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SI CHUN CHOI N. J. WILDBERGER

Parabolic Triangles, Poles and Centroid Relations

Parabolic Triangles, Poles and Centroid Relations ABSTRACT

We investigate affine properties of centroids formed by three points on a parabola together with the polar triangle formed from the tangents. And we make a wide ranging conjecture about the extension of these results to general conics.

Key words: affine geometry, parabola, centroids, conics

MSC2010: 51N20, 14H50, 14Rxx

Trokuti upisani paraboli, polovi i težišta SAŽETAK

Istražujemo afina svojstva težišta koja određuju tri točke na paraboli i vrhovi njihovog polarnog trokuta. Donosimo pretpostavku o proširenju dobivenih rezultata na sve konike.

Ključne riječi: afina geometrija, parabola, težišta, konike

1 Introduction

The parabola has been intensively studied since antiquity, see for example [2], [4], [5], [6], [7] and [8]. In this paper we explore purely affine properties of a parabola, and in particular the configuration formed by three points A_1,A_2 and A_3 on the parabola, and the corresponding duals, or poles X_1,X_2 and X_3 of the three lines formed by those points. We are particularly interested in the relations between centroids of various triangles formed by the six points A_1,A_2,A_3,X_1,X_2 and X_3 . An example of such a result is the following, which was discussed by D. Liu [3].

Theorem: If G_{123} is the centroid of the triangle $\overline{A_1A_2A_3}$ and G^{123} is the centroid of the triangle $\overline{X_1X_2X_3}$, then the line $G_{123}G^{123}$ is parallel to the axis direction of the parabola.

We note that while a distinguished axis of a parabola is a metrical object, the axis direction itself is not, as it is simply the direction corresponding to the unique point at infinity on the parabola, so this is indeed a result purely of affine geometry—independent of any metrical, such as Euclidean, structure. We will show that in fact these six points support a rich fabric of results. In the final paragraph of the paper, we will propose a large-scale generalization of this phenomenon: in fact we conjecture that all the theorems in fact hold for a generic conic in the plane. However our proofs depend heavily on explicit computation, following the framework of [1]. We consider the tabulation of various points and lines a useful tool for further additional research in this direction.

2 Affine geometry and the standard parabola

We will find it convenient to use projective coordinates for points and lines, see for example [2] and [7], even in the affine situation. The underlying number system is a general field, and our arguments will try to avoid any particular assumptions about this field.

An **affine point** is an expression of the form [x, y], and in projective coordinates it will here be written A = [1 : x : y]. A general projective point is then [z : x : y] where by agree-

ment we can re-scale by any non-zero number, so that

$$[z:x:y] = [rz:rx:ry]$$

for any $r \neq 0$.

An **affine line** is a proportion (c : a : b) of numbers, with at least one of *a* and *b* non-zero. The point $A \equiv [x, y]$ **lies on** the line $l \equiv (c : a : b)$ precisely when

c + ax + by = 0

which we also call the **equation** of the line. Equivalently we say l **passes through** A, or that l and A are **incident**. In projective coordinates, the point [z : x : y] is incident with the line (c : a : b) precisely when

$$cz + ax + by = 0.$$

If $A_1 = [z_1 : x_1 : y_1]$ and $A_2 = [z_2 : x_2 : y_2]$ are two distinct points in projective coordinates then their **join** is the unique line A_1A_2 passing through them both, and in projective coordinates is formed from a cross product:

$$A_1A_2 = (x_1y_2 - x_2y_1 : y_1z_2 - y_2z_1 : x_2z_1 - x_1z_2).$$

Similarly if $l_1 = (k_1 : l_1 : m_1)$ and $l_2 = (k_2 : l_2 : m_2)$ are two distinct lines in projective coordinates then their **meet** is the unique point l_1l_2 lying on them both, which is formed from the same cross product, and is

$$l_1 l_2 = [l_1 m_2 - l_2 m_1 : k_2 m_1 - k_1 m_2 : k_1 l_2 - k_2 l_1].$$

This approach treats points and lines symmetrically, and provides good motivation for us to use projective coordinates even in affine geometry. It also brings the crucial computations of joins and meets together. We will move freely from affine to projective coordinates as needed in this paper.

We recall various standard results from [1], reformulated in this projective framework, and simplified somewhat. The general parabola with equation

$$y^2 = 4\lambda x$$

may be re-scaled by sending [x, y] to $[\lambda x, \lambda y]$. in which case it becomes the simpler equation

$$y^2 = 4x$$

or in projective coordinates

 $y^2 = 4xz.$

We will call this now the **standard parabola**, recognizing that the formulas of may be transformed to this simpler setting just by setting $\lambda = 1$ throughout. Using projective coordinates, let $[1:t^2:2t]$ denote a general point on this parabola, and suppose that

$$A \equiv \begin{bmatrix} 1 : a^2 : 2a \end{bmatrix}$$
 and $B \equiv \begin{bmatrix} 1 : b^2 : 2b \end{bmatrix}$

are two specific points on the parabola.

The chord *AB* in Cartesian form is given in projective coordinates by a cross product, which after re-scaling is

$$\begin{split} & \left[1, a^2, 2a \right] \times \left[1, b^2, 2b \right] \\ & = \left(2ab \left(a - b \right) : 2 \left(a - b \right) : - \left(a - b \right) \left(a + b \right) \right) \\ & = \left(2ab : 2 : -a - b \right). \end{split}$$

The tangent to the parabola at A may be simply obtained by setting a = b in the equation of the above chord, giving

$$(2a^2:2:-2a) = (a^2:1:-a).$$

The external point of the chord AB is the meet of the tangents at A and B, which after a cross product and re-scaling is

$$(a^{2}, 1, -a) \times (b^{2}, 1, -b)$$

= $[a - b : a^{2}b - ab^{2} : a^{2} - b^{2}]$
= $[1 : ab : a + b].$

3 Parabolic triangles and Liu's result

Let A_1 , A_2 and A_3 be three distinct points on the standard parabola, forming what we call a **parabolic triangle** $\overline{A_1A_2A_3}$ and which we denote by Δ_{123} . Without loss of generality we assume that

$$A_1 \equiv [1:a_1^2:2a_1], A_2 \equiv [1:a_2^2:2a_2], A_3 \equiv [1:a_3^2:2a_3]$$

with the numbers a_1, a_2 and a_3 distinct. The lines of Δ_{123} are then

$$A_1A_2 = (2a_1a_2:2:-a_1-a_2)$$

$$A_2A_3 = (2a_2a_3:2:-a_2-a_3)$$

$$A_1A_3 = (2a_3a_1:2:-a_3-a_1).$$

The centroid of $\overline{A_1A_2A_3}$ is, switching to affine coordinates and averaging,

$$G_{123} = \frac{1}{3} \left[a_1^2, 2a_1 \right] + \frac{1}{3} \left[a_2^2, 2a_2 \right] + \frac{1}{3} \left[a_3^2, 2a_3 \right]$$
$$= \left[\frac{1}{3} a_1^2 + \frac{1}{3} a_2^2 + \frac{1}{3} a_3^2, \frac{2}{3} a_1 + \frac{2}{3} a_2 + \frac{2}{3} a_3 \right]$$
$$= \left[3 : a_1^2 + a_2^2 + a_3^2 : 2 \left(a_1 + a_2 + a_3 \right) \right].$$

The external points, or poles, of the sides $\overline{A_2A_3}$, $\overline{A_1A_3}$ and $\overline{A_1A_2}$ are respectively

$$X_1 \equiv [1:a_2a_3:a_2+a_3], X_2 \equiv [1:a_1a_3:a_1+a_3],$$

 $X_3 \equiv [1:a_1a_2:a_1+a_2].$

C123

We denote the external triangle $\overline{X_1X_2X_3}$ by Δ^{123} . Its centroid is

$$= \frac{1}{3}[a_2a_3, a_2 + a_3] + \frac{1}{3}[a_1a_3, a_1 + a_3] + \frac{1}{3}[a_1a_2, a_1 + a_2]$$

= $\left[\frac{1}{3}a_1a_2 + \frac{1}{3}a_2a_3 + \frac{1}{3}a_1a_3, \frac{2}{3}a_1 + \frac{2}{3}a_2 + \frac{2}{3}a_3\right]$
= $[3:a_1a_2 + a_2a_3 + a_1a_3: 2(a_1 + a_2 + a_3)].$

We now give a proof of the observation of D. Liu.

Theorem 1 The line $G_{123}G^{123}$ is always parallel to the axis direction of the parabola, and in this case has equation

$$y = \frac{2}{3} \left(a_1 + a_2 + a_3 \right).$$

Proof. The axis direction for the standard parabola is just the *x*-axis, so this is the statement that $G_{123}G^{123}$ is horizontal. We compute that

$$G_{123}G^{123}$$

$$= [3, a_1^2 + a_2^2 + a_3^2, 2(a_1 + a_2 + a_3)]$$

$$\times [3, a_1a_2 + a_2a_3 + a_1a_3, 2(a_1 + a_2 + a_3)]$$

$$= (2(a_1 + a_2 + a_3): 0: -3)$$

with equation

$$y = \frac{2}{3} \left(a_1 + a_2 + a_3 \right).$$

Another approach would be to compute the vector

$$\overline{G_{123}G^{123}} = \frac{1}{3} [a_1a_2 + a_2a_3 + a_1a_3, 2(a_1 + a_2 + a_3)] - \frac{1}{3} [a_1^2 + a_2^2 + a_3^2, 2(a_1 + a_2 + a_3)] = -\frac{1}{3} (a_1^2 + a_2^2 + a_3^2 - a_1a_2 - a_2a_3 - a_1a_3) (1,0).$$

4 More general centroids

The six points A_1, A_2, A_3, X_1, X_2 and X_3 determine also mixed triangles, such as for example

$$\overline{A_1 A_2 X_3} = \Delta_{12}^3$$
 or $\overline{A_1 X_2 X_3} = \Delta_1^{23}$

where the lower indices record A_i vertices, and the raised indices record X_j vertices, and where we agree to arrange indices always in ascending order.

Define G_{ij}^k to be the centroid of Δ_{ij}^k , for all allowed index combinations *i*, *j* and *k*. Then we may compute, using projective coordinates, that

$$\begin{split} G_{12}^{1} &= \begin{bmatrix} 3:a_{1}^{2} + a_{2}^{2} + a_{2}a_{3}: 3a_{2} + 2a_{1} + a_{3} \end{bmatrix} \\ G_{12}^{2} &= \begin{bmatrix} 3:a_{1}^{2} + a_{2}^{2} + a_{1}a_{3}: 3a_{1} + 2a_{2} + a_{3} \end{bmatrix} \\ G_{23}^{2} &= \begin{bmatrix} 3:a_{2}^{2} + a_{3}^{2} + a_{1}a_{3}: 3a_{3} + 2a_{2} + a_{1} \end{bmatrix} \\ G_{23}^{3} &= \begin{bmatrix} 3:a_{2}^{2} + a_{3}^{2} + a_{1}a_{2}: 3a_{3} + 2a_{2} + a_{1} \end{bmatrix} \\ G_{13}^{1} &= \begin{bmatrix} 3:a_{1}^{2} + a_{3}^{2} + a_{2}a_{3}: 3a_{3} + 2a_{1} + a_{2} \end{bmatrix} \\ G_{13}^{3} &= \begin{bmatrix} 3:a_{1}^{2} + a_{3}^{2} + a_{2}a_{3}: 3a_{3} + 2a_{1} + a_{2} \end{bmatrix} \\ G_{13}^{3} &= \begin{bmatrix} 3:a_{1}^{2} + a_{3}^{2} + a_{1}a_{2}: 3a_{1} + 2a_{3} + a_{2} \end{bmatrix} \end{split}$$

while

$$\begin{split} G_1^{12} &= \begin{bmatrix} 3:a_1a_3 + a_2a_3 + a_1^2: 3a_1 + 2a_3 + a_2 \end{bmatrix} \\ G_2^{12} &= \begin{bmatrix} 3:a_1a_3 + a_2a_3 + a_2^2: 3a_2 + 2a_3 + a_1 \end{bmatrix} \\ G_1^{13} &= \begin{bmatrix} 3:a_1a_2 + a_2a_3 + a_1^2: 3a_1 + 2a_2 + a_3 \end{bmatrix} \\ G_3^{13} &= \begin{bmatrix} 3:a_1a_2 + a_2a_3 + a_3^2: 3a_3 + 2a_2 + a_1 \end{bmatrix} \\ G_2^{23} &= \begin{bmatrix} 3:a_1a_3 + a_1a_2 + a_2^2: 3a_2 + 2a_1 + a_3 \end{bmatrix} \\ G_3^{23} &= \begin{bmatrix} 3:a_1a_2 + a_1a_3 + a_2^2: 3a_3 + 2a_1 + a_2 \end{bmatrix} \end{split}$$

In fact the definitions also make sense for a degenerate triangle such as $\overline{A_1X_2X_3}$: even though these points are collinear, their centroid G_1^{23} is well-defined and altogether we have six such points, namely

$$\begin{split} G_{23}^1 &= \begin{bmatrix} 3:a_2^2 + a_3^2 + a_2a_3: 3(a_2 + a_3) \end{bmatrix} \\ G_{1}^{23} &= \begin{bmatrix} 3:a_1(a_1 + a_2 + a_3): 4a_1 + a_2 + a_3 \end{bmatrix} \\ G_{13}^2 &= \begin{bmatrix} 3:a_1^2 + a_3^2 + a_1a_3: 3(a_1 + a_3) \end{bmatrix} \\ G_{2}^{13} &= \begin{bmatrix} 3:a_2(a_1 + a_2 + a_3): a_1 + 4a_2 + a_3 \end{bmatrix} \\ G_{12}^3 &= \begin{bmatrix} 3:a_1^2 + a_2^2 + a_1a_2: 3(a_1 + a_2) \end{bmatrix} \\ G_{3}^{12} &= \begin{bmatrix} 3:a_3(a_1 + a_2 + a_3): a_1 + a_2 + 4a_3 \end{bmatrix}. \end{split}$$

Theorem 2 For any distinct indices $1 \le i < j \le 3$



Figure 1: Mixed centroids of a parabolic triangle

Proof. We consider the case i = 1 and j = 2, the other cases are similar. Using affine coordinates, we have that

$$\overline{A_1A_2} = A_2 - A_1 = [a_2^2, 2a_2] - [a_1^2, 2a_1]$$
$$= (a_2^2 - a_1^2, 2a_2 - 2a_1)$$
$$= (a_2 - a_1) (a_1 + a_2, 2).$$

But also

$$\overline{G_1^{12}G_2^{12}} = \frac{1}{3} \left[a_1a_3 + a_2a_3 + a_2^2, 3a_2 + 2a_3 + a_1 \right] - \frac{1}{3} \left[a_1a_3 + a_2a_3 + a_1^2, 3a_1 + 2a_3 + a_2 \right] = \frac{1}{3} \left(a_2^2 - a_1^2, 2a_2 - 2a_1 \right) = \frac{1}{3} \left(a_2 - a_1 \right) \left(a_1 + a_2, 2 \right).$$

The result follows.

Theorem 3 Let I_1 be the meet of the lines $G_1^{12}G_2^{12}$ and $G_1^{13}G_3^{13}$, I_2 be the meet of the lines $G_1^{12}G_2^{12}$ and $G_2^{23}G_3^{23}$, and I_3 be the meet of the lines $G_1^{13}G_3^{13}$ and $G_2^{23}G_3^{23}$. Then for $1 \le i < j \le 3$

$$\overline{I_{i}I_{j}} = \frac{1}{2}\overline{A_{i}A_{j}}.$$

Figure 2: An affinely similar triangle $\overline{I_1I_2I_3}$ to $\overline{A_1A_2A_3}$

Proof. The equations of the relevant lines through the centroids are

$$\begin{aligned} G_1^{12}G_2^{12} &= \left(a_1^2 + a_2^2 + 4a_1a_2 : 6 : -3(a_1 + a_2)\right) \\ G_2^{23}G_3^{23} &= \left(a_2^2 + a_3^2 + 4a_2a_3 : 6 : -3(a_2 + a_3)\right) \\ G_3^{13}G_1^{13} &= \left(a_3^2 + a_1^2 + 4a_3a_1 : 6 : -3(a_3 + a_1)\right). \end{aligned}$$

Then the meets of these lines can calculated as follows $I_{1} = (G_{1}^{12}G_{2}^{12}) (G_{1}^{13}G_{2}^{13})$

$$I_{1} = (G_{1} G_{2}) (G_{1} G_{3})$$

$$= [6: a_{1}a_{2} + a_{2}a_{3} + a_{3}a_{1} + 3a_{1}^{2}: 2(a_{2} + a_{3} + 4a_{1})]$$

$$I_{2} = (G_{1}^{12}G_{2}^{12}) (G_{2}^{23}G_{3}^{23})$$

$$= [6: a_{1}a_{2} + a_{2}a_{3} + a_{3}a_{1} + 3a_{2}^{2}: 2(a_{3} + a_{1} + 4a_{2})]$$

$$I_{3} = (G_{1}^{13}G_{3}^{13}) (G_{2}^{23}G_{3}^{23})$$

$$= \left| 6: a_1a_2 + a_2a_3 + a_3a_1 + 3a_3^2: 2(a_1 + a_2 + 4a_3) \right|.$$

Then for the case i = 1 and j = 2, $\overline{I_1I_2}$ can be calculated using affine coordinates as

$$\begin{split} \overline{I_1I_2} &= I_2 - I_1 \\ &= \frac{1}{6} \left[a_1a_2 + a_2a_3 + a_3a_1 + 3a_2^2, 2(a_3 + a_1 + 4a_2) \right] \\ &\quad - \frac{1}{6} \left[a_1a_2 + a_2a_3 + a_3a_1 + 3a_1^2, 2(a_2 + a_3 + 4a_1) \right] \\ &= \left(\frac{1}{2} \left(a_2^2 - a_1^2 \right), (a_2 - a_1) \right) \\ &= \frac{1}{2} \left(a_2 - a_1 \right) \left(a_1 + a_2, 2 \right). \end{split}$$

The result for $\overrightarrow{I_1I_2}$ follows, and the other cases are symmetrically similar.

Recall that in affine geometry the **signed area** of a triangle $\overrightarrow{A_1A_2A_3}$ with oriented sides $\overrightarrow{A_1A_2} = (a,b)$ and $\overrightarrow{A_1A_3} = (c,d)$ is

$$s\left(\overrightarrow{A_1A_2A_3}\right) \equiv \frac{1}{2}\left(ad - bc\right)$$

 \square

again independent of any metrical structure.

Corollary 1 The triangles Δ_{123} and $\overline{I_1I_2I_3}$ have parallel sides and so are affinely similar, and so their signed areas satisfy

$$s(\overrightarrow{A_1A_2A_3}):s(\overrightarrow{I_1I_2I_3})=4:1.$$

Proof. This follows from the relation between the vectors making the sides of the triangles. \Box





Figure 3: A centre of perspectivity G^{123}

Proof. Using our formula for joins we compute that

$$A_1I_1 = (2(a_1a_2a_3 - a_1^3) : 2(a_2 + a_3 - 2a_1))$$

: $3a_1^2 - a_1a_2 - a_2a_3 - a_3a_1).$

To check that the centroid

$$G^{123} = [3:a_1a_2 + a_2a_3 + a_1a_3:2(a_1 + a_2 + a_3))$$

is incident with this line, verify that

$$2(a_1a_2a_3 - a_1^3) + 2(a_2 + a_3 - 2a_1)(a_1a_2 + a_2a_3 + a_1a_3) + (3a_1^2 - a_1a_2 - a_2a_3 - a_3a_1) + (3a_1^2 - a_1a_2 - a_2a_2 - a_3a_1) + (3a_1^2 - a_1a_2 - a_1a_2 - a_2a_2 - a_2a_1) + (3a_1^2 - a_1a_2 - a_2a_1) + (3a_1^2 - a_1a_2 - a_2a_2 - a_2a_1) + (3a_1^2 - a_1a_2 - a_2a_2 - a_2a_1) + (3a_1^2 - a_1a_2 - a_1a_2 - a_1a_2) + (3a_1^2 - a_1a_2 - a_1a_$$

Since this argument also holds for A_2I_2 and A_3I_3 , all three lines are concurrent at the centroid G^{123} .

In particular it follows that the triangles $\overline{I_1I_2I_3}$ and Δ_{123} are in perspective via the centroid G^{123} of triangle Δ^{123} .

Corollary 2 The points I_1 , I_2 , and I_3 are the midpoints of $\overline{A_3G^{123}}$, $\overline{A_2G^{123}}$ and $\overline{A_1G^{123}}$ respectively.

Proof. Since $A_1 \equiv [1:a_1^2:2a_1]$ and $G^{123} \equiv [3:a_1a_2+a_2a_3+a_1a_3:2(a_1+a_2+a_3)]$ we can check directly that the midpoint of $\overline{A_1G^{123}}$ is

 $I_1 = \left[6:a_1a_2 + a_2a_3 + a_3a_1 + 3a_1^2: 2(4a_1 + a_2 + a_3)\right].$

Similarly, the midpoints of $\overline{A_2G^{123}}$ and $\overline{A_3G^{123}}$ are I_2 and I_3 respectively.

Theorem 5 The three lines $G_{23}^2G_{23}^3$, $G_{13}^1G_{13}^3$ and $G_{12}^1G_{12}^2$ are parallel to the tangents to the parabola at A_1 , A_2 and A_3 respectively. These three lines are concurrent at the point G_{123} .



Figure 4: Parallel lines to tangents concurrent at G₁₂₃

Proof. From the formulas for the centroids we get

 $G_{12}^{1}G_{12}^{2} = \left(a_{3}^{2} + 2a_{2}a_{3} + 2a_{1}a_{3} - a_{1}^{2} - a_{2}^{2} : 3 : -3a_{3}\right).$

This is parallel to the tangent to the parabola at A_3 which is

$$X_1 X_2 = \left(a_3^2 : 1 : -a_3\right)$$

To check that the centroid

$$G_{123} = \left[3:a_1^2 + a_2^2 + a_3^2:2(a_1 + a_2 + a_3)\right]$$

is incident with this line we compute

$$(a_3^2 + 2a_2a_3 + 2a_1a_3 - a_1^2 - a_2^2) 3 + 3 (a_1^2 + a_2^2 + a_3^2) - 3a_32 (a_1 + a_2 + a_3) = 0.$$

By symmetry G_{123} also lies on the lines $G_{23}^2 G_{23}^3$ and $G_{13}^1 G_{13}^3$, hence all three lines are concurrent.

Theorem 6 The three lines $G_1^{12}G_1^{13}$, $G_2^{23}G_2^{12}$ and $G_3^{13}G_3^{23}$ are parallel to the tangents to the parabola at A_1 , A_2 and A_3 respectively. These three lines are concurrent at the point G^{123} .



Figure 5: Another concurrency of parallel lines at G^{123}

Proof. From the formulas for the centroids we get

$$G_1^{12}G_1^{13} = \left(2a_1^2 + a_1a_2 + a_1a_3 - a_2a_3 : 3 : -3a_1\right).$$

This is again parallel to $X_2X_3 = (a_1^2 : 1 : -a_1)$. And the centroid $G^{123} = [3 : a_1a_2 + a_2a_3 + a_3a_1 : 2(a_1 + a_2 + a_3)]$ is incident with $G_1^{12}G_1^{13}$ since

$$(2a_1^2 + a_1a_2 + a_1a_3 - a_2a_3) 3 + 3(a_1a_2 + a_2a_3 + a_3a_1) - 3a_12(a_1 + a_2 + a_3) = 0.$$

By symmetry G^{123} also lies on the lines $G_2^{23}G_2^{12}$ and $G_3^{13}G_3^{23}$, hence all three lines are concurrent.

