we know the three linear factors of P_3 to be as follows, with respect to (18):

$$\begin{aligned} P_3(x,y) &= \frac{1}{k_1}x(y-k_1x)(k_1y+x) \\ &= -x^3 + \left(\frac{1}{k_1} - k_1\right)yx^2 + xy^2 \end{aligned}$$

and it follows that

$$A = -1, \quad B = \left(\frac{1}{k_1} - k_1\right), \quad C = 1, \quad D = 0. \tag{17}$$

If we include in consideration the three intersections of the curve with the x-axis, which tells us that the equation (12) contains the expression $(x-p_0)(x^2-x_a^2)$ we can deduce that

$$A = -1, \quad E = p_0, \quad H = x_a^2, \quad L = -p_0x_a^2. \tag{18}$$

This leaves us to determine the remaining three coefficients F, G, K. We used a particular curve and solved the system of equations from the condition of the incenter and excenter being on the curve and calculated the following values:

$$F = 0, \quad G = p_0, \quad K = -\left(\frac{1}{k_1} - k_1\right)x_a^2. \tag{19}$$

Hence, the equation of the cubic, in (case a) and (case b), can be written as

$$\left(\frac{1}{k_1} - k_1\right)y(x^2 - x_a^2) + y^2(x + p_0) = (x - p_0)(x^2 - x_a^2).$$

If the line p intersects outside the segment $\overline{AB}$ (case b) the curve has three open branches and an oval (see Fig. 14). If the line p intersects the segment $\overline{AB}$ inside (case a), then an open branch is stretched along the vertical asymptote which the cubic intersects and converges to the asymptote from different directions and different sides.

In (case c), if $p \parallel c$ then $C_0 = C_\infty$, i.e., there is only one degenerate triangle from which we can conclude that one asymptote of the cubic is parallel to the line c and the line at infinity is a tangent of the cubic with the tangent point at the ideal point of the axis y. The cubic has a parabolic asymptote. The equation of the cubic is

$$-\frac{l}{2}y^2 + (x^2 - x_a^2)\left(y - \frac{l}{2}\right) = 0$$

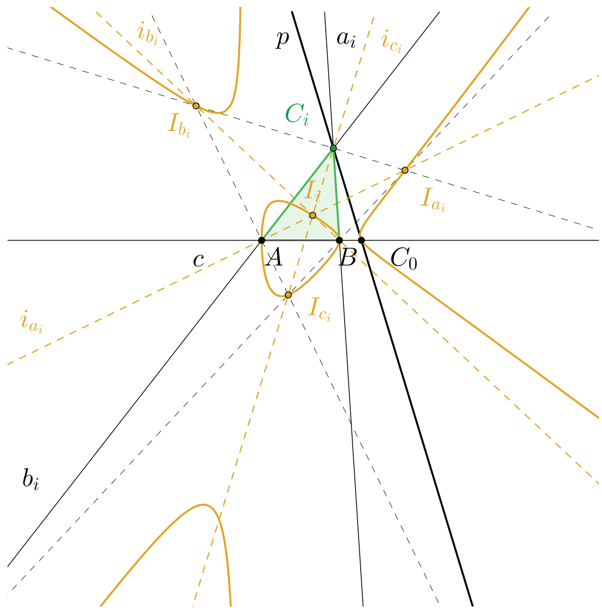

Figure 14: *The locus of incenters and excenters when p intersects $\overline{AB}$ outside.*

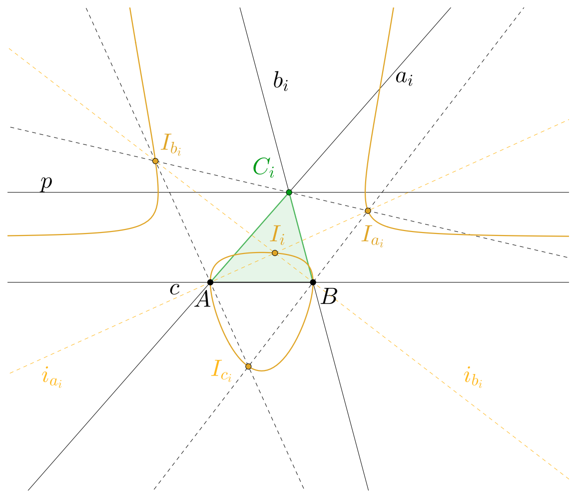

Figure 15: *The locus of incenters and excenters when $p \parallel c$.*

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