Theorem 7 Let J_1 be the meet of the lines $G_{13}^3G_{23}^3$ and $G_{12}^2G_{23}^2$, J_2 be the meet of the lines $G_{12}^1G_{13}^1$ and $G_{13}^3G_{23}^3$, and J_3 be the meet of the lines $G_{12}^1G_{13}^1$ and $G_{12}^2G_{23}^2$. Then for $1 \le i < j \le 3$

$$\overrightarrow{J_iJ_j} = -\frac{1}{2}\overrightarrow{A_iA_j}.$$

Proof. The equations of the relevant lines are

$$\begin{split} G_{12}^{1}G_{13}^{1} &= \\ \left(\left(a_{2}^{2} + a_{3}^{2} - 2a_{1}^{2} + 2a_{1}a_{2} + 2a_{2}a_{3} + 2a_{3}a_{1} \right) : 6 : -3\left(a_{2} + a_{3}\right) \right) \\ G_{12}^{2}G_{23}^{2} &= \\ \left(\left(a_{1}^{2} + a_{3}^{2} - 2a_{2}^{2} + 2a_{1}a_{2} + 2a_{2}a_{3} + 2a_{3}a_{1} \right) : 6 : -3\left(a_{1} + a_{3}\right) \right) \\ G_{13}^{3}G_{23}^{3} &= \\ \left(\left(a_{1}^{2} + a_{2}^{2} - 2a_{3}^{2} + 2a_{1}a_{2} + 2a_{2}a_{3} + 2a_{3}a_{1} \right) : 6 : -3\left(a_{1} + a_{2}\right) \right) . \end{split}$$

The meets of the lines can then be computed to be

$$\begin{aligned} J_1 &= \left(G_{13}^3 G_{23}^3\right) \left(G_{12}^2 G_{23}^2\right) \\ &= \left[6: \left(2a_2^2 + 2a_3^2 - a_1^2 + a_1a_2 + a_2a_3 + a_3a_1\right): 6\left(a_2 + a_3\right)\right] \\ J_2 &= \left(G_{12}^1 G_{13}^1\right) \left(G_{13}^3 G_{23}^3\right) \\ &= \left[6: \left(2a_1^2 + 2a_3^2 - a_2^2 + a_1a_2 + a_2a_3 + a_3a_1\right): 6\left(a_1 + a_3\right)\right] \\ J_3 &= \left(G_{12}^1 G_{13}^1\right) \left(G_{12}^2 G_{23}^2\right) \\ &= \left[6: \left(2a_1^2 + 2a_2^2 - a_3^2 + a_1a_2 + a_2a_3 + a_3a_1\right): 6\left(a_1 + a_2\right)\right] \end{aligned}$$

Then using affine coordinates

$$\begin{aligned} \overrightarrow{J_1J_2} &= \left[\frac{1}{6} \left(2a_1^2 + 2a_3^2 - a_2^2 + a_1a_2 + a_2a_3 + a_3a_1 \right), (a_1 + a_3) \right] \\ &- \left[\frac{1}{6} \left(2a_2^2 + 2a_3^2 - a_1^2 + a_1a_2 + a_2a_3 + a_3a_1 \right), (a_2 + a_3) \right] \\ &= -\frac{1}{2} \left(a_2 - a_1 \right) \left(a_1 + a_2, 2 \right) \\ &= -\frac{1}{2} \overrightarrow{A_1A_2}. \end{aligned}$$

The other cases are symmetrically similar.

Corollary 3 *The triangles* Δ_{123} *and* $\overline{J_1J_2J_3}$ *have parallel sides and so are similar, and so the signed areas satisfy*

$$s(\overrightarrow{A_1A_2A_3}):s(\overrightarrow{J_1J_2J_3})=-4:1.$$

Proof. The argument is the same as previously, except now the orientations are reversed. \Box

5 The point P

Theorem 8 $\overline{J_1J_2J_3}$ is in perspective with Δ_{123} . We denote the point of perspectivity by *P*. Then *P* divides each of the sides $\overline{A_1J_1}$, $\overline{A_2J_2}$ and $\overline{A_3J_3}$ in the ratio 2 : 1, that is



Figure 6: Perspectivity between $\overline{J_1J_2J_3}$ and $\overline{A_1A_2A_3}$ via P

Proof. We compute that the meet of A_1J_1 and A_2J_2 is the point

$$P = \left[9: 2a_1^2 + 2a_2^2 + 2a_3^2 + a_1a_2 + a_2a_3 + a_3a_1: 6(a_1 + a_2 + a_3)\right]$$

Moving to affine coordinates, we compute the affine combination

$$\frac{1}{3}A_{1} + \frac{2}{3}J_{1} = \frac{1}{3}[a_{1}^{2}, 2a_{1}] \\ + \frac{2}{3} \cdot \frac{1}{6}[(2a_{2}^{2} + 2a_{3}^{2} - a_{1}^{2} + a_{1}a_{2} + a_{2}a_{3} + a_{3}a_{1}), 6(a_{2} + a_{3})] \\ = \frac{1}{9}[2a_{1}^{2} + 2a_{2}^{2} + 2a_{3}^{2} + a_{1}a_{2} + a_{1}a_{3} + a_{2}a_{3}, 6(a_{1} + a_{2} + a_{3})] \\ = P.$$

The other ratios

$$P = \frac{1}{3}A_2 + \frac{2}{3}J_2 = \frac{1}{3}A_3 + \frac{2}{3}J_3$$

are similar.

Corollary 4 The triangles $\overline{I_1I_2I_3}$ and $\overline{J_1J_2J_3}$ are affinely negatively congruent, by which we mean that corresponding vectors of sides are negatives of each other.

Proof. This follows directly from the relations between the vectors of the sides of the triangles. \Box

Theorem 9 The points G_{123} , G^{123} , and P are collinear. P also divides the side $\overline{G_{123}G^{123}}$ in the ratio 2 : 1, that is

$$P = \frac{1}{3}G^{123} + \frac{2}{3}G_{123}.$$

Proof. Using the affine coordinates of the centroids we find that

$$\frac{1}{3}G^{123} + \frac{2}{3}G_{123}$$

$$= \frac{1}{3} \left[\frac{(a_1a_2 + a_2a_3 + a_3a_1)}{3}, \frac{2(a_1 + a_2 + a_3)}{3} \right]$$

$$+ \frac{2}{3} \left[\frac{(a_1^2 + a_2^2 + a_3^2)}{3}, \frac{2(a_1 + a_2 + a_3)}{3} \right]$$

$$= \frac{1}{9} \left[2a_1^2 + 2a_2^2 + 2a_3^2 + a_1a_2 + a_1a_3 + a_2a_3, 6(a_1 + a_2 + a_3) \right]$$

$$= P.$$

Theorem 10 The triangles $\overline{I_1I_2I_3}$ and $\overline{J_1J_2J_3}$ are perspective from the point M which is the midpoint of G_{123} and G^{123} .

Proof. The midpoint of $\overline{G_{123}G^{123}}$ is

$$M =$$

 $\left[6:a_1^2+a_2^2+a_3^2+a_1a_2+a_2a_3+a_3a_1:4(a_1+a_2+a_3)\right].$

Computing the lines I_1J_1 , I_2J_2 and I_3J_3 we get

$$\begin{split} I_1 J_1 &= (-3a_1a_2^2 + 3a_1^2a_2 - 3a_1a_3^2 + 3a_1^2a_3 + 2a_1^3 - a_2^3 - a_3^3 \\ &: 6 \left(2a_1 - a_2 - a_3\right) : 3 \left(-2a_1^2 + a_2^2 + a_3^2\right)\right) \\ I_2 J_2 &= \left(3a_1a_2^2 - 3a_1^2a_2 - 3a_2a_3^2 + 3a_2^2a_3 - a_1^3 + 2a_2^3 - a_3^3 \\ &: 6 \left(-a_1 + 2a_2 - a_3\right) : 3 \left(a_1^2 - 2a_2^2 + a_3^2\right)\right) \\ I_3 J_3 &= \left(3a_1a_3^2 - 3a_1^2a_3 + 3a_2a_3^2 - 3a_2^2a_3 - a_1^3 - a_2^3 + 2a_3^3 \\ &: 6 \left(-a_1 - a_2 + 2a_3\right) : 3 \left(a_1^2 + a_2^2 - 2a_3^2\right)\right). \end{split}$$

By computation the meet of I_1J_1 and I_2J_2 is:

$$9(a_2 - a_3)(a_1 - a_3)(a_1 - a_2) \left[6:a_1^2 + a_2^2 + a_3^2 + a_1a_2 + a_1a_3 + a_2a_3: 4(a_1 + a_2 + a_3)\right]$$

which after re-scaling becomes

$$M = [6:a_1^2 + a_2^2 + a_3^2 + a_1a_2 + a_1a_3 + a_2a_3: 4(a_1 + a_2 + a_3)].$$

Theorem 11 The four points G_{23}^1 , G_{13}^2 , G_{12}^3 and P are collinear.



Figure 7: Collinearity of G_{23}^1 *,* G_{13}^2 *,* G_{12}^3 *and* P

Proof. The four points are

$$\begin{aligned} G_{23}^1 &= \left[3:a_2^2 + a_3^2 + a_2a_3: 3\left(a_2 + a_3\right)\right] \\ G_{13}^2 &= \left[3:a_1^2 + a_3^2 + a_1a_3: 3\left(a_1 + a_3\right)\right] \\ G_{12}^3 &= \left[3:a_1^2 + a_2^2 + a_1a_2: 3\left(a_1 + a_2\right)\right] \\ P &= \left[9:2a_1^2 + 2a_2^2 + 2a_3^2 + a_1a_2 + a_1a_3 + a_2a_3\right] \\ &: 6\left(a_1 + a_2 + a_3\right)\right]. \end{aligned}$$

The equation of the line

$$G_{23}^1 G_{13}^2$$
 is $[3: -(a_1+a_2+a_3): a_1a_2+a_1a_3+a_2a_3]$,

which passes through the points G_{12}^3 and P.

Theorem 12 The three points G_1^{23} , G_2^{13} and G_3^{12} are collinear on a line l, which is also parallel to the line

through the points G_{23}^1 , G_{13}^2 , G_{12}^3 and P. Moreover l is a tangent to the parabola at the point Q where it meets the central line $G_{123}G^{123}$. Furthermore Q divides the side $\overline{G^{123}G_{123}}$ in the ratio 1 : 2. So the four points G^{123} , Q, P and G_{123} are equally spaced on the central line.



Figure 8: Equal spacing of G^{123} , Q, P and G_{123} and tangency at Q

Proof. The three points are

 $\begin{aligned} G_1^{23} &= [3:a_1(a_1+a_2+a_3):4a_1+a_2+a_3] \\ G_2^{13} &= [3:a_2(a_1+a_2+a_3):a_1+4a_2+a_3] \\ G_3^{12} &= [3:a_3(a_1+a_2+a_3):a_1+a_2+4a_3]. \end{aligned}$

The equation of the line $G_1^{23}G_2^{13}$ is

$$\left[\left(a_1 + a_2 + a_3 \right)^2 : 9 : -3 \left(a_1 + a_2 + a_3 \right) \right],$$

which passes through the point G_3^{12} by computation

$$3(a_1 + a_2 + a_3)^2 + 9a_3(a_1 + a_2 + a_3) - 3(a_1 + a_2 + a_3)(a_1 + a_2 + 4a_3) = 0.$$

Hence the three points G_1^{23} , G_2^{13} and G_3^{12} are collinear. This line is also a tangent to the parabola since it is in the form of $(a^2 : 1 : -a)$. Moreover it is parallel to the line through the points G_{23}^1 , G_{13}^2 , G_{12}^3 and *P*:

 $(a_1a_2 + a_1a_3 + a_2a_3 : 3 : - (a_1 + a_2 + a_3)).$

The meet of the line $G_1^{23}G_2^{13}$ with the central line $G_{123}G^{123}$ is the point

$$Q = \left[9: (a_1 + a_2 + a_3)^2: 6(a_1 + a_2 + a_3)\right]$$

which lies on the parabola, and we can check that

$$\begin{aligned} &\frac{2}{3}G^{123} + \frac{1}{3}G_{123} \\ &= \frac{2}{3} \left[\frac{(a_1a_2 + a_2a_3 + a_3a_1)}{3}, \frac{2(a_1 + a_2 + a_3)}{3} \right] \\ &+ \frac{1}{3} \left[\frac{(a_1^2 + a_2^2 + a_3^2)}{3}, \frac{2(a_1 + a_2 + a_3)}{3} \right] \\ &= \frac{1}{9} \left[(a_1 + a_2 + a_3)^2, 6(a_1 + a_2 + a_3) \right] = Q. \end{aligned}$$

Theorem 13 *M* is the midpoint of the following pair of points: *P* and *Q*, G_{123} and G^{123} , I_1 and J_1 , I_2 and J_2 , I_3 and J_3 , G_{23}^1 and G_{13}^{23} , G_{13}^2 and G_{23}^{13} , G_{12}^3 and G_{12}^{12} , G_{12}^1 and G_{33}^{23} , G_{33}^3 and G_{12}^{12} , G_{12}^2 and G_{33}^{13} , G_{13}^3 and G_{22}^{12} , G_{23}^2 and G_{13}^{13} , G_{13}^3 and G_{22}^{12} , G_{23}^2 and G_{13}^{13} , G_{13}^3 and G_{22}^{12} , G_{23}^2 and G_{13}^{13} , G_{13}^3 and G_{22}^{12} .

Proof. From the explicit forms of the various points that we have so far determined, the midpoint of $\overline{I_1J_1}$ is

$$\begin{aligned} &\frac{1}{2} \left[\frac{a_1 a_2 + a_2 a_3 + a_3 a_1 + 3a_1^2}{6}, \frac{2(a_2 + a_3 + 4a_1)}{6} \right] \\ &+ \frac{1}{2} \left[\frac{\left(2a_2^2 + 2a_3^2 - a_1^2 + a_1 a_2 + a_1 a_3 + a_2 a_3\right)}{6}, \frac{6(a_2 + a_3)}{6} \right] \\ &= \left[\frac{a_1^2 + a_2^2 + a_3^2 + a_1 a_2 + a_1 a_3 + a_2 a_3}{6}, \frac{4(a_1 + a_2 + a_3)}{6} \right] \\ &= M \end{aligned}$$

and similarly we can verify that the midpoint of all the other pairs are also M.

Corollary 5 The lines PQ, $G_{123}G^{123}$, I_1J_1 , I_2J_2 , I_3J_3 , $G_{23}^{1}G_{23}^{23}$, $G_{13}^{2}G_{23}^{13}$, $G_{13}^{2}G_{23}^{12}$, $G_{12}^{3}G_{33}^{12}$, $G_{23}^{1}G_{12}^{12}$, $G_{23}^{2}G_{13}^{12}$, $G_{12}^{2}G_{33}^{33}$, $G_{13}^{3}G_{22}^{12}$, $G_{23}^{2}G_{1}^{13}$ and $G_{13}^{1}G_{23}^{23}$ are all concurrent at the point M.

Proof. Since *M* is the midpoint of all the intervals, clearly the lines are all concurrent at *M*. \Box

Corollary 6 *The quadrilateral formed by any pairs of the diagonals from the set*

$$\{PQ, G_{123}G^{123}, I_1J_1, I_2J_2, I_3J_3, G_{13}^{12}G_{13}^{23}, G_{13}^{2}G_{23}^{13}, G_{12}^{3}G_{3}^{12}, G_{12}^{3}G_{3}^{12}, G_{12}^{12}G_{3}^{12}, G_{12}^{12}G_{13}^{12}, G_{13}^{12}G_{23}^{12}, G_{13}^{12}G_{13}^{12}, G_{13}^{12}G_{23}^{12}, G_{13}^{12}, G_{13}^{12},$$

is a parallelogram.

Proof. Since all the diagonals from the set

$$\{ PQ, G_{123}G^{123}, I_1J_1, I_2J_2, I_3J_3, G_{13}^{1}G_{13}^{23}, G_{13}^{23}G_{23}^{13}, G_{12}^{3}G_{3}^{12}, G_{12}^{3}G_{3}^{12}, G_{12}^{3}G_{3}^{12}, G_{12}^{3}G_{23}^{13}, G_{13}^{1}G_{23}^{12}, G_{23}^{2}G_{13}^{13}, G_{13}^{1}G_{23}^{23} \}$$

share a common midpoint *M*, the quadrilateral formed by any pairs of the diagonals will result in a parallelogram. Counting we see that from this list there will be $14 \times 13/2 = 91$ different parallelograms.

6 Further directions

The results of this paper are affine results, because they rest on notions of midpoints and centroids, but no metrical structure. However it is not hard to generalize this to projective geometry if we remember that any line in the projective plane can act as the distinguished 'line at infinity' which characterizes affine geometry, and that then the usual midpoint construction is generalized to harmonic conjugation of the distinguished point at infinity on the line determined by two points. In this way we can restate the results above in projective geometry, but for a conic which is tangent to this line at infinity. Somewhat remarkably, our investigations suggest a rather big generalization of this, which we state as a conjecture.

Conjecture: All of the results of this paper that do not involve the axis of the parabola hold if the parabola is replaced by a generic conic.

However the relative ease of proof that we have been able to obtain with the standard coordinates on the parabola will have to be rethought.

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Steiner's Hat: a Constant-Area Deltoid Associated with the Ellipse

Steiner's Hat: a Constant-Area Deltoid Associated with the Ellipse

ABSTRACT

The Negative Pedal Curve (NPC) of the Ellipse with respect to a boundary point M is a 3-cusp closed-curve which is the affine image of the Steiner Deltoid. Over all M the family has invariant area and displays an array of interesting properties.

Key words: curve, envelope, ellipse, pedal, evolute, deltoid, Poncelet, osculating, orthologic

MSC2010: 51M04 51N20 65D18

Steinerova krivulja: deltoide konstantne površine pridružene elipsi

SAŽETAK

Negativno nožišna krivulja elipse s obzirom na neku njezinu točku M je zatvorena krivulja s tri šiljka koja je afina slika Steinerove deltoide. Za sve točke M na elipsi krivulje dobivene familije imaju istu površinu i niz zanimljivih svojstava.

Ključne riječi: krivulja, envelopa, elipsa, nožišna krivulja, evoluta, deltoida, Poncelet, oskulacija, ortologija

1 Introduction

Given an ellipse \mathcal{E} with non-zero semi-axes a, b centered at O, let M be a point in the plane. The Negative Pedal Curve (NPC) of \mathcal{E} with respect to M is the envelope of lines passing through points P(t) on the boundary of \mathcal{E} and perpendicular to [P(t) - M] [4, pp. 349]. Well-studied cases [7, 14] include placing M on (i) the major axis: the NPC is a two-cusp "fish curve" (or an asymmetric ovoid for low eccentricity of \mathcal{E}); (ii) at O: this yielding a four-cusp NPC known as Talbot's Curve (or a squashed ellipse for low eccentricity), Figure 1.

As a variant to the above, we study the family of NPCs with respect to points M on the *boundary* of \mathcal{E} . As shown in Figure 2, this yields a family of asymmetric, constant-area 3-cusped deltoids. We call these curves "Steiner's Hat" (or Δ), since under a varying affine transformation, they are the image of the Steiner Curve (aka. Hypocycloid), Figure 3. Besides these remarks, we've observed:

Main Results:

- The triangle T' defined by the 3 cusps P'_i has invariant area over M, Figure 7.
- The triangle *T* defined by the pre-images P_i of the 3 cusps has invariant area over *M*, Figure 7. The P_i are the 3 points on \mathcal{E} such that the corresponding tangent to the envelope is at a cusp.
- The *T* are a Poncelet family with fixed barycenter; their caustic is half the size of *E*, Figure 7.
- Let C_2 be the center of area of Δ . Then M, C_2, P_1, P_2, P_3 are concyclic, Figure 7. The lines $P_i C_2$ are tangents at the cusps.
- Each of the 3 circles passing through $M, P_i, P'_i, i = 1, 2, 3$, osculate \mathcal{E} at P_i , Figure 8. Their centers define an area-invariant triangle T'' which is a half-size homothety of T'.

The paper is organized as follows. In Section 3 we prove the main results. In Sections 4 and 5 we describe properties of the triangles defined by the cusps and their pre-images, respectively. In Section 6 we analyze the locus of the cusps. In Section 6.1 we characterize the tangencies and intersections of Steiner's Hat with the ellipse. In Section 7 we describe properties of 3 circles which osculate the ellipse at the cusp pre-images and pass through M. In Section 8 we describe relationships between the (constant-area) triangles with vertices at (i) cusps, (ii) cusp pre-images, and (iii) centers of osculating circles. In Section 9 we analyze a fixed-area deltoid obtained from a "rotated" negative pedal curve. The paper concludes in Section 10 with a table of illustrative videos. Appendix A provides explicit coordinates for cusps, pre-images, and osculating circle centers. Finally, Appendix B lists all symbols used in the paper.



Figure 1: The Negative Pedal Curve (NPC) of an ellipse \mathcal{E} with respect to a point M on the plane is the envelope of lines passing through P(t) on the boundary, and perpendicular to P(t) - M. Left: When M lies on the major axis of \mathcal{E} , the NPC is a two-cusp "fish" curve. **Right:** When M is at the center of \mathcal{E} , the NPC is 4-cusp curve with 2-self intersections known as Talbot's Curve [12]. For the particular aspect ratio a/b = 2, the two self-intersections are at the foci of \mathcal{E} .



Figure 2: Left: The Negative Pedal Curve (NPC, purple) of \mathcal{E} with respect to a boundary point M is a 3-cusped (labeled P'_i) asymmetric curve (called here "Steiner's Hat"), whose area is invariant over M, and whose asymmetric shape is affinely related to the Steiner Curve [12]. $\Delta_u(t)$ is the instantaneous tangency point to the Hat. **Right**: The tangents at the cusps points P'_i concur at C_2 , the Hat's center of area, furthermore, P_i, P'_i, C_2 are collinear. **Video:** [10, PL#01]



Figure 3: Two systems which generate the 3-cusp Steiner Curve (red), see [2] for more methods. Left: The locus of a point on the boundary of a circle of radius 1 rolling inside another of radius 3. **Right:** The envelope of Simson Lines (blue) of a triangle T (black) with respect to points P(t) on the Circumcircle [12]. Q(t) denotes the corresponding tangent. Nice properties include (i) the area of the Deltoid is half that of the Circumcircle, and (ii) the 9-point circle of T (dashed green) centered on X₅ (whose radius is half that of the Circumcircle) is internally tangent to the Deltoid [13, p.231].

2 Preliminaries

Let the ellipse \mathcal{E} be defined implicitly as:

$$\mathcal{E}(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad c^2 = a^2 - b^2$$

where a > b > 0 are the semi-axes. Let a point P(t) on its boundary be parametrized as $P(t) = (a \cos t, b \sin t)$.

Let $P_0 = (x_0, y_0) \in \mathbb{R}^2$. Consider the family of lines L(t) passing through P(t) and orthogonal to $P(t) - P_0$. Its envelope Δ is called *antipedal* or *negative pedal curve* of \mathcal{E} .

Consider the spatial curve defined by

$$\mathcal{L}(P_0) = \{(x, y, t) : \mathcal{L}(t, x, y) = \mathcal{L}'(t, x, y) = 0\}.$$

The projection $\mathcal{E}(P_0) = \pi(\mathcal{L}(P_0))$ is the envelope. Here $\pi(x, y, t) = (x, y)$. In general, $\mathcal{L}(P_0)$ is regular, but $\mathcal{E}(P_0)$ is a curve with singularities and/or cusps.

Lemma 1 The envelope of the family of lines L(t) is given by:

$$\begin{aligned} x(t) &= \frac{1}{w} [(ay_0 \sin t - ab)x_0 - by_0^2 \cos t - c^2 y_0 \sin(2t) \\ &+ \frac{b}{4} ((5a^2 - b^2) \cos t - c^2 \cos(3t))] \\ y(t) &= \frac{1}{w} [-ax_0^2 \sin t + (by_0 \cos t + c^2 \sin(2t))x_0 - aby_0 \\ &- \frac{a}{4} ((5a^2 - b^2) \sin t - c^2 \sin(3t)] \end{aligned}$$
(1)

where $w = ab - bx_0 \cos t - ay_0 \sin t$.

Proof. The line L(t) is given by:

$$(x_0 - a\cos t)x + (y_0 - b\sin t)y + a^2\cos^2 t + b^2\sin^2 t - ax_0\cos t - by_0\sin t = 0.$$

Solving the linear system L(t) = L'(t) = 0 in the variables *x*, *y* leads to the result.

Triangle centers will be identifed below as X_k following Kimberling's Encyclopedia [6], e.g., X_1 is the Incenter, X_2 Barycenter, etc.

3 Main Results

Proposition 1 The NPC with respect to $M_u = (a \cos u, b \sin u)$ a boundary point of \mathcal{E} is a 3-cusp closed curve given by $\Delta_u(t) = (x_u(t), y_u(t))$, where

$$x_{u}(t) = \frac{1}{a} \left(c^{2} (1 + \cos(t + u)) \cos t - a^{2} \cos u \right)$$

$$y_{u}(t) = \frac{1}{b} \left(c^{2} \cos t \sin(t + u) - c^{2} \sin t - a^{2} \sin u \right)$$
(2)

Proof. It is direct consequence of Lemma 1 with $P_0 = M_u$.

Expressions for the 3 cusps P'_i in terms of *u* appear in Appendix A.

Remark 1 As $a/b \rightarrow 1$ the ellipse becomes a circle and Δ shrinks to a point on the boundary of said circle.

Remark 2 Though Δ can never have three-fold symmetry, for M_u at any ellipse vertex, it has axial symmetry.

Remark 3 The average coordinates $\overline{C} = [\overline{x}(u), \overline{y}(u)]$ of Δ_u w.r.t. this parametrization are given by:

$$\bar{x}(u) = \frac{1}{2\pi} \int_0^{2\pi} x_u(t) dt = -\frac{(a^2 + b^2)}{2a} \cos u$$
$$\bar{y}(u) = \frac{1}{2\pi} \int_0^{2\pi} y_u(t) dt = -\frac{(a^2 + b^2)}{2b} \sin u$$
(3)

Theorem 1 Δ_u is the image of the 3-cusp Steiner Hypocycloid *S* under a varying affine transformation.

Proof. Consider the following transformations in \mathbb{R}^2 :

rotation:
$$R_u(x,y) = \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

translation: $U(x,y) = (x,y) + \bar{C}$
homothety: $D(x,y) = \frac{1}{2}(a^2 - b^2)(x,y)$.
linear map: $V(x,y) = (\frac{x}{a}, \frac{y}{b})$

The hypocycloid of Steiner is given by $S(t) = 2(\cos t, -\sin t) + (\cos 2t, \sin 2t)$ [7]. Then:

$$\Delta_u(t) = [x_u(t), y_u(t)] = (U \circ V \circ D \circ R_u)\mathcal{S}(t)$$
(4)

Thus, Steiner's Hat is of degree 4 and of class 3 (i.e., degree of its dual). \Box

Corollary 1 The area of $A(\Delta)$ of Steiner's Hat is invariant over M_u and is given by:

$$A(\Delta) = \frac{(a^2 - b^2)^2 \pi}{2ab} = \frac{c^4 \pi}{2ab}$$
(5)

Proof. The area of S(t) is $\int_S x dy = 2\pi$. The Jacobian of $(U \circ S \circ D \circ R_u)$ given by Equation 4 is constant and equal to $c^4/4ab$.

Noting that the area of \mathcal{E} is πab , Table 1 shows the aspect ratios a/b of \mathcal{E} required to produce special area ratios.

a/b	approx. a/b	$A(\Delta)/A(\mathcal{E})$
$\sqrt{2+\sqrt{3}}$	1.93185	1
$\phi = (1 + \sqrt{5})/2$	1.61803	1/2
$\sqrt{2}$	1.41421	1/4
1	1	0

Table 1: Aspect ratios yielding special area ratios of main ellipse \mathcal{E} to Steiner's Hat Δ .

It is well known that if M is interior to \mathcal{E} then the NPC is a 2-cusp or 4-cusp curve with one or two self-intersections.

Remark 4 It can be shown that when M is interior to \mathcal{E} the iso-curves of signed area of the NPC are closed algebraic curves of degree 10, concentric with \mathcal{E} and symmetric about both axes, see Figure 4.

It is remarkable than when M moves from the interior to the boundary of \mathcal{E} , the iso-curves transition from a degree-10 curve to a simple conic.

Remark 5 It can also be shown that when M is exterior to \mathcal{E} , the NPC is a two-branched open curve, see Figure 5.



Figure 4: The isocurves of signed area for the negative pedal curve when M is interior to the ellipse are closed algebraic curves of degree 10. These are shown in gray for an NPC with two cusps (left), and 4 cusps (right).



Figure 5: Left: When *M* is exterior to \mathcal{E} the NPC is a two-branched open curve. One branch is smooth and non-self-intersecting, and the other has 2 cusps and one self-intersection. **Right:** Let t_1, t_2 be the parameters for which MP(t) is tangent to \mathcal{E} . At these positions, the NPC intersects the line at infinity in the direction of the normal at $P(t_1), P(t_2)$, i.e., the lines through P(t) perpendicular to P(t) - M are asymptotes.

Proposition 2 Let C_2 be the center of area of Δ_u . Then $C_2 = \overline{C}$.

Proof.

The center of area is defined by

$$C_2 = \frac{1}{A(\Delta)} \left(\int_{int(\Delta)} x \, dx \, dy, \int_{int(\Delta)} y \, dx \, dy \right)$$

Using Green's Theorem, evaluate the above using the parametric in Equation (1). This yields the expression for \bar{C} in Equation 3. Alternatively, one can obtain the same result from the affine transformation defined in Theorem 1. \Box

Referring to Figure 6(left):

Corollary 2 The locus of C_2 is an ellipse always exterior to a copy of \mathcal{E} rotated 90° about O.

Proof. Equation 3 describes an ellipse. Since $a^2 + b^2 \ge 2ab$ the claim follows directly.



Figure 6: Left: The area center C_2 of Steiner's Hat coincides with the barycenter X'_2 of the (dashed) triangle T' defined by the cusps. Over all M, both the Hat and T' have invariant area. C_2 's locus (dashed purple) is elliptic and exterior to a copy of \mathcal{E} rotated 90° about its center (dashed black). **Right:** Let P'_i (resp. P_i), i = 1, 2, 3 denote the Hat's cusps (resp. their pre-images on \mathcal{E}), colored by i. Lines $P_iP'_i$ concur at C_2 .

Let *T* denote the triangle of the *pre-images* P_i on \mathcal{E} of the Hat's cusps, i.e. $P(t_i)$ such that $\Delta_u t_i$ is a cuspid. Explicit expressions for the P_i appear in Appendix A. Referring to Figure 7:

Theorem 2 The points M, C_2, P_1, P_2, P_3 are concyclic.

Proof. $\Delta_u(t)$ is singular at $t_1 = -\frac{u}{3}$, $t_2 = -\frac{u}{3} - \frac{2\pi}{3}$ and $t_3 = -\frac{u}{3} - \frac{4\pi}{3}$. Let $P_i = [a \cos t_i, b \sin t_i], i = 1, 2, 3$. The circle \mathcal{K} passing through these is given by:

$$\mathcal{K}(x,y) = x^2 + y^2 - \frac{c^2 \cos u}{2a}x + \frac{c^2 \sin u}{2b}y - \frac{1}{2}(a^2 + b^2) = 0.$$
(6)

Also, $\mathcal{K}(M) = \mathcal{K}(a\cos u, b\sin u) = 0.$

The center of \mathcal{K} is $(M + C_2)/2$. It follows that $C_2 \in \mathcal{K}$ and that MC_2 is a diameter of \mathcal{K} .



Figure 7: The cusp pre-images P_i define a triangle T (orange) whose area is invariant over M. Its barycenter X_2 is stationary at the center of \mathcal{E} , rendering the latter its Steiner Ellipse. Let C_2 denote the center of area of Steiner's Hat. The 5 points M, C_2, P_1, P_2, P_3 lie on a circle (orange), with center at X_3 (circumcenter of T). Over all M, the T are a constant-area Poncelet family inscribed on \mathcal{E} and tangent to a concentric, axis-aligned elliptic caustic (dashed orange), half the size of \mathcal{E} , i.e., the latter is the (stationary) Steiner Inellipse of the T. Note also that M is the Steiner Point X_{99} of T since it is the intersection of its Circumcircle with the Steiner Ellipse. Furthermore, the Tarry Point X_{98} of T coincides with C_2 , since it is the antipode of $M = X_{99}$ [6]. Video: [10, PL#02,#05].

In 1846, Jakob Steiner stated that given a point M on an ellipse \mathcal{E} , there exist 3 other points on it such that the osculating circles at these points pass through M [8, page 317]. This property is also mentioned in [4, page 97, Figure 3.26].

It turns out the cusp pre-images are said special points! Referring to Figure 8:

Proposition 3 *Each of the 3 circles* \mathcal{K}_i *through* $M, P_i, P'_i, i = 1, 2, 3$, *osculate* \mathcal{E} *at* P_i .

Proof. The circle \mathcal{K}_1 passing through M, P_1 and P'_1 is given by

$$\begin{aligned} \mathcal{K}_{1}(x,y) &= 2ab(x^{2}+y^{2}) - 4bc^{2}\cos^{3}\left(\frac{u}{3}\right)x - 4ac^{2}\sin^{3}\left(\frac{u}{3}\right)y \\ &+ ab\left(3c^{2}\cos\left(\frac{2u}{3}\right) - a^{2} - b^{2}\right) = 0. \end{aligned}$$

Recall a circle osculates an ellipse if its center lies on the evolute of said ellipse, given by [4]:

$$E^{*}(t) = \left[\frac{c^{2} \cos^{3} t}{a}, -\frac{c^{2} \sin^{3} t}{b}\right]$$
(7)

It is straightforward to verify that the center of \mathcal{K}_1 is $P_1'' = \mathcal{E}^*(-\frac{u}{3})$. A similar analysis can be made for \mathcal{K}_2 and \mathcal{K}_3 .



Figure 8: The circles passing through a cusp P'_i , its preimage P_i , and M osculate \mathcal{E} at the P_i . The centers P''_i of said circles define a triangle T'' (dashed black) whose area is constant for all M. X''_2 denotes its (moving) barycenter. **Video:** [10, PL#03,#05].

Since the area of \mathcal{E}^* is $A(\mathcal{E}^*) = \frac{3\pi c^4}{8ab}$, and the area of Δ is given in Equation 5:

Remark 6 *The area ratio of* Δ *and the interior of* \mathcal{E}^* *is equal to* 4/3.

3.1 Why is Δ affine to Steiner's Curve

Up to projective transformations, there is only one irreducible curve of degree 4 with 3 cusps. In a projective coordinate frame $(x_0 : x_1 : x_2)$ with the cusps as base points (1 : 0 : 0), (0 : 1 : 0) and (0 : 0 : 1) and the common point of the cusps' tangents as unit point (1:1:1), the quartic has the equation

$$x_0^2 x_1^2 + x_0^2 x_2^2 + x_1^2 x_2^2 - 2x_0 x_1 x_2 (x_0 + x_1 + x_2) = 0$$

At Steiner's three-cusped curve, the cusps form a regular triangle with the tangents passing through the center. Hence, whenever a three-cusped quartic has the meeting point of the cusps' tangents at the center of gravity of the cusps, it is affine to Steiner's curve, since there is an affine transformation sending the four points into a regular triangle and its center.

4 The Cusp Triangle

Recall $T' = P'_1 P'_2 P'_3$ is the triangle defined by the 3 cusps of Δ .

Proposition 4 The area A' of the cusp triangle T' is invariant over M and is given by:

$$A' = \frac{27\sqrt{3}}{16} \frac{c^4}{ab}$$

Proof. The determinant of the Jacobian of the affine transformation in Theorem 1 is $|J| = \frac{c^4}{4ab}$. Therefore, the area

of T' is simply $|J|A_e$, where A_e is the area of an equilateral triangle inscribed in a circle of radius 3 with side $3\sqrt{3}$. \Box

Referring to Figure 6:

Proposition 5 The barycenter X'_2 of T' coincides with the center of area C_2 of Δ .

Proof. Direct calculations yield $X'_2 = C_2$.

Referring to Figure 9:

Proposition 6 The Steiner Ellipse \mathcal{E}' of T' has constant area and is a scaled version of \mathcal{E} rotated 90° about O.

Proof. \mathcal{E}' passes through the vertices of T' and is centered on $C_2 = X'_2$. Direct calculations yield the following implicit equation for it:

$$a^{2}x^{2} + b^{2}y^{2} + (a^{2} + b^{2}) (a\cos ux + b\sin uy) - (a^{2} - 2b^{2}) (2a^{2} - b^{2}) = 0$$

Its semi-axes are $b' = \frac{3c^2}{2a}$ and $a' = \frac{3c^2}{2b}$. Therefore \mathcal{E}' is similar to a 90°-rotated copy of \mathcal{E} .

Remark 7 *This proves that* T' *can never be regular and* Δ *has never a three-fold symmetry.*

Corollary 3 The ratio of area of \mathcal{E}' and \mathcal{E} is given by $\frac{9}{4}\frac{c^4}{a^2b^2}$, and at $a/b = (1 + \sqrt{10})/3$, the two ellipses are congruent.



Figure 9: Left: The Steiner Ellipse \mathcal{E}' of triangle T' defined by the P'_i is a scaled-up and 90°-rotated copy of \mathcal{E} . Right: At $a/b = (1 + \sqrt{10})/3 \simeq 1.38743$, \mathcal{E}' and \mathcal{E} have the same area.

Proposition 7 The Steiner Point X'_{99} of T' is given by:

$$X'_{99}(u) = \left[\frac{(a^2 - 2b^2)}{a}\cos u, -\frac{(2a^2 - b^2)}{b}\sin u\right]$$

Proof. By definition X_{99} is the intersection of the circumcircle of $T(\mathcal{K})$ with the Steiner ellipse. The Circumcircle \mathcal{K}' of the triangle $T' = \{P'_1, P'_2, P'_3\}$ is given by:

$$\mathcal{K}'(x,y) = 8a^{2}b^{2}\left(x^{2}+y^{2}\right) + 2a\cos u\left(3a^{4}-2a^{2}b^{2}+7b^{4}\right)x$$

+2b sin u $\left(7a^{4}-2a^{2}b^{2}+3b^{4}\right)y$
- $\left(a^{2}+b^{2}\right)\left(c^{2}\left(a^{2}+b^{2}\right)\cos 2u+5a^{4}-14a^{2}b^{2}+5b^{4}\right)$
=0.

With the above, straightforward calculations lead to the coordinates of X'_{99} .

5 The Triangle of Cusp Pre-Images

Recall $T = P_1 P_2 P_3$ is the triangle defined by pre-images on \mathcal{E} to each cusp of Δ .

Proposition 8 The barycenter X_2 of T is stationary at O, *i.e.*, \mathcal{E} is is Steiner Ellipse [12].

Proof. The triangle *T* is an affine image of an equilateral triangle with center at 0 and $P_i = \mathcal{E}(t_i) = \mathcal{E}(-\frac{u}{3} - (i-1)\frac{2\pi}{3})$. So the result follows.

Remark 8 *M* is the Steiner Point X₉₉ of T.

Proposition 9 Over all M, the T are an N = 3 Poncelet family with external conic \mathcal{E} with the Steiner Inellipse of T as its caustic [12]. Futhermore the area of these triangles is invariant and equal to $\frac{3\sqrt{3}ab}{4}$.

Proof. The pair of concentric circles of radius 1 and 1/2 is associated with a Poncelet 1d family of equilaterals. The image of this family by the map $(x, y) \rightarrow (ax, by)$ produces the original pair of ellipses, with the stated area. Alternatively, the ratio of areas of a triangle to its Steiner Ellipse is known to be $3\sqrt{3}/(4\pi)$ [12, Steiner Circumellipse] which yields the area result.

6 Locus of the Cusps

We analyze the motion of the cusps P'_i of Steiner's Hat Δ with respect to continuous revolutions of M on \mathcal{E} . Referring to Figure 10:

Remark 9 The locus C(u) of the cusps of Δ is parametrized by:

$$C(u): \frac{3c^2}{2} \left[\frac{1}{a} \cos \frac{u}{3}, \frac{1}{b} \sin \frac{u}{3} \right] - \frac{a^2 + b^2}{2} \left[\frac{1}{a} \cos u, \frac{1}{b} \sin u \right]$$
(8)

This is a curve of degree 6, with the following implicit equation:

$$\begin{aligned} &-4a^{6}x^{6} - 4b^{6}y^{6} - 12a^{2}x^{2}b^{2}y^{2}\left(a^{2}x^{2} + b^{2}y^{2}\right) \\ &+12a^{4}\left(a^{4} - a^{2}b^{2} + b^{4}\right)x^{4} + 12b^{4}\left(a^{4} - a^{2}b^{2} + b^{4}\right)y^{4} \\ &+24a^{2}b^{2}\left(a^{4} - a^{2}b^{2} + b^{4}\right)x^{2}y^{2} \\ &-3a^{2}\left(2a^{2} - b^{2}\right)\left(a^{2} + b^{2}\right)\left(2a^{4} - 5a^{2}b^{2} + 5b^{4}\right)x^{2} \\ &+3b^{2}\left(a^{2} - 2b^{2}\right)\left(a^{2} + b^{2}\right)\left(5a^{4} - 5a^{2}b^{2} + 2b^{4}\right)y^{2} \\ &+\left(2a^{2} - b^{2}\right)^{2}\left(a^{2} - 2b^{2}\right)^{2}\left(a^{2} + b^{2}\right)^{2} = 0\end{aligned}$$

Proposition 10 It can be shown that over one revolution of M about \mathcal{E} , C_2 will cross the ellipse on four locations $W_j, j = 1, \dots, 4$ given by:

$$W_j = \frac{1}{2\sqrt{a^2 + b^2}} \left(\pm a\sqrt{a^2 + 3b^2}, \pm b\sqrt{3a^2 + b^2} \right)$$

At each such crossing, C_2 coincides with one of the preimages.

Proof. From the coordinates of C_2 given in equation (3) in terms of the parameter u, one can derive an equation that is a necessary and sufficient condition for $C_2 \in \mathcal{E}$ to happen, by substituting those coordinates in the ellipse equation $x^2/a^2 + y^2/b^2 - 1 = 0$. Solving for sinu and substituting back in the coordinates of C_2 , one easily gets the four solutions W_1, W_2, W_3, W_4 .

Now, assume that $C_2 \in \mathcal{E}$. The points M, P_1, P_2, P_3, C_2 must all be in both the ellipse $\mathcal E$ and the circumcircle $\mathcal K$ of $P_1P_2P_3$. Since the two conics have at most 4 intersections (counting multiplicity), 2 of those 5 points must coincide. It is easy to verify from the previously-computed coordinates that M can only coincide with the preimages P_1, P_2, P_3 at the vertices of \mathcal{E} . In such cases, owing to the symmetry of the geometry about either the x- or y-axis, the circle \mathcal{K} must be tangent to \mathcal{E} at *M*. Thus, that intersection will count with multiplicity (of at least) 2, so another pair of those 5 points must also coincide. Since P_1, P_2, P_3 must all be distinct, C_2 will coincide with one of the preimages. However, this can never happen, since if M is on one of the vertices of \mathcal{E} , C_2 won't be in \mathcal{E} . In any other case, since P_1, P_2, P_3 must be distinct and C_2 is diametrically opposed to M in \mathcal{K} , C_2 must coincide with one of the preimages. \square

Remark 10 C_2 will visit each of the preimages cyclically. Moreover, upon 3 revolutions (with 12 crossings in total), each P_i will have been visited four times and the process repeats.



Figure 10: The loci of the cusps of Δ (dashed line) is a degree-6 curve with 2 internal lobes with either 2, 3, or 4 self-intersections. From left to right, $a/b = \{1.27, \sqrt{2}, 1.56\}$. Note that at $a/b = \sqrt{2}$ the two lobes touch, i.e., the cusps pass through the center of \mathcal{E} . Also shown is the elliptic locus of C_2 (purple). Points Z_i (resp. W_i) mark off the intersection of the locus of the cusps (resp. of C_2) with \mathcal{E} . These never coincide **Video:** [10, PL#04].

6.1 Tangencies and Intersections of the Deltoid with the Ellipse

Definition 1 (Apollonius Hyperbola) Let M be a point on an ellipse \mathcal{E} with semi-axes a, b. Consider a hyperbola \mathcal{H} , known as the Apollonius Hyperbola of M [5]:

$$\mathcal{H}: \langle (x,y) - M, (y/b^2, -x/a^2) \rangle = 0.$$

Notice that for P on \mathcal{E} , only the points for which the normal at P points to M will lie on \mathcal{H} . See also [4, page 403]. Additionally, \mathcal{H} passes through M and O, and its asymptotes are parallel to the axes of \mathcal{E} .

Proposition 11 Δ *is tangent to* \mathcal{E} *at* $\mathcal{E} \cap \mathcal{H}$ *, at 1, 2 or 3 points* Q_i *depending on whether* M *is exterior or interior to the evolute* \mathcal{E}^* .

Proof. Δ is tangent to \mathcal{E} at some Q_i if the normal of \mathcal{E} at Q_i points to $M = (M_x, M_y)$, i.e., when \mathcal{H} intersects with \mathcal{E} . It can be shown that their *x* coordinate is given by the real roots of:

$$Q(x) = c^4 x^3 - c^2 M_x (a^2 + b^2) x^2 - a^4 (a^2 - 2b^2) x + a^6 M_x = 0$$
(9)

The discriminant of the above is:

$$-4c^4a^6(a^2-M_x^2)[(a^2-b^2)(a^2+b^2)^3M_x^2+a^4(a^2-2b^2)^3].$$

Let $\pm x^*$ denote the solutions to $(a^2 - b^2)(a^2 + b^2)^3 M_x^2 + a^4(a^2 - 2b^2)^3 = 0$. Assuming a > b, Equation 9 has three

real solutions when $|x| < x^*$. The intersections of the evolute \mathcal{E}^* with the ellipse \mathcal{E} are given by the four points $(\pm x^*, \pm y^*)$, where:

$$x^{*} = \frac{a^{2}\sqrt{a^{4} - b^{4}} \left(a^{2} - 2b^{2}\right)^{\frac{3}{2}}}{\left(a^{4} - b^{4}\right)\left(a^{2} + b^{2}\right)}$$
$$y^{*} = \frac{b^{2}\sqrt{a^{4} - b^{4}} \left(2a^{2} - b^{2}\right)^{\frac{3}{2}}}{\left(a^{4} - b^{4}\right)\left(a^{2} + b^{2}\right)}$$
(10)

For $M \in \mathcal{E} \cap \mathcal{E}^*$ two coinciding roots result in a 4-point contact between Δ and the ellipse.

Let $M = (M_x, M_y)$ be a point on \mathcal{E} and $\mathcal{I}(x)$ denote the following cubic polynomial:

$$\mathcal{I}(x) = (a^2 + b^2)^2 x^2 - 2M_x c^2 (a^2 + b^2) x - 4a^4 b^2 + M_x^2 (a^2 + b^2)^2$$
(11)

Proposition 12 Δ intersects \mathcal{E} at $Q(x)\mathcal{I}(x) = 0$, in at least 3 and up to 5 locations locations, where Q is as in Equation 9.

Proof. As before, $M = (M_x, M_y) = (a \cos u, b \sin u) \in \mathcal{E}$ and $P = (x, y) = (a \cos t, b \sin t)$. The intersection $\Delta_u(t)$ with \mathcal{E} is obtained by setting $\mathcal{E}(\Delta_u(t)) = 0$. Using Equation 1, obtain the following system:



Figure 11: Steiner's Hat Δ (purple, top cusp not shown) is tangent to \mathcal{E} at the intersections Q_i of the Apollonius Hyperbola \mathcal{H} (olive green) with \mathcal{E} , excluding M. Notice \mathcal{H} passes through the center of \mathcal{E} . Left: When M is exterior to the evolute \mathcal{E}^* (dashed purple), only one tangent Q_1 is present. **Right:** When M is interior to \mathcal{E}^* , three tangent points Q_i , i = 1, 2, 3 arise. The intersections of \mathcal{E}^* are given in Equation 10. Note: the area ratio of Δ -to- \mathcal{E}^* is always 4/3.

$$\begin{split} F(x,y) =& b^2 \left(a^2 - 2M_x^2\right) \left(a^2 + b^2\right) c^4 x^4 \\ &+ 2 a^2 M_x M_y \left(a^2 + b^2\right) c^4 x^3 y \\ &+ 2 a^2 b^2 M_x c^2 \left(a^4 + b^4\right) x^3 - 2 a^4 M_y c^2 \left(a^4 + b^4\right) x^2 y \\ &+ [-a^4 b^2 \left(a^2 + b^2\right) \left(3 a^4 - 4 a^2 b^2 + 2 b^4\right) \\ &+ a^2 b^2 c^2 M_x^2 \left(3 a^2 - b^2\right) \left(a^2 + b^2\right)] x^2 \\ &- 2 a^6 M_x M_y c^2 \left(a^2 + b^2\right) x y \\ &- 2 b^2 M_x a^6 \left(a^4 - a^2 b^2 + b^4\right) x \\ &+ 2 a^{12} M_y y + a^8 b^2 \left(2 a^4 - (a^2 + b^2) M_x^2\right) = 0 \\ \mathcal{E}(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \end{split}$$

By Bézout's theorem, the system $\mathcal{E}(x,y) = F(x,y) = 0$ has eight solutions, with algebraic multiplicities taken into account. Δ has three points of tangency with ellipse, some of which may be complex, which by Proposition 11 are given by the zeros of Q(x). Eliminating *y* by computing the resultant we obtain an Equation G(x) = 0 of degree 8 over *x*. Manipulation with a Computer Algebra System yields a compact representation for G(x):

 $G(x) = Q(x)^2 \mathcal{I}(x)$

with \mathcal{I} as in Equation 11. If $|M_x| \leq a$, the solutions of $\mathcal{I}(x) = 0$ are real and given by $J_x = [c^2 M_x \pm 2ab\sqrt{a^2 - M_x^2}]/(a^2 + b^2)$ and $|J_x| \leq a$.

Referring to Figure 10:

Proposition 13 When a cusp P'_i crosses the boundary of \mathcal{E} , it coincides with its pre-image P_i at $Z_i = \frac{1}{\sqrt{a^2+b^2}}(\pm a^2, \pm b^2)$.

Proof. Assume P'_i is on \mathcal{E} . Since P'_i is on the circle \mathcal{K}_i defined by M, P_i and P'_i which osculates \mathcal{E} at P_i , this circle intersects \mathcal{E} at P_i with order of contact 3 or 4. By construction, we have $MP_i \perp P_iP'_i$, so M and P'_i are diametrically opposite in \mathcal{K}_i . Thus, M and P'_i must be distinct. Since two conics have at most 4 intersections (counting multiplicities), we either have $P'_i = P_i$ or $P_i = M$. The second case will only happen when M is on one of the four vertices of the ellipse \mathcal{E} , in which case the osculating circle \mathcal{K}_i has order of contact 4, so P'_i could not also be in the ellipse in the first place. Thus, $P'_i = P_i$ as we wanted.

Substituting the parameterization of P_i in the equation of \mathcal{E} , we explicitly find the four points Z_i at which P'_i can intersect the ellipse \mathcal{E} .

7 A Triad of Osculating Circles

Recall \mathcal{K}_i are the circles which osculate \mathcal{E} at the pre-images P_i , see Figure 8. Define a triangle T'' by the centers P''_i of the \mathcal{K}_i . These are given explicit coordinates in Appendix A. Referring to Figure 12:

Proposition 14 *Triangles* T' *and* T'' *are homothetic at ratio* 2 : 1, *with* M *as the homothety center.* **Proof.** From the construction of Δ , for each i = 1, 2, 3 we have $MP_i \perp P_iP'_i$, that is, $\angle MP_iP'_i = 90^\circ$. Hence, MP'_i is a diameter of the osculating circle \mathcal{K}_i that goes through M, P_i , and P'_i as proved in Proposition 3. Thus, the center P''_i of \mathcal{K}_i is the midpoint of MP'_i and therefore P'_i is the image of P''_i under a homothety of center M and ratio 2.

Corollary 4 The area A'' of T'' is invariant over all M and is 1/4 that of T'.

Proof. This follows from the homothety, and the fact that the area of T' is invariant from Proposition 4.



Figure 12: Lines connecting each cusp P'_i to the center P''_i of the circle which osculates \mathcal{E} at the pre-image P_i concur at M. Note these lines are diameters of said circles. Therefore M is the perspector of T' and T'', i.e., the ratio of their areas is 4. This perspectivity implies C_2, X''_2 , M are collinear. Surprisingly, the X''_2 coincides with the circumcenter X_3 of the pre-image triangle T (not drawn).

Proposition 15 Each (extended) side of T' passes through an intersection of two osculating circles. Moreover, those sides are perpendicular to the radical axis of said circle pairs.

Proof. It suffices to prove it for one of the sides of T' and the others are analogous. Let M_1 be the intersection of \mathcal{K}_2 and \mathcal{K}_3 different than M and let $M_{1/2}$ be the midpoint of M and M_1 . Since MM_1 is the radical axis of \mathcal{K}_2 and \mathcal{K}_3 , the lines $P_2''P_3''$ and MM_1 are perpendicular and their intersection is $M_{1/2}$. Applying an homothety of center M and ratio 2, we get that the lines $P_2'P_3''$ and MM_1 (the radical axis) are perpendicular and their intersection is M_1 , as desired. \Box

Corollary 5 The Steiner ellipse \mathcal{E}'' of triangle T'' is similar to \mathcal{E}' . In fact, $\mathcal{E}'' = \frac{1}{2}\mathcal{E}'$.

Proof. This follows from the homothety of T' and T''. \Box

8 Relations between T, T', T''

As before we identify Triangle centers as X_k after Kimberling's Encyclopedia [6].

Proposition 16 *The circumcenter* X_3 *of* T *coincides with the barycenter* X_2'' *of* T''.

Proof: Follows from direct calculations using the coordinate expressions of P_i and P''_i . In fact,

$$X_2'' = X_3 = \frac{c^2}{4} \left[\frac{\cos u}{a}, -\frac{\sin u}{b} \right].$$

Corollary 6 *The homothety with center M and factor* 2 *sends* X_2'' *to* $X_2' = C_2$.

Proposition 17 The lines joining a cusp P'_i to its preimage P_i concur at Δ 's center of area C_2 .

Proof. From Theorem 2, points *M* and *C*₂ both lie on the circumcircle of *T* and form a diameter of this circle. Thus, for each i = 1, 2, 3, we have $\angle MP_iC_2 = 90^\circ$. By construction, $\angle MP_iP'_i = 90^\circ$, so P_i, P'_i , and C_2 are collinear as desired. \Box

Referring to Figure 14(left):

Corollary 7 $C_2 = X'_2$ is the perspector of T' and T.

Lemma 2 Given a triangle T, and its Steiner Ellipse Σ , the normals at each vertex pass through the Orthocenter of T, i.e., they are the altitudes.

Proof. This stems from the fact that the tangent to Σ at a vertex of \mathcal{T} is parallel to opposide side of \mathcal{T} [12, Steiner Circumellipse].

Referring to Figure 14(right):

Proposition 18 The orthocenter X_4 is the perspector of T and T''. Equivalently, a line connecting a vertex of T to the respective vertex of T'' is perpendicular to the opposite side of T.

Proof. Since *T* has fixed X_2 , \mathcal{E} is its Steiner Ellipse. The normals to the latter at P_i pass through centers P''_i since these are osculating circles. So by Lemma 2 the proof follows.

Definition 2 According to J. Steiner [3, p. 55], two triangles ABC and DEF are said to be orthologic if the perpendiculars from A to EF, from B to DF, and from C to DE are concurrent. Furthermore, if this holds, then the perpendiculars from D to BC, from E to AC, and from F to AB are also concurrent. Those two points of concurrence are called the centers of orthology of the two triangles [9].

Note that orthology is symmetric but not transitive [9, p. 37], see Figure 13 for a non-transitive example involving a reference, pedal, and antipedal triangles.



Figure 13: Consider a reference triangle \mathcal{T} (blue), and its pedal \mathcal{T}' (red) and antipedal \mathcal{T}'' (green) triangles with respect to some point Z. Construction lines for both pedal and antipedal (dashed red, dashed green) imply that Z is an orthology center simultaneousy for both $\mathcal{T}, \mathcal{T}'$ and $\mathcal{T}, \mathcal{T}''$, i.e. these pairs are orthologic. Also shown are H' and H", the 2nd orthology centers of said pairs (construction lines omitted). Non-transitivity arises from the fact that perpendiculars dropped from the vertices of \mathcal{T}' to the sides of \mathcal{T}'' (dashed purple, feet are marked X) are non-concurrent (purple diamonds mark the three disjoint intersections), i.e., $\mathcal{T}', \mathcal{T}''$ are not orthologic.

Lemma 3 Let $-P_i$ denote the reflection of P_i about $O = X_2$ for i = 1, 2, 3. Then the line from M to $-P_1$ is perpendicular to the line $P_2''P_3''$, and analogously for $-P_2$, and $-P_3$.

Proof. This follows directly from the coordinate expressions for points M, $P_i = \mathcal{E}(t_i)$ and $P''_i = \mathcal{E}^*(t_i)$. It follows that $\langle M + P_1, P''_2 - P''_3 \rangle = 0$.

Referring to Figure 14(right):

Theorem 3 Triangles T and T' are orthologic and their centers of orthology are the reflections X_{671} of M on X_2 and on X_4 .

Proof. We denote by X_{671} the reflection of $M = X_{99}$ on $O = X_2$. From Lemma 3, the line through M and $-P_1$ is perpendicular to $P''_2P''_3$. Reflecting about $O = X_2$, the line P_1X_{671} is also perpendicular to $P''_2P''_3$. Since $P''_2P''_3 \parallel P'_2P'_3$ from the homothety, we get that $P_1X_{671} \perp P'_2P'_3$. This means the perpendicular from P_1 to $P'_2P'_3$ passes through X_{671} . Analogously, the perpendiculars from P_2 to $P'_1P'_3$ and from P_3 to $P'_1P'_2$ also go through X_{671} . Therefore T and T' are orthologic and X_{671} is one of their two orthology centers.

Let X_h be the reflection of M on X_4 . From Proposition 18, the line through P_1 and P''_1 passes through the orthocenter X_4 of T, that is, P''_1 is on the P_1 -altitude of T. This means that the perpendicular from P''_1 to the line P_2P_3 passes through X_4 . Applying the homothety with center M and ratio 2, the perpendicular from X'_1 to X_2X_3 passes through X_h . Analogously, the perpendiculars from X'_2 to X_1X_3 and from X'_3 to X_1X_2 also pass through X_h . Hence, X_h is the second orthology center of T and T'.

Theorem 4 Triangles T and T'' are orthologic and their centers of orthology are X_4 and the reflection X_{671} of M on X_2 .

Proof. From Proposition 18, the perpendiculars from P_1'' to P_2P_3 , from P_2'' to P_1P_3 , and from P_3'' to P_1P_2 all pass through X_4 . Thus, triangles T and T'' are orthologic and X_4 is one of their two orthology centers.

As before, we denote by X_{671} the reflection of $M = X_{99}$ on $O = X_2$. Again, from Lemma 3, the line through M and $-P_1$ is perpendicular to $P_2''P_3''$, so reflecting it at X_2 , we get that $P_1X_{671} \perp P_2''P_3''$. Since the triangles T'' and T' have parallel sides, we get $P_1X_{671} \perp P_1'P_3' \parallel P_1''P_3''$. Thus, X_{671} is the second orthology center of T and T''.



Figure 14: Left: T and T' are perspective on X_2 . They are also orthologic, with orthology centers X_{671} and the reflection of X_{99} on X_4 . Right: T and T'' are perspective on X_4 . They are also orthologic, with orthology centers X_4 and X_{671} .

Theorem 5 (Sondat's Theorem) If two triangles are both perspective and orthologic, their centers of orthology and perspectivity are collinear. Moreover, the line through these centers is perpendicular to the perspectrix of the two triangles [11, 9].

Referring to Figure 15:

Theorem 6 The perspectrix of T and T' is perpendicular to the Euler Line of T.



Figure 15: The perspectrix of T, T'' is perpendicular to the line through X_4 and X_{671} . Compare with Figure 13: the perspectrix of T, T' is perpendicular to the Euler Line of T.

Proof. Since *T* and *T'* are both orthologic and perspective from Corollary 7 and Theorem 3, by Sondat's Theorem, their perspectrix is perpendicular to the line through their orthology centers (reflections of *M* at X_2 and X_4) and perspector ($X'_2 = C_2 = X_{98}$ =reflection of *M* at X_3). By applying a homothety of center *M* and ratio 1/2, this last line is parallel to the line through X_2 , X_3 , and X_4 , the Euler line of *T*. Therefore the perspectrix of *T* and *T'* is perpendicular to the Euler line of *T*.

Proposition 19 The perspectrix of T and T'' is perpendicular to the line X_4X_{671} (which is parallel to the line through M and X_{376} , the reflection of X_2 at X_3).

Proof. Since *T* and *T*["] are both orthologic and perspective from Proposition 18 and Theorem 4, by Sondat's Theorem, their perspectrix is perpendicular to the line through their orthology centers X_4 and X_{671} . Reflecting this last line at X_2 , we find that it is parallel to the line through *M* and the reflection of X_4 at X_2 , which is the same as the reflection of X_2 at X_3 .

Table 2 lists a few pairs of triangle centers numerically found to be common over T, T' or T, T''.

Т	T'	T''
X_3	-	X_2''
X_4	-	X_{671}''
X_5	-	X_{115}''
X_{20}	-	X_{99}''
X_{76}	-	$X_{598}^{''}$
X_{98}	X'_2	-
<i>X</i> ₁₁₄	$X_{230}^{\overline{\prime}}$	_
X382	_	X_{148}''
X_{548}	_	X_{620}''
X_{550}	-	$X_{2482}^{\prime\prime}$

Table 2: Triangle Centers which coincide T, T' or T, T''.

9 Addendum: Rotated Negative Pedal Curve

The Negative Pedal Curve is the envelope of lines L(t) passing through P(t) and perpendicular to P(t) - M. Here we consider the envelope Δ_{θ}^* of the L(t) rotated clockwise a fixed θ about P(t); see Figure 16.

Proposition 20 Δ_{θ}^* is the image of the NPC Δ under the similarity which is the product of a rotation about *M* through θ and a homothety with center *M* and factor $\cos \theta$.

Proof. For variable parameter *t*, the lines L(t) and $L_{\theta}^{*}(t)$ perform a motion which sends *P* along \mathcal{E} , while the line through *P* orthogonal to L(t) slides through the fixed point *M*. Due to basic results of planar kinematics [1, p. 274], the instantaneous center of rotation *I* lies on the normal to \mathcal{E} at *P* and on the normal to *MP* at *M*. We obtain a rectangle with vertices *P*, *M* and *I*. The fourth vertex is the enveloping point *C* of L(t). The enveloping point C^* of L_{θ}^* is the pedal point of *I*. Since the circumcircle of the rectangle with diameter *MC* also passes through C^* , we see that C^* is the image of *C* under the stated similarity, Figure 17.

This holds for all points on Δ , including the cusps, but also for the center C_2 . At poses where *C* reaches a cusp P'_i of Δ , then for all lines $L^*_{\theta}(t)$ through *P* the point C^* is a cusp of the corresponding envelope. Then the point is the so-called return pole, and the circular path of *C* the return circle or cuspidal circle [1, p. 274].

Corollary 8 The area of Δ_{θ}^* is independent of M and is given by:

$$A = \frac{c^4 \cos^2 \theta \,\pi}{2ab}.$$

Note this is equal to $\cos^2 \theta$ of the area of Δ , see Equation 5.

Remark 11 For variable θ between -90° and 90° , the said similarity defines an equiform motion where each point in the plane runs along a circle through M with the same

angular velocity. For each point, the configuration at $\theta = 0$ and M define a diameter of the trajectory.



Figure 16: From left to right: for a fixed M, the line passing through P(t) and perpendicular to the segment P(t) - M (dashed) purple is rotated clockwise by $\theta = 30,45,60$ degrees, respectively (dashed olive green). For all P(t) these envelop new constant-area crooked hats Δ^* (olive green) whose areas are $\cos(\theta)^2$ that of Δ . For the θ shown, these amount to 3/4, 1/2, 1/4 of the area of Δ (purple). As one varies θ , the center of area C_2^* of Δ^* sweeps the circular arc between C_2 and M with center at angle 2 θ . The same holds for the cusps running along the corresponding osculating circles (shown dashed red, green, blue), which are stationary and independent of θ . **Video:** [10, PL#06]



Figure 17: The construction of points C, C*of the envelopes Δ (red) and Δ^* (blue) with the help of the instant center of rotation I reveals that the rotation about M through θ and scaling with factor $\cos \theta$ sends Δ to Δ^* (Proposition 20).

Recall the pre-images P_i of the cusps of Δ have vertices at $\mathcal{E}(t_i)$, where $t_1 = -\frac{u}{3}$, $t_2 = -\frac{u}{3} - \frac{2\pi}{3}$, $t_3 = -\frac{u}{3} - \frac{4\pi}{3}$, see Theorem 2.

Corollary 9 The cusps P_i^* of Δ_{θ}^* have pre-images on \mathcal{E} which are invariant over θ and are congruent with the $P_{i,i} = 1, 2, 3$.

Corollary 10 Lines $P_i P_i^*$ concur at C_2^* .

Corollary 11 C_2^* is a rotation of C_2 by 2 θ about the center X_3 of K. In particular, When $\theta = \pi/2$, $C_2^* = M$, and Δ_{θ}^* degenerates to point M.

10 Conclusion

Before we part, we would like to pay homage to eminent swiss mathematician Jakob Steiner (1796–1863), discoverer of several concepts appearing herein: the Steiner Ellipse and Inellipse, the Steiner Curve (or hypocycloid), the Steiner Point X_{99} . Also due to him is the concept of orthologic triangles and the theorem of 3 concurrent osculating circles in the ellipse. Hats off and vielen dank, Herr Steiner!

Some of the above phenomena are illustrated dynamically through the videos on Table 3.

PL#	Title	Narrated
01	Constant-Area Deltoid	no
02	Properties of the Deltoid	yes
03	Osculating Circles at the Cusp Pre-Images	yes
04	Loci of Cusps and C_2	no
05	Concyclic pre-images, osculating circles,	no
	and 3 area-invariant triangles	
06	Rotated Negative Pedal Curve	yes

 Table 3: Playlist of videos. Column "PL#" indicates the entry within the playlist.

Appendix A. Explicit Expressions for the P_i, P'_i, P''_i

$$\begin{split} P_1 &= \left[a \cos \frac{u}{3}, -b \sin \frac{u}{3}, \right] \\ P_2 &= \left[-a \sin \left(\frac{u}{3} + \frac{\pi}{6} \right), -b \cos \left(\frac{u}{3} + \frac{\pi}{6} \right) \right] \\ P_3 &= \left[-a \cos \left(\frac{u}{3} + \frac{\pi}{6} \right), b \sin \left(\frac{u}{3} + \frac{\pi}{6} \right) \right] \\ P_1' &= \left[\frac{3c^2}{2a} \cos \frac{u}{3} - \frac{(a^2 + b^2)}{2a} \cos u, \frac{3c^2}{2b} \sin \frac{u}{3} - \frac{(a^2 + b^2)}{2b} \sin u \right] \\ P_2' &= \left[-\frac{3c^2}{4a} \cos \frac{u}{3} - \frac{3\sqrt{3}c^2}{4a} \sin \frac{u}{3} - \frac{(a^2 + b^2)}{2a} \cos u, \frac{3\sqrt{3}c^2}{4b} \cos \frac{u}{3} - \frac{3c^2}{4b} \sin \frac{u}{3} - \frac{(a^2 + b^2)}{2b} \sin u \right] \\ P_3' &= \left[-\frac{3c^2}{4a} \cos \frac{u}{3} + \frac{3\sqrt{3}c^2}{4a} \sin \frac{u}{3} - \frac{(a^2 + b^2)}{2a} \cos u, -\frac{3\sqrt{3}c^2}{4b} \cos \frac{u}{3} - \frac{3c^2}{4b} \sin \frac{u}{3} - \frac{(a^2 + b^2)}{2b} \sin u \right] \\ P_1'' &= \left[\frac{3c^2}{4a} \cos \frac{u}{3} + \frac{c^2}{4a} \cos u, \frac{3c^2}{4b} \sin \frac{u}{3} + \frac{c^2}{4b} \sin u \right] \\ P_2'' &= \left[-\frac{3c^2}{8a} \cos \frac{u}{3} - \frac{3\sqrt{3}c^2}{8a} \sin \frac{u}{3} + \frac{c^2}{4a} \cos u, \frac{3\sqrt{3}c^2}{8b} \cos \frac{u}{3} - \frac{3c^2}{8b} \sin \frac{u}{3} - \frac{c^2}{4b} \sin u \right] \\ P_3'' &= \left[-\frac{3c^2}{8a} \cos \frac{u}{3} + \frac{3\sqrt{3}c^2}{8a} \sin \frac{u}{3} + \frac{c^2}{4a} \cos u, \frac{3\sqrt{3}c^2}{8b} \cos \frac{u}{3} - \frac{3c^2}{8b} \sin \frac{u}{3} - \frac{c^2}{4b} \sin u \right] \end{split}$$

symbol	meaning	note
${\mathcal E}$	main ellipse	
a,b	major, minor semi-axes of $\mathcal E$	
С	half the focal length of $\mathcal E$	$c^2 = a^2 - b^2$
0	center \mathcal{E}	
M, M_u	a fixed point on the boundary of $\mathcal E$	$[a\cos u, b\sin u],$
		perspector of $T', T'', = X_{99}$
P(t)	a point which sweeps the boundary of $\mathcal E$	$[a\cos t, b\sin t]$
L(t)	Line through $P(t)$ perp. to $P(t) - M$	
Δ, Δ_u	Steiner's Hat, negative pedal curve	invariant area
	of \mathcal{E} with respect to M	
$\Delta^*_{m{ heta}}$	envelope of $L(t)$ rotated θ about $P(t)$	invariant area
\bar{C}	average coordinates of Δ	$=C_2=X_2'$
C_2, C_2^*	area center of Δ, Δ^*	$C_2 = X'_2 = X_{98}$
P_i', P_i, P_i''	The cusps of Δ , their pre-images,	
	and centers of K_i (see below)	
P_i^*	cusps of Δ^*	$P_i P_i^*$ concur at C_2^*
T,T',T''	triangles defined by the P_i, P'_i, P''_i	invariant area over M
A,A',A''	areas of T, T', T''	A'/A'' = 4 for any M, a, b
S	Steiner's Curve	aka. Hypocycloid and Triscupoid
\mathcal{E}'	Steiner Circumellipse	centered at C_2
	of cusp (P'_i) triangle	
a',b'	major, minor semi-axes of \mathcal{E}'	invariant, axis-parallel
		and similar to 90°-rotated ${\cal E}$
${\mathcal K}$	Circumcircle of <i>T</i>	center X_3 , contains M, P_i, C_2, C_2^*
\mathcal{K}'	Circumcircle of T'	
\mathcal{K}_{i}	Circles osculating \mathcal{E} at the P_i	contain P_i, P'_i, M
\mathcal{E}^*	evolute of \mathcal{E}	the K_i lie on it
${\mathcal H}$	Apollonius Hyperbola of \mathcal{E} wrt M	Δ is tangent to \mathcal{E} at $\mathcal{H} \cap \mathcal{E}$
X_3	circumcenter of T	$=X_{2}''$
X_4	perspector of T and T''	
X_{99}	Steiner Point of T	=M
X_{98}	Tarry Point of <i>T</i>	$=C_2$
X'_2	centroid of T'	$= C_2$, and perspector of T, T'
X'_{99}	Steiner Point of T'	
X_2''	centroid of <i>T</i> "	$=X_3$

Appendix B.Table of Symbols

Table 4:All Symbols used.

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Distance Product Cubics

Distance Product Cubics

ABSTRACT

The locus of points that determine a constant product of their distances to the sides of a triangle is a cubic curve in the projectively closed Euclidean triangle plane. In this paper, algebraic and geometric properties of these distance product cubics shall be studied. These cubics span a pencil of cubics that contains only one rational and non-degenerate cubic curve which is known as the Bataille acnodal cubic determined by the product of the actual trilinear coordinates of the centroid of the base triangle. Each triangle center defines a distance product cubic. It turns out that only a small number of triangle centers share their distance product cubic with other centers. All distance product cubics share the real points of inflection which lie on the line at infinity. The cubics' dual curves, their Hessians, and especially those distance product cubics that are defined by particular triangle centers shall be studied.

Key words: triangle cubic, elliptic cubic, rational cubic, trilinear distance, constant product, Steiner inellipse, triangle centers

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1 Introduction

Cubics occur frequently in triangle geometry. Sometimes, cubics are defined as locus of points satisfying certain geometric or algebraic conditions. There are many well-known cubics such as the Neuberg cubic, the Thomson cubic, the Darboux cubic to name just the most prominent examples. These triangle cubics are known to carry some triangle centers together with points related to the triangle, and besides their (in principle) Euclidean generation, some of them allow for a projective generation, cf. [8]. In many cases, these cubics pass through the vertices of the base triangle: For example, the Thomson cubic K_{002} (sometimes called seventeen-point cubic, illustrated in Figure 1) passes

Kubike konstantnog umnoška udaljenosti SAŽETAK

U projektivno zatvorenoj Euklidskoj ravnini trokuta geometrijsko mjesto točaka trokuta kojima je umnožak udaljenosti od stranica trokuta konstantan je jedna kubika. Proučavat će se algebarska i geometrijska svojstva tih kubika konstantnog umnoška udaljenosti. Takve kubike čine pramen kubika koje sadrže samo jednu racionalnu nedegeneriranu kubiku poznatu kao Batailleova kubika s izoliranom točkom, a koja je određena umnoškom pravih trilinearnih koordinata težišta temeljnog trokuta. Svaka točka trokuta određuje jednu kubiku konstantnog umnoška udaljenosti. Ispostavlja se da mali broj točaka trokuta međusobno dijele kubiku konstantnog umnoška udaljenosti. Sve kubike konstantnog umnoška udaljenosti dijele realne točke infleksije koje leže na pravcu u beskonačnosti. Proučavat će se dualne krivulje kubike, njihove Hessianove matrice i posebno one kubike konstantnog umnoška udaljenosti koje su određene poznatim točkama trokuta.

Ključne riječi: kubika trokuta, eliptična kubika, racionalna kubika, trilinearna udaljenost, konstantni umnožak, Steinerova upisana elipsa, točke trokuta

through the vertices of the base triangle and carries the triangle centers X_i with Kimberling indices

 $i \in \{1, 2, 3, 4, 6, 9, 57, 223, 282, 1073 \\ 1249, 3341, 3342, 3343, 3344, 3349, 3350, \\ 3351, 3352, 3356, 14481, 39161, 39162\},$

the midpoints of Δ 's sides, the midpoints of Δ 's altitudes, the vertices of the Thomson triangle, and the excenters (which are actually 38 points), see [4]. The numbering of triangle centers follows the exhaustive *Encyclopedia of Triangle Centers* by CLARK KIMBERLING, see [6, 7]. For example, the triangle centers X_{39161} and X_{39162} are the real foci of the inscribed Steiner ellipse *e*. On the other hand, the names and numbers of triangle cubics are taken from BERNARD GIBERT's pages [2].

Further, \mathcal{K}_{002} is a self-isogonal cubic with the centroid X_2 of Δ as its pivot point. 26 geometric definitions of the Thomson cubic can be found on GIBERT's page [4], dedicated exclusively to the Thomson cubic.



Figure 1: The Thomson cubic K_{002} with 23 triangle centers on it.

The cubics which shall be studied here, do not pass through the vertices of the triangle. Moreover, the number of triangle centers located on these cubics is rather small except in one case. In many cases, it is impossible to find more than one triangle center on such a cubic. Nevertheless, it is surprising that no one has payed attention to the set of points forming a constant product of distances to the triangle sides.

What is the reason for the interest especially in these curves? It is well-known that elliptic cubics carry a group structure. The operation on the set of points on an elliptic cubic can be seen as an addition. Furthermore, it is well known that these groups contain finitely generated subgroups, cf. [10]. Generators of these groups of finite order are highly sought after. Once, rational (or polynomial) points on elliptic curves are known, many more of them can be generated by simply doubling the initial points. Until now, only a few examples of finitely generated groups on elliptic curves are known. Within the huge amount of triangle cubics carrying rational points, it may be possible to find some more examples.

The paper is organized as follows: In the remaining part of this section, the equation of the distance product cubics are determined. Further, some geometric properties of these particular cubics are deduced. Then, the equations of the dual curves and the curves in the Hessian pencil are given. For the sake of completeness, the Weierstraß normal form of the distance product cubics is derived. Section 2 deals with the very special distance product cubics defined by triangle centers. A complete list (as to November 2020) of groups of triangle centers sharing their distance product cubics is given. It is described how these centers on such cubics can be found in an efficient way and attention is paid to special configurations of triangle centers on their respective distance product cubics. Then, in Section 3, some (until now) unknown triangle centers on some distance product cubics that contain only one known triangle center are given. Only triangle centers with a relatively short trilinear center function (homogeneous polynomial in the three side lengths a, b, c) shall be listed. Finally, Section 4 will outline future work and discusses computational problems and challenges. The present paper is an extension and completion of [9].

1.1 Prerequisites

In triangle geometry, trilinear coordinates proved useful. For that purpose, the vertices *A*, *B*, and *C* of the base triangle Δ (with side lengths $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$) are described by the homogeneous coordinates

$$A = (1,0,0), B = (0,1,0), C = (0,0,1),$$

i.e., the vectors of the canonical basis in \mathbb{R}^3 . The projective frame shall be completed by choosing

$$X_1 = (1, 1, 1)$$

as the unit point. At this point, it shall be said that the centroid X_2 of Δ (like any other center) can also serve as the unit point. With X_2 as the unit point, barycentric coordinates of points in the plane of the triangle are well-defined, cf. [6]. In the following, trilinear coordinates are preferred, since distances of points to the sides of the base triangle are involved.

Each point X in the plane of Δ can be uniquely determined by its homogeneous trilinear coordinates

$$X = (\xi, \eta, \zeta) \neq (0, 0, 0),$$

which are the ratios of the oriented distances of *X* to Δ 's oriented side lines. The side lines are oriented as *AB* (from *A* to *B*), *BC*, and *CA*. From homogeneous trilinear coordinates, the (inhomogeneous) *actual trilinear coordinates* (ξ^a , η^a , ζ^a) consisting of the three oriented distances of *X* to Δ 's side lines can be computed by

$$(\xi^a, \eta^a, \zeta^a) = \frac{2F}{a\xi + b\eta + c\zeta}(\xi, \eta, \zeta), \tag{1}$$

where F equals the area of the triangle. This normalization fails if

$$\omega: a\xi + b\eta + c\zeta = 0.$$

This is the equation of the *ideal line* ω (line at infinity) and all points (ξ , η , ζ) on it are *ideal points* (points at infinity). These points shall be excluded from the following considerations (although there are more than one thousand triangle centers on the ideal line), cf. [7].

1.2 Basic properties

If one multiplies the actual trilinear coordinates of a point in the plane of the triangle and sets this product equal to a constant $\delta \in \mathbb{R}$, it is possible to state:

Theorem 1 The locus of points X in the (Euclidean) plane of the triangle Δ that form a constant product $\delta \in \mathbb{R} \setminus \{0\}$ of distances to the side lines of Δ is a planar cubic curve with the equation

$$k_{\delta}: 8F^{3}\xi\eta\zeta - \delta(a\xi + b\eta + c\zeta)^{3} = 0$$
⁽²⁾

in terms trilinear coordinates.

Proof. The equation (2) is obtained by multiplying ξ^a , η^a , ζ^a from (1)

$$\xi^a \eta^a \zeta^a = \frac{8F^3}{(a\xi + b\eta + c\zeta)^3}$$

and, subsequently, setting this product equal to $\delta \in \mathbb{R} \setminus \{0\}$. The equation (2) is homogeneous and of degree three, and thus, it describes a planar cubic curve in the projective plane. A simple computation shows that if (2) is fulfilled by the actual trilinear coordinates of a point, then it is also fulfilled by an arbitrary multiple of these coordinates of the same point, and vice versa.

The equations (2) of the distance product cubics depend linearly on one parameter $\delta \in \mathbb{R} \setminus \{0\}$. Thus, the distance product cubics form a pencil of cubics. Replacing the inhomogeneous parameter δ in (2) by a homogeneous parameter $\delta = \delta_1 \delta_0^{-1}$ (with $\delta_0 : \delta_1 \neq 0 : 0$), shows that there are two degenerate cubics in the pencil:

(i) If δ_0 : $\delta_1 = 1$: 0, the equations of the cubics simplify to

 $\xi\eta\zeta=0,$

which is the equation of the union of Δ 's side lines. (ii) In the case $\delta_0 : \delta_1 = 0 : 1$, remainder of (2) equals

$$(a\xi + b\eta + c\zeta)^3 = 0,$$

which is the equation of the ideal line ω with multiplicity 3. Figure 2 shows a projective view of a few cubics from the pencil together the three common (collinear) real points of inflection I_1 , I_2 , I_3 .

From the fact that the equations of the distance product cubics are linear combinations of $\xi\eta\zeta = 0$ and $(a\xi + b\eta + c\zeta)^3 = 0$, it is clear that the ideal points of Δ 's side lines are the ideal points of the distance product cubics. Furthermore, the intersection of either side line with the each cubic in the pencil is of multiplicity three: For example, $\xi = 0$ yields $(b\eta + c\zeta)^3 = 0$, and therefore, $I_1 = (0, -c, b)$ as the intersection point with multiplicity three.





Figure 2: A projective view onto the pencil of distance product cubics shows the three real points of inflection I₁, I₂, I₃ on ω.

Theorem 2 The distance product cubics (2) share the three real inflection points, which are at the same time the three ideal points of the cubis. The homogeneous trilinear coordinates of the points of inflection are

$$I_1 = (0, -c, b), I_2 = (c, 0, -a), I_3 = (-b, a, 0).$$

The *harmonic polar* of a regular point $P \in k$ with respect to a non-degenerate cubic curve k is defined in the following way: Let P be a point on the cubic k and let l be a line through P different from the tangent of k at P. Then, in general, l meets the cubic in two further points, say Q and R. Provided, that $Q \neq R$ (l is not tangent to k at some point off P) and $Q, R \neq P$ (l is not an inflection tangent), then there exists exactly one point S which is the harmonic conjugate of P with respect to Q and R. The locus of S for all l in the pencil about P is called the harmonic polar of P with respect to k. The harmonic polars of the inflection points on cubics are straight lines, cf. [1]. In the case of the distance product cubics, the three harmonic polars corresponding to the three real inflection points have a special geometric meaning:

Theorem 3 The harmonic polars of the three inflection points of the distance product cubics are the medians of the base triangle independent of the choice of δ .

Proof. The lines *l* pencil about $I_1 = (0, -c, b)$ can be parametrized by

$$\mathbf{l}(\lambda,\mu) = \lambda(0,-c,b) + \mu(u,v,w)$$

with $\lambda : \mu \neq 0 : 0$, where it means no restriction to assume that Q = (u, v, w) is a further point on *k* with equation (2). Now, the intersection *R* of any *l* with the cubics equals

$$R = (bcu, c^2w, b^2v)$$

Hence, the harmonic conjugate of I_1 with respect to k is the point

$$S = (2bcu, c(bv + cw), b(bv + cw)).$$

The point *S* lies on the line $b\eta - c\zeta = 0$, which is the median through *A*. In the same way it can be shown that the harmonic polars of I_2 and I_3 are the medians through *B* and *C*, respectively. Obviously, the harmonic polar of I_1 is independent of δ , and so are the harmonic polars of I_2 and I_3 .

Since the three harmonic polars corresponding to the real inflection points common to all cubics are the medians, the centroid X_2 must have a special meaning for the distance product cubics. Now, the following can be shown:

Theorem 4 The distance product cubic k_2 through the centroid of Δ is the only rational cubic (among the regular ones) in the pencil and the centroid is an isolated node on k_2 .

Remark 1 Rational cubics are often referred to as singular cubics, because a rational cubic needs to have a singularity. We would like to put emphasis on the fact that rational cubics are regular except in one point and the term singular often indicates degeneracy which is definitely not the case here.

Proof. The equation of the cubic k_2 is determined by inserting the (homogeneous) trilinear coordinates of

$$X_2 = (bc, ca, ab)$$

(see [6, 7]) into (2). This yields the corresponding parameter (in the pencil of cubics)

$$\delta_2 = \frac{8F^3}{27abc} \tag{3}$$

and the equation of the cubic by inserting (3) into (2):

$$k_2: 27abc\xi\eta\zeta - (a\xi + b\eta + c\zeta)^3 = 0.$$
(4)

Now it is easily verified that X_2 is the only singular point on k_2 . (Compute the gradient of k_2 with respect to (ξ, η, ζ) at X_2 and recall that a non-degenerate cubic cannot have more than one singularity.) The tangents to k_2 at X_2 are given by the equation

$$\sum_{\text{cyclic}} a^2 \xi^2 - bc \, \eta \zeta = 0. \tag{5}$$

In order to show that k_2 is the only singular (nondegenerate) cubic in the pencil (2), the singular points of all cubics in the pencil are computed. For that purpose, first the gradient grad k is computed. Second, the equation (2) is used to eliminate all variables but one, say ξ . In an intermediate step, the factor $a\xi + b\eta$ is cut out from two resultants. This is admissible, since together with the third resultant $a\xi + b\eta = 0$ implies $\xi = 0$, and then $\eta = 0$ which does not yield a proper point on any of the cubics. In the last elimination step, the final resultant is obtained and reads

$$27a^{3}bc\delta(27abc\delta-8F^{3})^{2}(27abc\delta+64F^{3})\xi^{4}$$

which can only be zero if either

or

$$27abc\delta + 64F^3 = 0.$$

 $27abc\delta - 8F^3 = 0$

since $a, b, c \neq 0$ (and hence $F \neq 0$, otherwise there is no triangle) and $\xi = 0$ only yields the inflection point $I_2 \in \omega$. The first equation leads precisely to δ_2 and k_2 , while the second equation yields

$$\delta = -\frac{64F^3}{27abc}$$

which determines a regular elliptic cubic. The choice of the variable to be eliminated does not matter. \Box

Among the cubics (2), the cubic k_2 is the only cubic that can be found on BERNARD GIBERT's page [3], where it is labeled as \mathcal{K}_{656} and has the name *Bataille acnodal cubic*. Figure 3 shows an example of the Bataille acnodal cubic.



Figure 3: The Bataille acnodal cubic K_{656} with the 16 known centers on it.

It is easily verified that the Bataille acnodal cubic \mathcal{K}_{656} carries 16 known and labeled triangle centers. These are the centers X_i with the Kimberling numbers

see also [3]). This is by far the highest number of known triangle centers on a distance product cubic.

The cubic $k_2 = \mathcal{K}_{656}$ admits the surprisingly simple parametrization

$$(bcm^3, -ca(m+n)^3, abn^3)$$
 with $m: n \neq 0:0$.

In terms of homogeneous barycentric coordinates, the equation of k_2 becomes very simple and reads

$$27\xi\eta\zeta - (\xi + \eta + \zeta)^3 = 0.$$

1.3 Dual curves, Hessian pencils, and Weierstraß form

It is well-known that elliptic cubics are of class 6, while rational cubics are of class 4 or 3, depending on whether the singularity is a node (isolated or not) or a cusp. The latter case cannot occur: According to Theorem 4, the curve k_2 is only singular distance product cubic. The equations of the tangents d_1 and d_2 at the double point X_2 are the two (complex conjugate) linear factors of the singular quadratic form (5) and read

$$d_1: 2a\xi - b(\sqrt{3}i+1)\eta + c(i\sqrt{3}-1)\zeta = 0,$$

$$d_2: 2a\xi + b(\sqrt{3}i-1)\eta - c(i\sqrt{3}+1)\zeta = 0.$$

Only in the case $d_1 = d_2$, X_2 becomes a cusp. Since $d_2 = \overline{d_1}$ would imply $d_1 = \overline{d_1}$, and thus, both tangents would have to be real, which is the case only if b = c = 0. Hence, we have:

Theorem 5 The distance product cubics (2) are of class 6 if $\delta \neq \delta_2$. If $\delta = \delta_2$, the corresponding distance product cubic is rational and of class 4.

Especially in the case of $k_2 = \mathcal{K}_{656}$ there are remarkable connections between the dual k_2^* of the distance product cubic k_2 and some conics deduced from the base triangle Δ :

Theorem 6 The dual curve k_2^* of k_2 is the isogonal image of the Steiner inellipse e and the isotomic image of the triangle's inellipse i with the third Brocard point X_{76} for its Brianchon point (after the canonical identification of homogeneous line and point coordinates).

Proof. The dual curve of k_2 (or \mathcal{K}_{656}) has the equation

$$k_2^{\star}: \sum_{\text{cyclic}} (a^2 u_1 u_2 - 2bc u_0^2) u_1 u_2 = 0$$
 (6)

which can be found by eliminating ξ , η , ζ , and ρ from the following system of equations:

$$\operatorname{grad} k_2 = \rho \cdot (u_0, u_1, u_2)$$

(Note that ξ , η , and ζ are subject to (2) which has to be taken into account during the elimination process.) In order to verify that (6) is the isogonal image of the Steiner inellipse *e*, homogeneous line coordinates $u_0 : u_1 : u_2 \neq 0:0:0$ are identified with homogeneous point coordinates $\xi : \eta : \zeta \neq 0:0:0$. Then, the substitution of $u_0 = x_1x_2$

(cyclic, cf. [5]) into (6) indeed yields the equation of the Steiner inellipse

$$\iota(k_2^{\star}) = e : \sum_{\text{cyclic}} a^2 x_0^2 - 2bcx_1 x_2 = 0$$

after canceling the (cyclic symmetric) factor $x_0^2 x_1^2 x_2^2$ that describes the sides of Δ 's sides (each with multiplicity 2). Finally, it remains to show that the curve k_2^* is also the isotomic image of the inellipse with Brianchon point X_{76} (the 3^{rd} Brocard point) and center X_{141} (the complement of the Symmedian point X_6). Applying the isotomic transformation to k_2^* means to substitute $u_0 = b^2 c^2 x_1 x_2$ (cyclic, cf. [5]) into (6). In doing so, one finds

$$\tau(k_2^*) = i : \sum_{\text{cyclic}} a^6 x_0^2 - 2b^3 c^3 x_1 x_2 = 0.$$

Like in the previous cases, the factor $x_0^2 x_1^2 x_2^2$ is cut out. It is a rather elementary task, to determine the Brianchon point and the center of *i*, see [5].

The curve k_2^{\star} has three ordinary cusps at the vertices of Δ which correspond to the three inflection tangents. Again, homogeneous line coordinates are interpreted as homogeneous point coordinates in the plane of the base triangle and in the underlying projective coordinate system $(A, B, C; X_1)$.



Figure 4: The cubic k_2 , its dual k_2^* (interpreted as a point curve), the Steiner inellipse e as the isogonal conjugate $e = \iota(k_2^*)$, and the inellipse i as the isotomic conjugate $i = \tau(k_2^*)$.

Figure 4 shows the curves e, i, k_2 , and k_2^{\star} mentioned in Theorem 6.

The computation of the dual curves of the non-rational distance product cubics is much more complicated. In contrast to the case of the rational curve k_2 , one cannot rely on a parametrization of the curve. Thus, the homogeneous point coordinates ξ , η , and ζ have to be eliminated from the following system of equations

$$\partial_{\xi}k_{\delta} = u_{0}\rho, \ \partial_{\eta}k_{\delta} = u_{1}\rho, \ \partial_{\zeta}k_{\delta} = u_{2}\rho,$$
$$8F^{3}\xi\eta\zeta - \delta(a\xi + b\eta + c\zeta)^{3} = 0.$$

in order to obtain the implicit equation of the dual curves of (2) (in terms of homogeneous line coordinates $u_0 : u_1 : u_2$). This yields the sextic curves

$$k_{\delta}^{\star}: 27\delta^{2} \prod_{\text{cyclic}} (bu_{2} - cu_{1})^{2} - 16\delta F^{3} \prod_{\text{cyclic}} (au_{1}u_{2} + bu_{0}u_{2} - 2cu_{0}u_{1}) =$$
(7)
= $64F^{6}u_{0}^{2}u_{1}^{2}u_{2}^{2}$

whose equations depend quadratically on the parameter $\delta \in \mathbb{R} \setminus \{0\}$. The sextic curves (7) have three real and six complex cusps corresponding to the three real and six complex points of inflection on the cubics (2). The nine cusps of (7) form a Hesse configuration (9₄,12₃) if $u_0 : u_1 : u_2$ are viewed as homogeneous point coordinates.

1.4 Hessian pencil

The equation of the Hessian curve Hc of an algebraic curve c with the implicit homogeneous equation $F(x_0, x_1, x_2) = 0$ is given by

$$Hc: \det(\partial_{ij}F) = 0. \tag{8}$$

It intersects c at ordinary inflection points with multiplicity one (while it intersects c at its singularities with multiplicities larger than 6). Clearly, the Hessian curves of cubics are again cubics.

In the particular case of distance product cubics, one can show:

Theorem 7 *The Hessian curves of the distance product cubics* (2) *form a pencil of cubics which is spanned by the degenerate cubics* $e \cup \omega$ *(union of the Steiner inellipse e and the line at infinity* ω *) and the three side lines of* Δ *.*

Proof. The equations of the Hessian curves of the cubics (2) are computed via (8). This results in

$$Hk_{\delta}: \ \delta\underbrace{\left(\sum_{\text{cyclic}} a\xi\right)}_{\omega} \cdot \underbrace{\left(\sum_{\text{cyclic}} a\xi(2b\eta - a\xi)\right)}_{\text{Steiner inellipse}} = \frac{8}{3}F^{3}\xi\eta\zeta.$$
(9)

Since the equations (9) of the Hessian curves are linear in δ , they form a pencil of cubics like the distance product cubics do. The choice of $\delta = 0$ yields $\xi \eta \zeta = 0$ which is the equation of the three side lines of Δ . Replacing the affine parameter δ by the homogeneous parameter $\delta_0 : \delta_1 \neq 0 : 0$ and setting $\delta_1 = 0$ (while $\delta_0 \neq 0$) yields the right-hand side

of (9) which factors into the equation of the ideal line ω and the equation of the Steiner inellipse *e*.

The fact that one factor of the right-hand side of (9) is the equation of ω clearly shows that for each $\delta \in \mathbb{R} \setminus \{0\}$ the corresponding distance product cubic k_{δ} and its Hessian curve Hk_{δ} intersect in the ideal points of Δ 's side lines (to mention only the real points). This again shows that the three real points of inflection of the distance product cubics (2) are the ideal points of Δ 's side lines (cf. Theorem 2).

1.5 Weierstraß form

The treatment of elliptic cubic curves is usually done in an affine setting. The choice of an affine coordinate frame properly attached to the cubic curve transforms the cubic's equation into the Weierstraß normal form, cf. [10]. Based on this normal form, many computations – especially those related to the group structure on the curve – can be performed in a very simple way.

We set $\zeta = 1$ and substitute

$$\begin{split} \xi &=\; \frac{2F^3 - 3abc\delta}{6a^2b\delta} - \frac{1}{64F^6a^2b\delta} (4F^3X + Y), \\ \eta &=\; \frac{2F^3 - 3abc\delta}{6ab^2\delta} - \frac{1}{64F^6ab^2\delta} (4F^3X - Y). \end{split}$$

into (2). This yields the Weierstraß normal form of distance product cubics:

$$k_W: Y^2 = X^3 + \frac{2^8}{3} (3abc\delta - F^3)F^9X + \frac{2^{10}}{3^3} (3^3(abc\delta)^2 - 6^2abc\delta F^3 + 2^3F^6)F^{12}.$$
(10)

The *j*-invariant j(e) of an elliptic curve

$$e: y^2 = x^3 + 3px + 2q$$

is computed via

$$j(e) = \frac{2^6 \cdot 3^3 \cdot p^3}{p^3 + q^2}$$

The *j*-invariant of all distance product cubics k_{δ} equals

$$j(k_{\delta}) = \frac{2^{12}F^3(F^3 - 3abc\delta)^3}{(abc\delta)^3(2^3F^3 - 3^3abc\delta)}$$

and becomes undetermined if, and only if, $\delta = \delta_2$ from (3).

2 Triangle centers with equal distance product

The triangle centers listed in KIMBERLING's *Encyclopedia of Triangle Centers* [6, 7] determine cubic curves as loci of points with the equal product of trilinear distances. Finding triangle centers located on the same cubic curve is equivalent to finding triangle centers with the same product of trilinear distances. This would be another classification of triangle centers. Surprisingly, among the many known, listed, and in principle arbitrarily numbered triangle centers, there is only a small number of triangle centers that gather on the same cubic.

Until now (as to November 2020), only the following groups of triangle centers located on the same cubic are known:

Theorem 8 The groups of triangle centers with equal distance product to the sides of a triangle are given in Table 1.

```
(1,764), (4,5489), (6,22260), (8,21132),
(2,3081,6545,8027 - 8032, 23610 - 23616),
(25,394), (42,321,8034), (55,40166), (57,200),
(75,21143),(76,23099),(86,21131),(99,14444),
(145,23764),(324,418),(455,40144),(459,3079),
(649,693), (669,850,32320), (671,14443),
(903,14442), (875,4375,4444), (756,8042),
(1022,3251), (1026,3675), (1422,40212),
(1641, 14423), (1647, 17780), (1648, 5468),
(1649,5466), (1650,4240), (2501,3265),
(3051,8024), (3227,14441), (3233,12079),
(3234,15634), (3239,3676), (3572,27855),
(3733,4036), (4024,7192), (4358,8661),
(4500,4507), (6358,40213), (6384,8026),
(6544,6548), (6557,15519), (8013,8025),
(8023, 8039), (14163, 14164), (14214, 14215),
(14401,34767), (15630,15631), (15632,15635),
(16748,21820), (20696,20700), (21140,23354),
(21438,23655), (27919,40217), (36414,40146),
(40149, 40152)
```



Proof. In order to verify the results given in the above theorem, it is sufficient to insert the trilinear representations of the respective centers into to the equations of the cubic curves. \Box

Just inserting trilinear representations of triangle centers into the equations of a particular distance product cubic is not a very efficient search for triangle centers on a cubic. It requires the presence of trilinear representations of triangle centers which is not the case for some triangle centers, cf. [7]. Complicated algebraic expressions involving cube roots or nested square roots can hardly be handled properly with computer algebra systems.

We can improve the search by recalling the following facts: A cubic curve is a *triangle cubic* if its equations in terms of homogeneous (trilinear or barycentric) coordinates is invariant under the cyclic substitution

$$a \to b \to c \to a \text{ and } \xi \to \eta \to \zeta \to \xi.$$

According to this, the cubics (2) and (9) are triangle cubics. Once a center *C* on a triangle cubic *k* is known, one can immediately find a new triangle center *R* as the intersection of *k*'s tangent T_Ck (at *C*) with *k*. The point *R* shall henceforth be called the *tangential remainder* or simply *remainder* of *C*. This yields – besides the contact point *C* with multiplicity 2 – exactly one further point *R*, provided that *C* is not a point of inflection.



Figure 5: A sequence of (rational) points on an elliptic cubic k. Each successor i + 1 is the intersection $(\neq i)$ of the tangent T_ik at i with k. Only for points of finite order, such chains are closed.

Once a rational point 1 on the elliptic cubic k is known, a sequence of further rational points can be generated (see Figure 5; note that the chain depicted there is not closed). Only in some rare cases, periodic sequences of remainders (or closed chains) are known and correspond to the groups of finite order on the elliptic curve, see [10].

On the distance product cubics mentioned in Theorem 8, one can observe the following relations between centers and their remainders:

Type 1: For example, on k_1 , the center $R_1 = X_{764}$ is the intersection of k_1 with the tangent T_1 at X_1 , see Figure 6.



Figure 6: The distance product cubic k_1 defined by the incenter X_1 . The center $X_{764} \in k_1$ is the intersection of k_1 's tangent at X_1 .

Note that

$$\delta_1 = r^3$$
,

i.e., the trilinear distance product of the points on the cubic k_1 equals the cube of Δ 's inradius r. Hence, the triangle center X_{764} is another triangle center with trilinear distance product $\delta_{764} = r^3$.

Type 2: The cubic k_{875} is an example of a distance product cubic with three collinear centers on it. These are the centers

$$X_{875}, X_{4375}, X_{4444}$$

which are shown together with k_{875} in Figure 7. In this case the search for further (already known) triangle centers on the cubic fails. None of the tangential remainders is a known triangle center.



Figure 7: A triple of three collinear centers on k_{875} : X_{875} , X_{4375} , and X_{4444} .

The two centers X_{649} and X_{693} also form the same distance

product, and thus, they both lie on the curve k_{649} . Clearly, the line

$$\mathcal{L}_{649,693} := [X_{649}, X_{693}]$$

meets k_{649} in a further center R with trilinear center function

$$\alpha_R = b^2 c^2 (b-c) (a^2 - bc)^3.$$

Unfortunately, this point cannot be found in KIMBER-LING's encyclopedia (cf. [7]) although it has a relatively simple algebraic representation compared to other centers.

Type 3: The cubic k_{42} , also hosts three (known) triangle centers. However, the points

$$X_{42}, X_{321}, X_{8034}$$

on k_{42} are not collinear. The centers X_{42} and X_{321} have the same remainder

$$R_{42} = R_{321} = X_{8034},$$

see Figure 8. Therefore, we could expect to find more triangle centers sending their tangents to X_{8034} . Unfortunately, the corresponding polynomial equation of degree 6 has only two rational solutions leading to the already known centers X_{42} and X_{321} .



Figure 8: The tangents to k_{42} at X_{42} and X_{321} meet in $X_{8034} \in k_{42}$.

Remark 2 At this point it shall be said that the assignment of numbers (Kimberling numbers) to triangle centers is done rather arbitrarily. Therefore, the configuration of known centers on their particular distance product cubic has no deeper geometric meaning.

Table 2 collects triangle centers and their tangential remainders on their respective distance product cubics. Table 3 gives the trilinear center function of the tangential remainders common to two different (known) triangle centers on their respective distance product cubic. These points do not occur in KIMBERLING's encyclopedia [7].

Xi	R_i	Xi	R_i
1	764	1641	14423
4	5489	3051, 8024	R ₃₀₅₁
6	22260	3227	14441
8	21132	3733, 4036	R ₃₇₃₃
25, 394	R ₂₅	4240	1650
42, 321	8034	4358	8661
55, 40166	R ₅₅	5466	1649
57, 200	R ₅₇	5468	1648
75	21143	6358, 40213	R ₆₃₅₈
76	23099	6384, 8026	R ₆₃₈₄
86	21131	6548	6544
99	14444	6557, 15519	R ₆₅₅₇
145	23764	8013, 8025	R ₈₀₁₃
324, 418	<i>R</i> ₃₂₄	8023, 8039	R ₈₀₂₃
455, 40144	R ₄₅₅	17780	1647
459, 3079	R ₄₅₉	20696, 20700	R ₂₀₆₉₆
671	14443	16748, 21820	<i>R</i> ₁₆₇₄₈
756, 8042	R ₇₅₆	23354	21140
903	14442	27919, 40217	<i>R</i> ₂₇₉₁₉
1022	3251	34767	14401
1026	3675	36414, 40146	<i>R</i> ₃₆₄₁₄
1422, 40212	<i>R</i> ₁₄₂₂	40149, 40152	R ₄₀₁₄₉

Table 2: The tangent of the triangle center X_i meets the cubic k_i at a further triangle center given in the column R_i . If this remainder is a known triangle center, then its number is given.

remainder of	trilinear center function
25, 394	$a(b^2-c^2)^3(a^2-b^2-c^2)^2$
55, 40166	$a(b-c)^{3}(a-b-c)(ab+ca-b^{2}-c^{2})^{3}$
57, 200	$(b-c)^3(a-b-c)^2$
324, 418	$a^{3}(b^{2}-c^{2})^{3}(a^{2}-b^{2}-c^{2})^{5}(a^{2}b^{2}+a^{2}c^{2}+2b^{2}c^{2}-b^{4}-c^{4})$
459, 3079	$bc(b^2-c^2)^3(a^2-b^2-c^2)^2(3a^4-2a^2(b^2+c^2)-(b^2-c^2))^2$
756, 8042	$(b-c)(b^2-c^2)^2(a^2-bc)^3$
3051, 8024	$a^3(b^2+c^2)(b^2-c^2)^3$
3733, 4036	$a(b-c)(b+c)^{2}(a^{3}(b+c)+a^{2}(b^{2}+c^{2})-a(b^{3}+c^{3})-b^{4}-c^{4})^{3}$
6358, 40213	$a(b+c)^{2}(b-c)^{3}(a-b-c)^{2}(a^{2}-b^{2}+bc-c^{2})^{3}$
6384, 8026	$a(b-c)^3(ab+ac-bc)^2$
6557, 15519	$bc(b-c)^{3}(a-b-c)(3a-b-c)^{2}$
8013, 8025	$bc(b-c)(b^2-c^2)^2(2a+b+c)$
8023, 8039	$a^{7}(b^{4}+c^{4})(b^{2}-c^{2})(b^{2}+c^{2})^{3}$
16748, 21820	$a^{4}(b-c)(b^{2}-c^{2})^{2}(ab+ac+2bc)$
36414, 40166	$a^{3}(b^{2}-c^{2})^{3}(b^{2}+c^{2})^{3}(a^{4}-b^{4}-c^{4})^{2}$
40149, 40152	$a(b+c)(b-c)^{3}(a-b-c)^{2}(a^{2}-b^{2}-c^{2})^{2}$
27919, 40217	$bc(b-c)^{3}(ab+ca-b^{2}-c^{2})(a^{2}-bc)^{2}$

 Table 3: Tangential remainders common to two centers on distance product cubics. Only those center functions (first trilinear coordinates) of reasonable length are given.

3 Tangential remainders for triangle centers

Table 2 contains a subset of Table 1 and gives a list of triangle centers with their tangential remainders on their distance product cubics. Known centers are given by their Kimberling numbers, while unknown tangential remainders of centers X_i are labeled with R_i .

Table 3 gives the first trilinear center functions of some of the unknown remainders R_i mentioned in Table 2, provided that these remainders are common to at least two centers and that the respective center function is of reasonable length.

For the remainders of some of those centers (not appearing in Theorem 8), Table 4 presents the first trilinear center functions together with the respective numeric search value for the triangle

$$(a,b,c) = (6,9,13)$$

(in order to simplify the identification and search on [7]).

4 Outlook, future work, computational problems

The search of rational points on elliptic triangle cubics, not necessarily distance product cubics, sometimes involves quadratic or cubic field extensions. In the beginning, *i.e.*, for triangle centers with small Kimberling number, most of the triangle centers have trilinear coordinates that are polynomials in *a*, *b*, *c* with integer coefficients. The trilinear representations of the centers X_{13}, \ldots, X_{18} involve $\sqrt{3}$ which does not cause problems in symbolic computations.

Square roots show up in the trilinear representations that involve half-angle functions. In order to handle expressions that involve the area function

$$F = \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}$$

of the base triangle, we add *F* as a further element of the coefficient ring. We have to add the square roots of 3 and 5 to the ring of coefficients if multiples of angles of $\frac{\pi}{3}$ and $\frac{\pi}{5}$ are ingredients of the construction of some center: *F* as well as $\sqrt{3}$ appear in the trilinear representations of the centers

$$X_{13}, \ldots, X_{18}$$

(1st and 2nd isogonic center, 1st and 2nd isodynamic point, 1st and 2nd Napoleon point). The trilinear representations of X_{1139} and X_{1140} (Outer and Inner Pentagon point) involve $\sqrt{5}$.

The triangle centers

$$X_{173}, X_{174}, X_{258}, X_{351}, \ldots, X_{364}$$

(related with isoscelizers points), involve square roots of a, b, c and sine and cosine of half angles. The trilinear

representations of the Square Root point and its isogonal conjugate

$$X_{365}$$
 and X_{366}

involve even \sqrt{a} , \sqrt{b} , and \sqrt{c} .

Triangle centers whose trilinear coordinate functions involve cube roots are also not tested whether or not they share their trilinear distance prodcut with others: These are centers like

$$X_{356}, X_{357}, X_{358}$$

(Morley point, 1st and 2nd Morley-Taylor-Marr center), and the Burgess point

$$X_{1133}, \ \alpha_{1133} = \sin \frac{\pi - A}{3} \operatorname{cosec} \frac{\pi + A}{3}$$

Here and in the following, the letter *A* denotes the measure of the interior angle at the vertex *A*. Thirds of angles are equivalent to roots of cubic polynomials, and thus, to field extensions of degree 3.

There are triangle centers that could be termed *transcendental*, for example:

$$X_{359}$$
 and X_{360} ,

i.e., Hofstaedter One point and the Hofsteadter Zero point with the trilinear center functions

$$\alpha_{359} = \frac{a}{A}$$
 and $\alpha_{360} = \frac{A}{a}$.

Their trilinear distance product are not compared with that of other centers, since they will hardly produce the same product as a polynomial center will do. This is also true for the Pure Angles center, the isogonal conjugate of the Point Algenib, the point Algenib, and the Exterior Angle Curvature Centroid, *i.e.*, for the centers

$$X_{1049}, X_{1085}, X_{1028}, \text{ and}, X_{1115}$$

with the respective trilinear triangle center functions

$$\alpha_{1049} = A, \ \alpha_{1085} = A^2,$$

 $\alpha_{1028} = A^{-2}, \ \alpha_{1115} = \frac{\pi - A}{a}$

The trilinear coordinates of triangle centers X_{40297} , ..., X_{40305} which are related to the power curve involve even logarithms, and thus, their trilinear distance products will not be equal to that of algebraic centers. Besides, X_{40297} , X_{40298} , X_{40299} are points at infinity.

Future work is guaranteed, since the ETC is growing continuously. Every day a few new triangle centers are added, awaiting to be tested whether or not they share their trilinear distance product with other centers.

Tangential remainders					
i	trilinear center function	search-6-9-13 value	i	trilinear center function	search-6-9-13 value
3	$a(b^2-c^2)^3(a^2-b^2-c^2)^4$	62.32822092189367	44	$(b-c)^{3}(2a-b-c)$.	
				$(a-2b-2c)^{3}$	14.27293379474638
5	$bc(a^2-b^2-c^2)^3$.		45	$(b-c)^{3}(a-2b-2c)$.	
	$\cdot (a^2(b^2+c^2)-(b^2-c^2)^2)$	5.08075321118240		$\cdot (2a-b-c)^3$	17.54515704225141
7	$bc(a-b-c)^{2}(b-c)^{3}$	-7.25734135716562	55	$a(b-c)^3(a-b-c)$	
				$\cdot (ab+ac-b^2-c^2)^3$	10.54263482016639
9	$(b-c)^{3}(a-b-c)^{4}$	-34.83523851439501	56	$a(b-c)^3(a-b-c)^2$	
				$(ab+ac+b^2+c^2)^3$	-2.26993220290925
10	$\frac{bc(b^2-c^2)(b-c)^2}{bc(b-c)^2}$	-2.36106585447966	57	$(b-c)^{3}(a-b-c)^{2}$	-3./103449382293/
11	$bc(a-b-c)(b-c)^{3}$	10 57220084201265	58	$a(b-c)(b^2-c^2)^2$	1 457054772220064
- 20	$(ab+ac-b^2-c^2)^3$	10.57329984291265	(2)	$(ab+ac+b^2+bc+c^2)^3$	-1.45/954//222064
20	$bc(3a^2-2a^2(b^2-c^2)-(b^2-c^2))$	176 00015604001600	63	$(b-c)^{3}(a^{2}-b^{2}-c^{2})$	57 20126911096199
21	-(b-c)	1/0.82313034331023	65	$\frac{(a-b-c)^2}{(b+a)^2(a-b-a)^2}$	-37.29420844980488
21	(b-c)(a-b-c) $(b^2-c^2)^2(a^2-b^2-c^2)^3$	20 10736086425203	0.5	$(b+c)(b-c)^{*}(a-b-c)^{*}$ $(a^{2}+ab+ac+2bc)^{3}$	-4 33523005169485
22	$a(b^2-c^2)^3(a^4-b^4-c^4)$	27.17750700425205	66	$bc(b^4-c^4)^3(a^4-b^4-c^4)^2$	77.10999885446101
	$(a^2-b^2-c^2)^3$	-223.89401674331905	00		,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,
23	$a(b^2-c^2)^3(a^2-b^2-c^2)^3$		75	$a(b-c)^{3}$	-0.86891361128715
	$\cdot (a^4 - b^4 + b^2 c^2 - c^4)$	-28.06151358971226			
24	$a(b^2-c^2)^3(a^2-b^2-c^2)^2$		76	$a^3(b^2-c^2)^3$	-0.5829933417003
	$\cdot (a^4 - 2a^2(b^2 + c^2) + b^4 + c^4)$	14.62241731719719			
25,	$(12 \ 2)3(2 \ 12 \ 2)2$	5 00100005057790	81	$(b+c)^2(b-c)^3$	-1.7256277235825
394	$a(b^{-}-c^{-})^{-}(a^{-}-b^{-}-c^{-})^{-}$	-5.02152505557785	02	$(h - z)^3(h^2 + z^2)^2$	
21	$bc(b+c)^{-}(b-c)^{+}$ $(a^{2}-b^{2}-c^{2})^{2}$	-23.86418264611576	82	$(b-c)^{*}(b^{2}+c^{2})^{-1}$ $\cdot(a^{2}+b^{2}+bc+c^{2})^{3}$	-2.7552998535293
28	$(b+c)^2(b-c)^3(a^2-b^2-c^2)^2$	-7.98316487555603	83	$bc(b^2+c^2)^2(b^2-c^2)^3$	-6.1052017302889
31	$a^{2}(b-c)^{3}(b^{2}+bc+c^{2})^{3}$	-1.14992786985712	85	$a(b-c)^{3}(a-b-c)^{5}$	-52.6095855034850
32	$a^{3}(b^{2}-c^{2})^{3}(b^{2}+c^{2})^{3}$	-0.94646193660754	86	$bc(b+c)^{2}(b-c)^{3}$	-2.8159081633291
37	$a^{3}(b+c)(b-c)^{3}$	-0.49015639611069	87	$(b-c)^3(ab+ac-bc)^5$	-0.0012017084959
38	$(b^2+c^2)(b-c)^3(a^2-bc)^3$	-35.23250675370688	88	$(b-c)^3(2a-b-c)^2$	3.8941592730472
39	$a^{7}(b^{2}+c^{2})(b^{2}-c^{2})^{3}$	-0.19164804132159	89	$(b-c)^3(a-2b-2c)^2$	-2.2543301181918
43	$(b-c)^3(ab+ac-bc)$	-0.19597779819792	94	$a^{3}(b^{2}-c^{2})^{3}$	
				$(a^2-b^2-bc-c^2)^2$	-17.011141733428
98	$bc(b^2-c^2)^3$.	56 616001423347	99	$bc(b^2-c^2)^2(2a^2-b^2-c^2)^3$	14.834689935205
100	$(b-c)^2(ab+ac-b^2-c^2)^3$	10.408092032867	105	$(b-c)^3(ab+ac-b^2-c^2)^2$	10.776490698571

 Table 4: Tangential remainders (not listed in KIMBERLING's encyclopedia of some triangle centers. The remainders of centers are added to this list only if their trilinear center function is of reasonable length.

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Generalized Helices on a Lightlike Cone in 3-Dimensional Lorentz-Minkowski Space

Generalized Helices on a Lightlike Cone in 3-Dimensional Lorentz-Minkowski Space

ABSTRACT

In this paper we provide characterizations and give some properties of generalized helices in 3-dimensional Lorentz-Minkowski space that lie on a lightlike cone. Furthermore, by analyzing their projections, which turn out to be Euclidean or Lorentzian logarithmic spiral, we present their parametrizations. In particular, we also analyze planar generalized helices, that is planar intersections of a lightlike cone.

Key words: Lorentz-Minkowski space, generalized helix, curve of constant slope, lightlike cone

MSC2010: 53A35, 53B30

Opće zavojnice na svjetlosnom stošcu u trodimenzionalnom Lorentz-Minkowskijevom prostoru

SAŽETAK

U radu je dana karaketrizacija općih zavojnica u 3dimenzionalnom Lorentz-Minkowskijevom prostoru koje leže na svjetlosnom stošcu te su predstavljena njihova svojstva. Nadalje, analizirajući njihove projekcije, koje su euklidske ili Lorentzove logaritamske spirale, odredili smo i njihove parametrizacije. Kao specijalan slučaj analizirane su ravninske opće zavojnice, tj. presjeci ravnine i svjetlosnog stošca.

Ključne riječi: Lorentz-Minkowskijev prostor, opća zavojnica, krivulje konstantnog nagiba, svjetlosni stožac

1 Introduction

Generalized helices or curves of constant slope in Euclidean space are curves making a constant angle with a fixed straight line. Hence, they are isogonal trajectories of rulings of a cylindrical surfaces whose rulings are parallel to a fixed direction. A classical Euclidean result states that a curve is a generalized helix if and only if the ratio of its torsion to curvature is constant. Of particular interest it is also to study generalized helices with some additional property, such as lying on certain surfaces, for instance, spheres. In Euclidean space, spherical generalized helices have a property that their orthogonal projections onto a plane normal to their axis appear as epicycloids.

In this paper we are interested to analyze the analogous problem of generalized helices in Lorentz-Minkowski 3-space with additional property that they lie on a lighlike cone. A lightlike cone LC(p) with a vertex at a point p is, from Euclidean perspective, a rotational cone with vertex p

and "vertical" axis. We perform this analysis by analyzing their projection curves.

In Lorentz-Minkowski space, generalized helices are studied in e.g. [4], where their analogous characterization to the Euclidean counterparts are presented. Null helices are studied in [2]. Curves on a lightlike cone are described in terms of their intrinsic curvatures in [3].

2 Preliminaries

Let \mathbb{R}^3_1 be a pseudo-Euclidean or Lorentz-Minkowski space, that is, the vector space \mathbb{R}^3 equipped with the indefinite symmetric bilinear form (a pseudo-scalar product)

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

A vector x in the Lorentz-Minkowski 3-space is called spacelike if $\langle x,x \rangle > 0$ or x = 0, timelike if $\langle x,x \rangle < 0$ and lightlike if $\langle x,x \rangle = 0$ and $x \neq 0$. A timelike vector $x = (x_1, x_2, x_3)$ is said to be positive (resp. negative) if $x_3 > 0$ ($x_3 < 0$). The pseudo-norm of a vector x is defined as a real number

$$||x|| = \sqrt{|\langle x, x \rangle|} \ge 0. \tag{1}$$

Angle between vectors in \mathbb{R}^3_1 is introduced as follows:

$$\begin{cases}
\cosh \varphi = -\frac{\langle x, y \rangle}{||x|| ||y||}, & \text{if } x, y \text{ are positive (negative)} \\
& \text{timelike vectors,}
\end{cases}$$

 $\cos \varphi = \frac{\langle x, y \rangle}{||x|| ||y||}$, if x, y are spacelike vectors that span a spacelike plane,

$$\cosh \varphi = \frac{|\langle x, y \rangle|}{||x|| ||y||}, \text{ if } x, y \text{ are spacelike vectors}$$

that span a timelike plane,
$$|\langle x, y \rangle|$$

 $\sinh \phi = \frac{|\langle x, y \rangle|}{||x|| ||y||}, x \text{ is a spacelike, } y \text{ a positive}$ timelike vector.

(2)

The causal character (spacelike, timelike or lightlike) of a regular curve is determined by the causal character of its velocity vector. The local theory of curves in Lorentz-Minkowski space can be found in e.g. [4]. For our purposes, it is enough to provide it for *spacelike* curves only. Let $c: I \subset \mathbb{R} \to \mathbb{R}^3_1$ be a spacelike curve parametrized by the arc length. If c''(u) is not null, the Frenet frame and the Frenet equations are given by

$$\begin{pmatrix} T'\\N'\\B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0\\ -\epsilon \kappa & 0 & \tau\\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T\\N\\B \end{pmatrix},$$
 (3)

where $\varepsilon = \langle N, N \rangle = \pm 1$, $\kappa(u) = \|c''(u)\|$ is the curvature and $\tau(u) = \varepsilon \langle B'(u), N(u) \rangle$ is the torsion of *c*.

Let c''(u) be null for all $u \in I$. These curves are called *pseudo-null curves*. The principal normal vector is defined as N(u) = c''(u). The binormal vector B(u) is the null vector orthogonal to *T* satisfying $\langle N, B \rangle = 1$. Then the Frenet equations are

$$\begin{pmatrix} T'\\N'\\B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0\\ 0 & \tau & 0\\ -\kappa & 0 & -\tau \end{pmatrix} \begin{pmatrix} T\\N\\B \end{pmatrix}.$$
 (4)

Here $\tau = \langle N', B \rangle$ is the so-called pseudo-torsion, [4, 5]. The curvature κ can be either 0 (a curve is a straight line), or $\kappa = 1$.

3 Curves lying on a lightlike cone

A lightlike cone LC(p) with the vertex p in Lorentz-Minkowski space \mathbb{R}^3_1 is a quadratic surfaces defined as

$$LC(p) = \{q \in \mathbb{R}^3_1 : \langle q - p, q - p \rangle = 0\}.$$

It is a (Euclidean) rotational cone with the axis x_3 , and a ruled surface with lightlike generators (rulings). As a surface in \mathbb{R}^3_1 it inherits a degenerated metric of rank 1 (an isotropic metric), therefore it is a lightlike surface. The lightlike cone with the vertex at 0 is simply denoted by *LC*. If a curve c lies on a lightlike cone LC(p), then $\langle c(s) - p, c(s) - p \rangle = 0$ and therefore c(s) - p is a lightlike (null) vector. This implies $\langle c(s) - p, c'(s) \rangle = 0$, that is, c'(s) is orthogonal to a lightlike vector c(s) - p. Then c'(s)is either lightlike or spacelike (excluding zero vector due to regularity of a curve). In the case when c'(s) is lightlike, it needs to be collinear to c(s) - p, which would imply that c is a null straight line (the only lightlike vectors orthogonal to a lightlike vector are collinear vectors). In the case when c'(s) is a spacelike vector, we can assume that it is parametrized by arc length, $\langle c', c' \rangle = 1$. The vector c''(s)can be either spacelike, timelike or lightlike. Curves with the last property are pseudo-null curves.

For curves lying on LC(p), the following result holds.

Theorem 1 A curve c lies on a cone LC(p) centered at p if and only if

1. c is a spacelike curve with c" spacelike or timelike that satisfies (when parametrized by arc length)

$$\rho\tau = \pm \rho', \tag{5}$$

where $\rho = \frac{1}{\kappa}$;

- 2. c is a spacelike curve with c" lightlike if and only if c is a Lorentzian circle in a lightlike plane (that is, a Euclidean parabola with the axis parallel to the lightlike direction);
- *3. c* is a lightlike curve if and only if c is a generator line of a ruled surface LC(p).

Proof. To prove the first statement, we recall that $\langle c(s) - p, c(s) - p \rangle = 0$ implies $\langle c(s) - p, T(s) \rangle = 0$. Differentiating the previous equation and using the Frenet formulas (3) we get

$$\langle c(s) - p, N(s) \rangle = -\rho(s), \ \langle c(s) - p, B(s) \rangle = \varepsilon \frac{\rho'}{\tau}.$$

Therefore, if we write $c(s) - p = \alpha T(s) + \beta N(s) + \gamma B(s)$, we can conclude

$$c(s) - p = -\rho N(s) - \varepsilon \frac{\rho'}{\tau} B.$$

Furthermore, since c(s) - p is a null vector, we have

$$\rho^2 = \left(\frac{\rho'}{\tau}\right)^2$$

which was to prove. For the second statement we recall that a pseudo-null curve is planar and lies in a lightlike plane, [1]. A lightlike plane contains only one null direction, that is, a direction parallel to one null ruling of LC(p). Therefore, a curve *c* is planar section of LC(p) yielding as an intersection a Euclidean parabola with the axis parallel to the null direction, that is, a Lorentzian circle in a lightlike plane. Finally, the last statement is obvious.

Remark 1 Notice that among the curves described by the case (1) in the previous theorem, there are also other planar sections beside Euclidean parabola described in (2), that is curves that satisfy $\tau = 0$. By the condition (5) their curvature is $\kappa = \text{const.} \neq 0$. Therefore, they are Euclidean (resp. Lorentzian) circles in spacelike (resp. timelike) planes, that is, from Euclidean perspective, ellipses and equilateral hyperbolas (see Figure 1).



Figure 1: Lorentzian circle in spacelike, timelike and lightlike plane, respectively.

4 Generalized helices and their plane projections

A generalized helix is a regular unit speed curve (parametrized by the arc length or a pseudo-arc length) for which there exists a constant vector $u \in \mathbb{R}^3_1$, $u \neq 0$, such that

$$\langle T, u \rangle = const. =: \alpha.$$
 (6)

If a spacelike or a timelike curve c is a generalized helix then

$$\frac{\tau}{\kappa} = const. =: A, \tag{7}$$

and conversely, for a curve c with non-lightlike normal vectors, [4]. A direction determined by u is called the axis of a helix.

We allow also curves satisfying A = 0, that is, planar curves, to belong to the class of generalized helices. The constant vector *u* from the definition is the unit normal of the plane in which the curves lie, therefore $\alpha = 0$.

In this section we are interested in relation of a generalized helix and its projection curve onto a certain plane in Lorentz-Minkowski space. For this purpose we consider general non-lightlike planes as projection planes. Let a plane π be determined by its unit normal vector u, which is spacelike (resp. timelike) when a plane π is timelike (resp. spacelike). We denote $\delta = \langle u, u \rangle = \pm 1$. Then a projection curve \tilde{c} of c onto a plane π is given by

$$\tilde{c} = c - \delta \langle c, u \rangle u. \tag{8}$$

If the initial curve *c* is spacelike, the following result holds:

Theorem 2 Let c be a (unit speed) spacelike generalized helix with respect to a unit spacelike or timelike vector u. Let \tilde{c} be the projection of c onto a plane orthogonal to u. Then \tilde{c} has a constant speed. Furthermore, if the principal normals of c and \tilde{c} are of the same causal non-null character, the curvature of c and of \tilde{c} are related by

$$\tilde{\kappa}^2 (1 - \delta \alpha^2)^2 = \kappa^2. \tag{9}$$

Proof. Let *s* be the arc length parameter of *c* and \tilde{s} of \tilde{c} . Differentiating (8) with respect to *s* yields

$$T = \tilde{T}\frac{d\tilde{s}}{ds} + \delta\langle T, u \rangle u.$$
⁽¹⁰⁾

Therefore, since \tilde{T} and u are orthogonal,

$$\langle T,T\rangle = \langle \tilde{T},\tilde{T}\rangle (rac{d\tilde{s}}{ds})^2 + \delta lpha^2$$

Moreover, we have and $\langle T, T \rangle = 1$, and $\langle \tilde{T}, \tilde{T} \rangle = \pm 1$, since the causal character of \tilde{c} is not known. Therefore we have

$$\pm (\frac{d\tilde{s}}{ds})^2 = 1 - \delta \alpha^2. \tag{11}$$

Moreover, the speed of \tilde{c} is given by

$$||\frac{d\tilde{c}}{ds}|| = ||\tilde{T}\frac{d\tilde{s}}{ds}|| = \sqrt{|1 - \delta\alpha^2|} = const.$$
(12)

which implies that \tilde{c} is of constant speed (with respect to *s*).

To prove equation (9), we proceed as follows. First we notice that by (6) from the definition of a generalized helix we have

$$\langle N, u \rangle = 0$$

Now, by differentiating (10) and using the previous equation, we get

$$\kappa(s)N(s) = \tilde{\kappa}(\tilde{s})\tilde{N}(\tilde{s})(\frac{d\tilde{s}}{ds})^2,$$

where $\tilde{\kappa}$ is the curvature of \tilde{c} (as a curve in Euclidean or Lorenzian plane, \mathbb{R}^2 or \mathbb{R}^2_1). We can notice that the previous equation implies that *N* and \tilde{N} should be collinear, and therefore of the same causal character. In that case, by taking the pseudo-scalar square of the previous equation we get

$$\kappa(s)^2 = \tilde{\kappa}(\tilde{s})^2 (\frac{d\tilde{s}}{ds})^4.$$

Now (11) implies that the equation (9) holds. Note that otherwise, when N and \tilde{N} are not of the same causal character, it would not be possible for the relation (9) to be satisfied in real numbers.

Remark 2 The relation (9) between the curvatures of an initial curve and its projection is valid only under the assumption that their principal normals are of the same causal character. For example, the relation holds if a curve c with $\kappa = \text{const.} \neq 0$, $\tau = 0$, that is a circle in \mathbb{R}^3_1 , is projected onto the plane of the same character as the plane in which it lies: if c is a circle in a spacelike plane (an ellipse from the Euclidean perspective), then its projection onto xy-plane is a "real" circle in Euclidean plane, having constant curvature $\tilde{\kappa}$.

In the next result we discuss the causal character of the projection of a spacelike generalized helix.

Theorem 3 Let c be a (unit speed) spacelike generalized helix with respect to a unit spacelike or timelike vector u. Then \tilde{c} is:

- 1. spacelike, if u timelike or if u is spacelike such that u and T span a spacelike plane;
- 2. *timelike, if u is spacelike such that u and T span a timelike plane.*

Proof. Statements follow from the analysis of the speed of a projection, $\langle \tilde{c}', \tilde{c}' \rangle = 1 - \delta \alpha^2$, where derivative is taken with respect to *s*. If *u* is a timelike unit vector, then $\delta = -1$, which implies $\langle \tilde{c}', \tilde{c}' \rangle = 1 + \alpha^2$. Hence the projection \tilde{c} is spacelike (which is actually obvious as \tilde{c} is a projection onto a spacelike plane). To prove other statements, we relate the constant α in the definition (6) to the angle between the initial curve *c* and *u*, see (2). If *u*, *T* are spacelike vectors that span a spacelike plane, then we can interprete α as $\alpha = \cos \varphi$, for some real number φ ; if they span a timelike plane, then $|\alpha| = \cosh \varphi$. Therefore, in the first case of the Theorem we have $\langle \tilde{c}', \tilde{c}' \rangle = 1 - \alpha^2 = 1 - \cos^2 \varphi = \sin^2 \varphi > 0$, and in the second case $\langle \tilde{c}', \tilde{c}' \rangle = 1 - \alpha^2 = 1 - \cosh^2 \varphi = -\sinh^2 \varphi < 0$. By these relations the causal character of \tilde{c} is determined. **Remark 3** Let us discuss the case when a curve c is lightlike. The lightlike (null) generalized helices are those having a pseudo-torsion (lightlike curvature) constant [2, 4]. There are three types of null helices parametrized by pseudo-arc parameter, those congruent to $c(s) = (\frac{1}{a^2}\cos(as), \frac{1}{a^2}\sin(as), \frac{s}{a}), c(s) =$ $(-\frac{s}{a}, \frac{1}{a^2}\cosh(as), \frac{1}{a^2}\sinh(as))$ or to so-called null cubics $c(s) = (\frac{s^3}{4} - \frac{s}{3}, \frac{s^2}{2}, \frac{s^3}{4} + \frac{s}{3})$. Their axes are timelike, spacelike and lightlike, respectively. None of them are curves lying on a lightlike cone in Lorentz-Minkowski space.

5 Generalized helices lying on a lightlike cone LC(p)

Theorem 4 Let c be a (unit-speed) generalized helix with respect to a unit vector u that lies on a lightlike cone LC(p). Then the curvature and the torsion of c are given by

$$\kappa(s) = \pm \frac{1}{As}, \ \tau(s) = \pm \frac{1}{s}.$$
(13)

The curvature $\tilde{\kappa}(\tilde{s})$ of a projection \tilde{c} of c on the plane orthogonal to u is given by

$$\tilde{\kappa}(\tilde{s}) = \frac{1}{a\tilde{s}}, \ i.e. \ \tilde{\rho}(\tilde{s}) = a\tilde{s},$$
(14)

where $a^2 = A^2 |1 - \delta \alpha^2|$.

In particular, if A = 0, that is, if c is planar, then $\kappa(s) = const. \neq 0, \tau = 0$, and therefore $\tilde{\kappa}(\tilde{s}) = const. \neq 0$.

Proof. If *c* is a unit-speed on LC(p), then *c* is a spacelike curve whose curvature and torsion satisfy (5). Then $\rho'^2 = A^2$, that is, $\rho = \pm As + c$ and the assertion follows (by neglecting without a loss of generality a constant *c*). The curvature of a projection \tilde{c} is obtained from (9). Furthermore, in particular if A = 0, then $\alpha = 0$, (9) implies $\kappa(s) = \tilde{\kappa}(\tilde{s}) = const$.

Remark 4 *Pseudo-null curves lying on a lightlike cone, that is Euclidean parabolas in lightlike planes (with pseudo-torsion* $\tau = 0$ *), are also curves of constant slope by definition, but we do not consider their projection since they have null principal normals.*

Our next goal is to analyze plane projections of generalized helices lying on a lightlike cone in Lorentz-Minkowski 3space. Because of a special position of a lightlike cone LC(p) (having x_3 as its axis, usually depicted as vertical), we are interested in projections onto a spacelike *xy*-plane and Lorentzian *xz*-plane. Let *c* be a curve on a lightlike cone (spacelike curve parametrized by arc length) and *u* a timelike vector ($\delta = -1$). The constant α can be interpreted in terms of an angle as $\alpha = \langle T, u \rangle = \sinh \varphi$. Then we have $\langle \tilde{c}', \tilde{c}' \rangle = 1 - \delta \alpha^2 = \cosh^2 \varphi$. The projection \tilde{c} onto a spacelike plane orthogonal to *u* is a spacelike curve with the arc length parameter $\tilde{s} = (\cosh \varphi)s$. The curvature $\tilde{\kappa}$ is given by (14) as

$$\tilde{\kappa}(\tilde{s}) = \pm \frac{1}{(A\cosh\varphi)\tilde{s}} = \frac{1}{a\tilde{s}},$$

where $a = \pm A \cosh \varphi$.

We can reconstruct the projection curve \tilde{c} from its natural equation (14). In the case when the projection plane is *xy*-plane, we introduce

$$t(\tilde{s}) = \int_{\tilde{s}_0}^{\tilde{s}} \tilde{\kappa}(\tilde{s}) d\tilde{s} = \int_{\tilde{s}_0}^{\tilde{s}} \frac{1}{a\tilde{s}} d\tilde{s} = \frac{1}{a} \ln|b\tilde{s}|,$$
(15)

where *b* is a real constant. The curve \tilde{c} can be written as

$$\tilde{c}(\tilde{s}) = \left(\int_{\tilde{s}_0}^{\tilde{s}} \cos t(\tilde{s}) d\tilde{s}, \int_{\tilde{s}_0}^{\tilde{s}} \sin t(\tilde{s}) d\tilde{s}\right)$$

By calculating these integrals and using (15) we get the following parametrization of the curve \tilde{c}

$$x(t) = \frac{a}{b(1+a^2)}e^{at}(a\cos t + \sin t),$$
$$y(t) = \frac{a}{b(1+a^2)}e^{at}(-\cos t + a\sin t)$$

We can notice that this curve is a logarithmic spiral (see Figure 2, right). Now, having a projection, the remaining coordinate of a curve c lying on a lightlike cone LC is calculated as

$$z(t) = \frac{a}{b\sqrt{1+a^2}}e^{at}.$$

See Figure 2.



Figure 2: A generalized helix on a cone and its projection on xy-plane

Now, consider *u* to be a spacelike vector ($\delta = 1$), in particular u = (0, 1, 0). The projection plane is a timelike *xz*-plane.

If c' and u span a spacelike plane, a projection curve \tilde{c} is spacelike. Then α can be interpreted as $\cos \varphi$, $\varphi \in \mathbb{R}$ (see (2)). Then we have $\langle \tilde{c}', \tilde{c}' \rangle = \sin^2 \varphi$, and for the arc length parameters the following holds $\tilde{s} = \sin \varphi s$. The curvature $\tilde{\kappa}$

satisfies (14) with $a = A \sin \varphi$. The spacelike curve \tilde{c} in a timelike plane can be reconstructed from (see [4])

$$\tilde{c}(\tilde{s}) = \left(\int_{\tilde{s}_0}^{\tilde{s}} \cosh t(\tilde{s}) d\tilde{s}, \int_{\tilde{s}_0}^{\tilde{s}} \sinh t(\tilde{s}) d\tilde{s}\right),\$$

where $t(\tilde{s})$ is given by (15). By calculating these integrals and using (15) we get the following parametrization of the curve \tilde{c} with $a \neq 1$

$$x(t) = \frac{a}{b(a^2 - 1)}e^{at}(-\sinh t + a\cosh t),$$

$$z(t) = \frac{a}{b(a^2 - 1)}e^{at}(a\sinh t - \cosh t).$$

It would be a *Lorentzian logarithmic spiral* (see Figure 3, right). Furthermore, for the initial curve we need the coordinate

$$y(t) = \frac{a}{b\sqrt{1-a^2}}e^{at}$$

See Figure 3.



Figure 3: A generalized helix on a cone and its projection on xz-plane

In the case when a = 1, b = 1 we have

$$x(t) = \frac{1}{2}(\frac{e^{2t}}{2} + t),$$
$$z(t) = \frac{1}{2}(\frac{e^{2t}}{2} - t).$$

The coordinate *y* is given by

$$y^2(t) = -\frac{1}{2}e^{2t}t$$

which restricts to t < 0. See Figure 4.



Figure 4: A generalized helix on a cone and its projection on xz-plane

Finally, if *u* is spacelike, and *c'* and *u* span a timelike plane, then by (2) we have $|\alpha| = \cosh \varphi$, $\varphi \in \mathbb{R}$. Then we have $\langle \tilde{c}', \tilde{c}' \rangle = -\sinh^2 \varphi$, and for the arc length parameters $\tilde{s} = (\sinh \varphi)s$. The curvature $\tilde{\kappa}$ satisfies (14) with $a = \pm A \sinh \varphi$. The timelike curve \tilde{c} in a timelike plane can be reconstructed from

$$\tilde{c}(\tilde{s}) = \left(\int_{\tilde{s}_0}^{\tilde{s}} \sinh t(\tilde{s}) d\tilde{s}, \int_{\tilde{s}_0}^{\tilde{s}} \cosh t(\tilde{s}) d\tilde{s}\right),$$

where $t(\tilde{s})$ is given by (15). Lorentzian logarithmic spirals obtained in this case are the same as in the previous, but with the *x* and *z* coordinates interchanged.

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Finally, as we have already commented, when projections curves are circles in *xy*- or *xz*-planes, that is, curves having a constant curvature $\tilde{\kappa}$, then by (7), their torsion τ is also constant, meaning they are helices lying on a *circular* cylinder as well. Then they should be planar curves, that is planar intersections of a lighlike cone (Euclidean circles or hyperbolas).

We can also note, that contrary to Euclidean case, where the projections of cone helices are logarithmic spirals and circles only, in Lorentz-Minkowski case we have other classes of cone helices as well, those whose projections are Lorentzian logarithmic spirals and hyperbolas, and those lying in lightlike planes (parabolas).

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The Feuerbach Theorem and Cyclography in Universal Geometry

The Feuerbach Theorem and Cyclography in Universal Geometry

ABSTRACT

We have another look at the Feuerbach theorem with a view to extending it in an oriented way to finite fields using the purely algebraic approach of rational trigonometry and universal geometry. Our approach starts with the tangent lines to three rational points on the unit circle, and all subsequent formulas involve the three parameters that define them. Tangency of incircles is treated in the oriented setting via a simplified form of cyclography. Some interesting features of the finite field case are discussed.

Key words: Feuerbach theorem, incircles, universal geometry, cyclography, finite fields

MSC2010: 51N10, 51E26, 01A55

Feuerbachov teorem i ciklografija u univerzalnoj geometriji

SAŽETAK

Dajemo drugačiji pogled na Feuerbachov teorem s ciljem da ga se orijentirano proširi na konačna polja koristeći isključivo algebarski pristup racionalne trigonometrije i univerzalne geometrije. Naš pristup počinje s tangentama u tri racionalne točke na jediničnoj kružnici, i sve naknadne formule uključuju tri parametra koja ih definiraju. Tangencijalnost upisanih kružnica promatra se u orijentiranom okruženju koristeći pojednostavljene forme ciklografije. Promatraju se neka zanimljiva događanja u slučaju konačnih polja.

Ključne riječi: Feuerbachov teorem, upisane kružnice, univerzalna geometrija, ciklografija, konačna polja

1 Introduction

This paper looks to show that the Feuerbach Theorem, on the tangency of the nine-point circle of a triangle with the four incircles/excircles, holds for triangle geometry over a general field, once the existence of an incircle is established. While such an assumption is implicit in many geometric constructions, algebraically it requires a solution to a quadratic equation, involving number theoretic conditions. This is because incentres rely on bilines of a triangle, which are the rational equivalents of angle bisectors in Rational Trigonometry ([9], [10]). Note also that we adopt the four-fold symmetry towards the incentre hierarchy of [6], where all four incentres /excentres are just referred to as *incentres*. Over a general field, there is not a good distinction between these.

Our approach starts with the unit circle as an incircle, choosing three rational points on it with parameters t_1, t_2

and t_3 , and then creating the basic starting triangle via the tangents to these points. We are then able to make numerous calculations of important points and lines in terms of t_1, t_2 and t_3 , culminating with the novel result that all four incentre quadrances are indeed squares in the underlying field.

This then allows us to apply a simple form of the classical theory of *cyclography*, which connects oriented_circles to relativistic geometry, even in the context of general fields, and which yields a purely algebraic proof of the Feuerbach theorem. So our approach establishes the result even over finite fields.

Metrical geometry over finite fields is a subject still largely in its infancy, and this result allows us to investigate several novel features. One of them is what we might call *overlapping*: a given finite field might actually be too small to support distinct objects that may be familiar in the rational, or more generally unbounded, field situation. To gain some intuition in this direction, we examine some discrete versions of the Feuerbach result in some detail over the prime field \mathbb{F}_{17} to illustrate that new phenomenon really do appear in the finite field case. We end with an example in \mathbb{F}_{23} which contains all the points of the Feuerbach configuration without duplication.

2 History of the Feuerbach theorem

Recall that the nine-point circle of a triangle passes through the three midpoints of the sides, the three feet of the altitudes, and the three midpoints between the vertices and the orthocentre.

Theorem 1 (*Feuerbach*) *The nine-point circle of a triangle is tangent to the four incircles of that triangle.*

This celebrated theorem is one of the most challenging results in classical geometry, as its statement is simple and surprising, but proofs are highly non-trivial. It was published in 1822 by Karl Wilhelm Feuerbach in the book "*Eigenschaften einiger merkwürdigen Punkte des* geradlinigen Dreiecks und mehrerer durch sie bestimmten Linien und Figuren. Eine analytisch-trigonometrische Abhandlung" ([3]). Since Feuerbach's initial exposition, many authors have attempted proofs. Almost all of them have relied on distances and/or angles, making them difficult to extend to more general situations, for example geometry over finite fields.

John Casey's 1864 proof ([1]) used inversion and a form of Ptolemy's theorem for collinear points, along with a suitable inversion and comparison of distance ratios of the four incircles and the nine-point circle. Somewhat similar proofs appeared in books of William M'Clelland (1891) ([7]) and Robert Lachlan (1893) ([5]) as semi-completed exercises for the reader. However these proofs only covered the tangency of the interior incircle with the ninepoint circle, with the proof for the tangency of the exterior incircles omitted, and relied on arguments involving distances, angles and concyclic quadrangles.

Coxeter and Greitzer's 1967 proof ([2]) builds upon Casey's proof and streamlines it, using a sequence of inversions and applications of Heron's formula to set up and transform different distance ratios to check what constructions can be preserved under a given inversion.

Franz Hofbauer's 2016 proof ([4]) currently stands as perhaps the simplest proof, and uses the same skeleton as Michael Scheer's 2011 proof ([8]), except that where Scheer used barycentric coordinates and the law of sines, Hofbauer uses vector geometry and Heron's formula.

Hofbauer's proof starts with a triangle \overline{ABC} with a circumradius *R* and side lengths *a*, *b* and *c*, chosen such that the circumcentre *O* of the triangle is centred at the origin. By assigning position vectors to the vertices of the triangle, it becomes simple to find the position vectors of the ninepoint centre, interior incentre and an exterior incentre. After establishing some relationships between the distances in this set up, an application of Heron's formula proves the result.

Both the classical and more modern proofs can be critiqued from several directions, especially when we intend to extend the theorem to universal geometry, over general fields. Distances and angles are hard to generalize, constructions based on diagrams can be problematic, and do we have a consistent theory of inversion and cyclic quadrilaterals over general fields? When we examine in some detail some examples in finite fields at the end of this paper, another possible question emerges: how do we actually know that synthetic arguments create points and lines which are distinct? If further constructions require joins or meets of existing points, there are logical questions here that may be hidden in a proof based on a physical construction.

3 Projective coordinates and quadrances

We work over a general field, characteristic two excluded. Projective coordinates for points and lines are useful since cross-product of coordinates express both joins of points and meets of lines. So the affine point A = [a,b] will be expressed in projective coordinates as A = [1 : a : b], and the line *l* with equation r + sx + ty = 0 will be expressed in projective coordinates as l = (r,s,t). Then the incidence between *A* and *l* is expressed as the dot product between their respective coordinates being zero, that is r + sa + tb = 0. Then for distinct affine points $A = [a_1, a_2]$ and $B \equiv [b_1, b_2]$ we have their join

$$A_1A_2 = [1:a_1:a_2] \times [1:b_1:b_2] = (a_1b_2 - a_2b_1:a_2 - b_2:b_1 - a_1)$$

and similarly for lines $l_1 : r_1 + s_1x + t_1y = 0$ and $l_2 : r_2 + s_2x + t_2y = 0$ their meet is

$$l_1 l_2 = (r_1 : s_1 : t_1) \times (r_2 : s_2 : t_2)$$

= [s_1 t_2 - s_2 t_1 : r_2 t_1 - r_1 t_2 : r_1 s_2 - r_2 s_1].

In each case only the usual three-dimensional cross product is involved, thus significantly reducing complexity of elementary calculations.

Using an elementary concept from rational trigonometry ([9]) the (perpendicular) quadrance from a point P = [a, b] to a line l = (r : s : t) with equation r + sx + ty = 0 is given

$$Q(P,l) = \frac{(r+sa+tb)^2}{s^2+t^2}$$

assuming the line is non-null, that is its normal vector (s,t) is not null, meaning that $Q((s,t)) = s^2 + t^2 \neq 0$.

A circle will be an equation of the form $(x-a)^2 + (y-b)^2 = Q$, with centre [a,b] and quadrance Q. We do

not assume that Q is necessarily a square in the field, so in general a circle does not have a well-defined "radius". When it does, then there will be two possible associated radii. This observation will play an important role when we discuss cyclography later.

4 Rational points on the unit circle

Since all starting incircles are equivalent under translation and dilation, (actually the latter might require a quadratic field extension) we can, without loss of generality, start with the unit circle $C_0: x^2 + y^2 = 1$ with centre $I_0 = [0,0]$ and quadrance $R_0 = 1$ which we will arrange to be an incircle of our basic triangle. We choose three distinct points on the unit circle

$$Z_i = \left[1 + t_i^2 : 1 - t_i^2 : 2t_i\right] = \left[\frac{1 - t_i^2}{1 + t_i^2}, \frac{2t_i}{1 + t_i^2}\right] \text{ for } i = 1, 2, 3$$

where each *t* value in the field is distinct. The geometrical meaning of the parameter values is shown in Figure 1.



Figure 1: Parameters t correspond to y-intercepts

Remark 1 In order to guarantee the existence of the points on the circle, our choice of t-values must be such that $t^2 + 1 \neq 0$. For the rational numbers, and a prime field F_p satisfying $p \equiv 3 \pmod{4}$ this condition holds automatically, but not so for a finite prime field where $p \equiv 1 \pmod{4}$.

Since the tangent to the point [r,s] on the unit circle has equation rx + sy = 1, with projective coordinates (-1:r:s), the tangent to C_0 at Z_i will be the line $z_i = (-(1+t_i^2):1-t_i^2:2t_i)$ for i = 1,2,3.

We can then define the vertex A_1 to be the meet of z_2 and z_3 , and similarly define A_2 and A_3 which gives us our basic

triangle $\overline{A_1A_2A_3}$ which a calculation shows has vertices

$$A_1 \equiv z_2 z_3 = [1 + t_2 t_3, 1 - t_2 t_3, t_2 + t_3]$$

$$A_2 \equiv z_3 z_1 = [1 + t_3 t_1, 1 - t_3 t_1, t_3 + t_1]$$

$$A_3 \equiv z_1 z_2 = [1 + t_1 t_2, 1 - t_1 t_2, t_1 + t_2].$$

To avoid having any vertices at infinity, we will further require that no two *t*-values have a product of -1, since tangents z_i and z_j are parallel precisely when the chosen points Z_i and Z_j are diametrically opposed, which amounts to the condition $t_i t_j = -1$. This construction guarantees that the unit circle C_0 will be tangent to the three sides of the triangle $\overline{A_1A_2A_3}$, meaning it is one of the incircles of the triangle.



Figure 2: If $t_1 = 1.8$, $t_2 = -0.2$ and $t_3 = -1.5$, C_0 is the interior incircle



Figure 3: If $t_1 = 0.8$, $t_2 = 0.5$ and $t_3 = -0.4$, C_0 is an exterior incircle

Remark 2 Over the rational numbers, different choices for the t-values will determine whether or not C_0 will be the interior incircle or one of the exterior incircles. Indeed, it can be shown that C_0 will be the interior incircle precisely when $(t_1t_2+1)(t_1t_3+1)(t_2t_3+1) < 0$. But over general fields this kind of consideration is reduced in importance.

5 The incentres

Since we know that I_0 is an incentre, it is determined as the meet of three of the bilines of the triangle $\overline{A_1A_2A_3}$, with one biline from each vertex. The biline b_{1a} is the join of $I_0 = [0,0]$ and A_1 so in projective coordinates it is

$$b_{1a} = [1:0:0] \times [1+t_2t_3:1-t_2t_3:t_2+t_3]$$

= (0:-(t_2+t_3):1-t_2t_3)

and similarly

$$b_{2a} = (0: -(t_3 + t_1): 1 - t_3 t_1)$$

and

$$b_{3a} = (0: -(t_1+t_2): 1-t_1t_2).$$

Since bilines come in pairs, meeting perpendicularly at their corresponding vertex, we can use the above equations to find the remaining three bilines of $\overline{A_1A_2A_3}$. For example: the biline perpendicular to b_{1a} at A_1 will have equation

$$b_{1b}: (1 - t_2 t_3) x + (t_2 + t_3) y = c$$

and substituting the coordinates of A_1 gives $\frac{(1-t_2t_3)^2}{1+t_2t_3} + \frac{(t_2+t_3)^2}{1+t_2t_3} = c$ which simplifies to $c = \frac{(1+t_2^2)(1+t_3^2)}{(1+t_2t_3)}$. Doing this for the pairs of bilines corresponding to each vertex yields the projective coordinates of the remaining bilines:

$$b_{1b} = \left(-\left(1+t_2^2\right)\left(1+t_3^2\right): (1-t_2t_3)\left(1+t_2t_3\right): (t_2+t_3)\left(1+t_2t_3\right)\right)$$

$$b_{2b} = \left(-\left(1+t_3^2\right)\left(1+t_1^2\right): (1-t_3t_1)\left(1+t_3t_1\right): (t_3+t_1)\left(1+t_3t_1\right)\right)$$

$$b_{3b} = \left(-\left(1+t_1^2\right) \left(1+t_2^2\right) : (1-t_1t_2) \left(1+t_1t_2\right) : (t_1+t_2) \left(1+t_1t_2\right) \right).$$



Figure 4: All six bilines of $\overline{A_1A_2A_3}$

These six bilines meet three at a time at the four incentres, which besides $I_0 = [0,0]$ are

$$I_{1} \equiv b_{2b}b_{3b} = [(1+t_{1}t_{2})(1+t_{3}t_{1}):(1+t_{1}^{2})(1-t_{2}t_{3}):(1+t_{1}^{2})(t_{2}+t_{3})]$$

$$I_{2} \equiv b_{1b}b_{3b} = [(1+t_{1}t_{2})(1+t_{2}t_{3}):(1+t_{2}^{2})(1-t_{3}t_{1}):(1+t_{2}^{2})(t_{3}+t_{1})]$$

$$I_{3} \equiv b_{1b}b_{2b} = [(1+t_{3}t_{1})(1+t_{2}t_{3}):(1+t_{3}^{2})(1-t_{1}t_{2}):(1+t_{3}^{2})(t_{1}+t_{2})]$$



Figure 5: Incentres I_0, I_1, I_2, I_3 of the triangle $\overline{A_1A_2A_3}$

6 Equations of the incircles

Now that we have the centres of the incircles, we may determine their quadrances, using the formula for the quadrance from a point to a line, to get

$$R_{1} \equiv Q(I_{1}, z_{1}) = \frac{(t_{1} - t_{2})^{2} (t_{1} - t_{3})^{2}}{(1 + t_{1}t_{2})^{2} (1 + t_{1}t_{3})^{2}}$$

$$R_{2} \equiv Q(I_{2}, z_{2}) = \frac{(t_{2} - t_{3})^{2} (t_{2} - t_{1})^{2}}{(1 + t_{1}t_{2})^{2} (1 + t_{2}t_{3})^{2}}$$

$$R_{3} \equiv Q(I_{3}, z_{3}) = \frac{(t_{3} - t_{1})^{2} (t_{3} - t_{2})^{2}}{(1 + t_{1}t_{3})^{2} (1 + t_{2}t_{3})^{2}}.$$

Remark 3 Since each incentre will be equidistant to the three sides of the triangle $\overline{A_1A_2A_3}$, we could actually use any of the tangents z_1 , z_2 or z_3 to compute the quadrances of the incircles.

As a consequence of these formulas, we may make the following deduction:

Theorem 2 If one of the incircles of a triangle has a quadrance which is a perfect square, then so do the others.

Proof. If one of the incircles has a quadrance which is a square, then we can position and dilate it to be the unit circle using only affine transformations involving the field, in which case the above formulas hold. Then translating back and performing an inverse dilation shows that the original incircles also have square quadrances.



Figure 6: The four incentres and incircles

7 Coordinates of the nine point centre

The nine-point circle of a triangle is the circumcircle of the three midpoints of the triangle. The following gives the affine coordinates for the centre N of this nine point circle in terms of the affine coordinates of three general vertices. The formulas are conveniently expressed using determinants of 3×3 matrices.

Theorem 3 In projective coordinates the centre of the nine-point circle of the triangle $\overline{G_1G_2G_3}$ where $G_1 = [1:a_1:b_1]$, $G_2 = [1:a_2:b_2]$ and $G_3 = [1:a_3:b_3]$, is $N = [z_N:x_N:y_N]$ where

$$\begin{aligned} z_N &= 4 \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} \\ x_N &= \begin{vmatrix} a_1^2 & b_1 & 1 \\ a_2^2 & b_2 & 1 \\ a_3^2 & b_3 & 1 \end{vmatrix} - 2 \begin{vmatrix} a_1b_1 & a_1 & 1 \\ a_2b_2 & a_2 & 1 \\ a_3b_3 & a_3 & 1 \end{vmatrix} + \begin{vmatrix} b_1 & b_1^2 & 1 \\ b_2 & b_2^2 & 1 \\ b_3 & b_3^2 & 1 \end{vmatrix} \\ y_N &= \begin{vmatrix} b_1^2 & a_1 & 1 \\ b_2^2 & a_2 & 1 \\ b_2^2 & a_2 & 1 \\ b_2^2 & a_3 & 1 \end{vmatrix} - 2 \begin{vmatrix} a_1b_1 & b_1 & 1 \\ a_2b_2 & b_2 & 1 \\ a_3b_3 & b_3 & 1 \end{vmatrix} + \begin{vmatrix} a_1 & a_1^2 & 1 \\ a_2 & a_2^2 & 1 \\ a_3 & a_3^2 & 1 \end{vmatrix}.$$

Proof. Since the nine-point circle of a triangle is the circumcircle of its median triangle, the nine-point centre is the intersection of the perpendicular bisectors of the median triangle. The median triangle has vertices

$$M_{1} = \frac{1}{2}G_{2} + \frac{1}{2}G_{3} = [2:a_{2} + a_{3}:b_{2} + b_{3}]$$

$$M_{2} = \frac{1}{2}G_{1} + \frac{1}{2}G_{3} = [2:a_{1} + a_{3}:b_{1} + b_{3}]$$

$$M_{3} = \frac{1}{2}G_{1} + \frac{1}{2}G_{2} = [2:a_{1} + a_{2}:b_{1} + b_{2}].$$

The perpendicular bisector of M_1M_2 has normal vector of $M_2 - M_1 = (a_1 - a_2, b_1 - b_2)$ and passes through the point $\frac{1}{2}M_1 + \frac{1}{2}M_2$ so we can write it as

$$\left[(a_1 - a_2, b_1 - b_2) (\frac{1}{4}(a_1 + a_2 + 2a_3), \frac{1}{4}(b_1 + b_2 + 2b_3))^T : \\ : a_1 - a_2 : b_1 - b_2 \right]$$

and, similarly, the perpendicular bisector of M_1M_3 can be written as

$$\left[(a_1 - a_3, b_1 - b_3) (\frac{1}{4}(a_1 + 2a_2 + a_3), \frac{1}{4}(b_1 + 2b_2 + b_3))^T : \\ : a_1 - a_3 : b_1 - b_3 \right].$$

The meet of these two lines may then be computed and re-expressed in terms of determinants as above. \Box

Theorem 4 For our basic triangle $\overline{A_1A_2A_3}$ determined by parameters t_1, t_2 and t_3 , the nine-point centre is $N = [z_N : x_N : y_N]$ where

$$z_N = 8 (t_2 t_3 + 1) (t_1 t_3 + 1) (t_1 t_2 + 1)$$

$$x_N = - (3t_1 t_2 t_3 + t_1 t_2 + t_1 t_3 + t_2 t_3 + t_1 + t_2 + t_3 + 3)$$

$$\cdot (3t_1 t_2 t_3 - t_1 t_2 - t_1 t_3 - t_2 t_3 + t_1 + t_2 + t_3 - 3)$$

$$y_N = 2 (t_1 + t_2 + t_3 + 3t_1 t_2 t_3) (t_1 t_2 + t_1 t_3 + t_2 t_3 + 3).$$

Proof. We attain this result by substituting the coordinates of the three points A_1, A_2 and A_3 into the expressions for the nine-point centre as determined by the previous theorem and simplifying the result.



Figure 7: The nine-point centre N of the triangle $\overline{A_1A_2A_3}$

8 Quadrance of the nine-point circle

We now come to our first somewhat remarkable calculation, that anticipates the subtlety of the Feuerbach theorem.



Figure 8: Tangency of the nine-point circle

Theorem 5 The quadrance of the nine-point circle is

$$R_N = \frac{\left(t_1^2 + 1\right)^2 \left(t_2^2 + 1\right)^2 \left(t_3^2 + 1\right)^2}{64 \left(t_1 t_2 + 1\right)^2 \left(t_1 t_3 + 1\right)^2 \left(t_2 t_3 + 1\right)^2}$$

Proof. Since the median points of a triangle also lie on the nine-point circle, we can use the nine-point centre and any of the median points of the triangle $\overline{A_1A_2A_3}$ to compute

the quadrance of the nine-point circle. A median point of $\overline{A_1A_2A_3}$ is

$$M_{1} = \frac{1}{2}A_{2} + \frac{1}{2}A_{3}$$
$$= \left[2:\frac{1-t_{1}t_{3}}{1+t_{1}t_{3}} + \frac{1-t_{1}t_{2}}{1+t_{1}t_{2}}:\frac{t_{1}+t_{3}}{1+t_{1}t_{3}} + \frac{t_{1}+t_{2}}{1+t_{1}t_{2}}\right]$$

So using the formula for the nine-point centre in the previous theorem, and after some miraculous simplification,

$$\begin{split} \mathcal{R}_{N} &= Q(I_{N}, M_{1}) = \\ & \left(\frac{(3t_{1}t_{2}t_{3} + t_{1}t_{2} + t_{1}t_{3} + t_{2}t_{3} + t_{1} + t_{2} + t_{3} + 3)}{(1 - \frac{(3t_{1}t_{2}t_{3} - t_{1}t_{2} - t_{1}t_{3} - t_{2}t_{3} + t_{1} + t_{2} + t_{3} - 3)}{8(t_{2}t_{3} + 1)(t_{1}t_{3} + 1)(t_{1}t_{2} + 1)} \\ & - \frac{\frac{1 - t_{1}t_{3}}{1 + t_{1}t_{3}} + \frac{1 - t_{1}t_{2}}{1 + t_{1}t_{2}}}{2} \right)^{2} \\ & + \left(\frac{(t_{1} + t_{2} + t_{3} + 3t_{1}t_{2}t_{3})(t_{1}t_{2} + t_{1}t_{3} + t_{2}t_{3} + 3)}{4(t_{2}t_{3} + 1)(t_{1}t_{3} + 1)(t_{1}t_{2} + 1)} \\ & - \frac{\frac{t_{1} + t_{3}}{1 + t_{1}t_{3}} + \frac{t_{1} + t_{2}}{1 + t_{1}t_{2}}}{2} \right)^{2} \\ & = \frac{(t_{1}^{2} + 1)^{2}(t_{2}^{2} + 1)^{2}(t_{3}^{2} + 1)^{2}}{64(t_{1}t_{2} + 1)^{2}(t_{1}t_{3} + 1)^{2}(t_{2}t_{3} + 1)^{2}}. \end{split}$$

Remark 4 Notice that this quadrance for the nine-point circle is also a square.

 \square

Our aim is to establish the Feuerbach theorem, but the previous results suggest that we can restate this in terms of the classical theory of oriented cycles, or cyclography.

9 Cyclography

We now introduce a very simple version of the 19th century theory of *cyclography*, but more generally over an arbitrary field \mathbb{F} . An **oriented_circle** (note that we employ the usage of one word) is an affine 3-point C = [a, b, r] in the affine space \mathbb{A}^3 over the field \mathbb{F} . The affine 2-point P = [a,b] in \mathbb{A}^2 is called the **centre** of *C*, while the number *r* is called the **oriented_radius** of *C*, and we will also write C = [P, r]. Two oriented_circles are equal precisely when both their centres and oriented_radii agree. A point [x, y]**lies on** C = [a, b, r] precisely when

$$(a-x)^{2} + (b-y)^{2} = r^{2}.$$

This is the equation of a circle *c* with centre also [a,b]and quadrance $Q = r^2$. Clearly both C = [a,b,r] and C' = [a,b,-r] are associated to the same circle *c*. Note however that a general circle cannot be expected to have associated oriented_circles: this will happen precisely when its quadrance is a square in \mathbb{F} . When r = 0 we say the oriented_circle is a **null circle**. Over the rational numbers such null oriented_circles have only one point lying on them. This gives us the **null plane** of null circles inside the three-dimensional space of points [a,b,r].

The space of oriented_circles is naturally identified with the three-dimensional affine space A^3 over \mathbb{F} . The geometry of oriented_circles however embues this threedimensional space \mathbb{A}^3 with a relativistic 2 + 1 quadratic form. In the rational case, non-horizontal lines (not in a plane of the form $r = r_0$) in this space \mathbb{A}^3 correspond to pencils of homothetic oriented_circles, with the meets of such lines with the plane of null circles giving the (oriented) homothetic centres.

To discuss homothetic centres, let's agree to the useful convention that if P = [a, b, c] is an affine point and v = [r, s, t] is a vector, then P + v is the affine point P + v = [a + r, b + s, c + t]. Then for $\alpha \neq 0$, the **dilation centred at** *P* by α is the bijection of the plane that sends P + v to $P + \alpha v$. The inverse is clearly the dilation centred at *P* by α^{-1} .

Proposition 1 *The unique homothetic centre for the oriented_circles* $C_1 = [a_1, b_1, r_1]$ *and* $C_2 = [a_2, b_2, r_2]$ *where* $r_1 \neq r_2$ *is*

$$P = \left[\frac{a_1r_2 - a_2r_1}{r_2 - r_1}, \frac{b_1r_2 - b_2r_1}{r_2 - r_1}\right].$$

If $r_1 \neq 0$ then the dilation in the plane with centre P by a factor of r_2/r_1 sends C_1 to C_2 , and if $r_2 \neq 0$ then the plane dilation with centre P by a factor of r_1/r_2 sends C_2 to C_1 .

Proof. If $C_1 = [a_1, b_1, r_1]$ and $C_2 = [a_2, b_2, r_2]$ are circles with $r_1 \neq r_2$, then the vector $w = (a_2 - a_1, b_2 - b_1, r_2 - r_1)$ is not horizontal, and so the line through those points of the parametric form

 $[a_1,b_1,r_1]+\lambda w$

will meet the null plane when $r_1 + \lambda(r_2 - r_1) = 0$. This occurs precisely when $\lambda = -\frac{r_1}{r_2 - r_1}$ and gives the point [P, 0] where *P* is the homothetic centre

$$P = \left[\frac{1}{r_2 - r_1} \left(a_1 r_2 - a_2 r_1\right), \frac{1}{r_2 - r_1} \left(b_1 r_2 - b_2 r_1\right)\right]. \qquad \Box$$

Thus we can write

$$[a_1, b_1, r_1] = [P, 0] + \frac{r_1}{r_2 - r_1} w$$
 and
 $[a_2, b_2, r_2] = [P, 0] + \frac{r_2}{r_2 - r_1} w$

which proves that if $r_1 \neq 0$ then the plane dilation centred at *P* by r_2/r_1 sends C_1 to C_2 , and similarly if $r_2 \neq 0$ then the dilation by r_1/r_2 sends C_2 to C_1 .

A point A_1 lying on C_1 and a point A_2 lying on C_2 are **homologous** if they are images of each other under these dilations. Now a dilation is an affine map, so directions are maintained, in the sense that the vector determined by the image of two points will be a multiple of the vector determined by the two points themselves. It follows that if A_1 lying on C_1 and a point A_2 lying on C_2 are homologous, then the vector $\overrightarrow{C_1A_1}$ will be a multiple of the vector $\overrightarrow{C_2A_2}$. This yields a direct construction of the homologous point A_2 to A_1 once the homothetic centre is determined.

Three points in \mathbb{F}^3 generally determine a plane, and the meet of such with the null plane will be a line, giving an easy proof of a theorem of Monge that the three homothetic centres of three oriented circles are in general collinear. However if we start with ordinary, unoriented circles that have square quadrances, then due to the choices of possible orientations there will be determined four such Monge lines.

Definition 1 The quadrance between oriented_circles $C_1 = [a_1, b_1, r_1]$ and $C_2 = [a_2, b_2, r_2]$ is

$$Q(C_1, C_2) \equiv (a_2 - a_1)^2 + (b_2 - b_1)^2 - (r_2 - r_1)^2$$

In the case of disjoint oriented_circles in the rational number plane which are not contained in each other, this quadrance has the interpretation of the quadrance along a common homothetic tangent, meaning a tangent which passes through the homothetic centre.

Definition 2 The oriented_circles $C_1 = [a_1, b_1, r_1]$ and $C_2 = [a_2, b_2, r_2]$ are **tangent** precisely when $r_1 \neq r_2$ and $Q(C_1, C_2) = 0$.

In other words, when the quadrance between their centres is equal to the square of this (non-zero) difference of their oriented_radii, that is when

$$(a_2-a_1)^2+(b_2-b_1)^2=(r_2-r_1)^2.$$

Example 1 The oriented_circles C = [0,0,1] and D = [2,0,1] are not tangent, even though their associated circles are. However the oriented_circles C = [0,0,1] and E = [2,0,-1] are tangent.

It is easy to see that if two oriented_circles are tangent, then so are their associated circles. One of the advantages of the cyclographic set-up is that the points of tangency for oriented circles can be obtained very directly. **Theorem 6 (Oriented tangency)** *If the oriented_circles* $C_1 = [a_1, b_1, r_1]$ and $C_2 = [a_2, b_2, r_2]$ are tangent then they meet at exactly one point, which is

$$J = \frac{r_2}{r_2 - r_1} [a_1, b_1] - \frac{r_1}{r_2 - r_1} [a_2, b_2]$$

= $[r_2 - r_1 : r_2 a_1 - r_1 a_2 : r_2 b_1 - r_1 b_2]$

Proof. If the two oriented_circles meet tangentially, they will do so on the line joining their centres, so we can consider a general affine combination

$$J = (1-t)[a_1,b_1] + t[a_2,b_2] = [ta_2 - a_1(t-1), tb_2 - b_1(t-1)]$$

Now in order for this to lie on C_1 we require that

$$(ta_2 - a_1(t-1) - a_1)^2 + (tb_2 - b_1(t-1) - b_1)^2 = r_1^2$$

which is just

$$((a_2-a_1)^2+(b_2-b_1)^2)t^2=r_1^2$$

But given that the two oriented_circles are tangent, we can rewrite this as

 $(r_2 - r_1)^2 t^2 = r_1^2$

so that

$$t=\pm\frac{r_1}{r_2-r_1}.$$

Similarly the condition that *J* lies on C_1 is

$$(ta_2 - a_1(t-1) - a_2)^2 + (tb_2 - b_1(t-1) - b_2)^2 = r_2^2$$

or
 $(r_2 - r_1)^2 (1-t)^2 = r_2^2$

so that $1 - t = \pm \frac{r_2}{r_2 - r_1}$.

Now if $t = \frac{r_1}{r_2 - r_1}$ then $1 - t = 1 - \frac{r_1}{r_2 - r_1} = \frac{1}{r_2 - r_1} (r_2 - 2r_1)$ and for this to equal $\pm \frac{r_2}{r_2 - r_1}$ we would require either $r_2 - 2r_1 = r_2$ which would imply $r_1 = 0$; or $r_2 - 2r_1 = -r_2$ which would imply $r_1 = r_2$ which is not allowed. It follows that $t = -\frac{r_1}{r_2 - r_1}$ so that $1 - t = 1 + \frac{r_1}{r_2 - r_1} = \frac{r_2}{r_2 - r_1}$.

10 A cyclographic proof of the Feuerbach theorem

We now show that the Incentre story we have developed so far fits into this cyclographic point of view. Recall the four incentres

$$I_{0} = [1:0:0]$$

$$I_{1} = \left[(1+t_{1}t_{2})(1+t_{1}t_{3}): (1+t_{1}^{2})(1-t_{2}t_{3}): (1+t_{1}^{2})(t_{2}+t_{3}) \right]$$

$$I_{2} = \left[(1+t_{1}t_{2})(1+t_{2}t_{3}): (1+t_{2}^{2})(1-t_{1}t_{3}): (1+t_{2}^{2})(t_{1}+t_{3}) \right]$$

$$I_{3} = \left[(1+t_{1}t_{3})(1+t_{2}t_{3}): (1+t_{3}^{2})(1-t_{1}t_{2}): (1+t_{3}^{2})(t_{1}+t_{2}) \right]$$

And now introduce associated oriented_radii, which are

$$r_{0} \equiv 1$$

$$r_{1} \equiv \frac{(t_{1} - t_{2})(t_{1} - t_{3})}{(1 + t_{1}t_{2})(1 + t_{1}t_{3})}$$

$$r_{2} \equiv \frac{(t_{2} - t_{3})(t_{2} - t_{1})}{(1 + t_{1}t_{2})(1 + t_{2}t_{3})}$$

$$r_{3} \equiv \frac{(t_{3} - t_{1})(t_{3} - t_{2})}{(1 + t_{1}t_{3})(1 + t_{2}t_{3})}$$

We have made some deliberate choices here. The nine-point centre is

$$N \equiv [x_N, y_N] = \left[8w : u^2 - v^2 : 2uv\right]$$

where

$$v = t_1 + t_2 + t_3 + 3t_1t_2t_3$$
, $u = t_1t_2 + t_1t_3 + t_2t_3 + 3$ and
 $w = (t_2t_3 + 1)(t_1t_3 + 1)(t_1t_2 + 1)$.

Introduce its associated oriented_radius (noting the negative sign):

$$r_N \equiv -\frac{\left(t_1^2+1\right)\left(t_2^2+1\right)\left(t_3^2+1\right)}{8\left(t_1t_2+1\right)\left(t_1t_3+1\right)\left(t_2t_3+1\right)} \\ = -\frac{\left(t_1^2+1\right)\left(t_2^2+1\right)\left(t_3^2+1\right)}{8w} = 1 - \frac{v^2+u^2}{8w}.$$

We have here used the identity

$$u^{2} + v^{2} =$$

(t₁² + 1) (t₂² + 1) (t₃² + 1) + 8 (t₂t₃ + 1) (t₁t₃ + 1) (t₁t₂ + 1).

Now we are able to both state and prove Feuerbach's theorem in both a stronger and a more general setting: valid over a general field, and with an oriented aspect.

Theorem 7 (Oriented Feuerbach theorem) *The* oriented_circle $C_N \equiv [N, r_N]$ *is tangent to each of the oriented_circles* $C_0 \equiv [I_0, r_0], C_1 \equiv [I_1, r_1], C_2 \equiv [I_2, r_2]$ and $C_3 \equiv [I_3, r_3].$

Proof. We first establish the easier tangency of C_N and C_0 by first introducing the coordinates of N and r_N as above as

$$N = [x_N, y_N] = \left[\frac{u^2 - v^2}{8w}, \frac{uv}{4w}\right] \text{ and } r_N = 1 - \frac{u^2 + v^2}{8w}$$

Then the cyclographic condition for tangency of C_N and C_0 is $x_N^2 + y_N^2 - (r_N - 1)^2 = 0$ which becomes

$$\left(\frac{u^2 - v^2}{8w}\right)^2 + \left(\frac{2uv}{8w}\right)^2 = \left(\frac{v^2 + u^2}{8w}\right)^2$$

which is automatic.

Now let's consider the more challenging tangency of C_N and C_1 , which amounts to the condition

$$\left(x_N - \frac{\left(1 + t_1^2\right) \left(1 - t_2 t_3\right)}{\left(1 + t_1 t_2\right) \left(1 + t_1 t_3\right)} \right)^2 + \left(y_N - \frac{\left(1 + t_1^2\right) \left(t_2 + t_3\right)}{\left(1 + t_1 t_2\right) \left(1 + t_1 t_3\right)} \right)^2$$

$$= \left(r_N - \frac{\left(t_1 - t_2\right) \left(t_1 - t_3\right)}{\left(1 + t_1 t_2\right) \left(1 + t_1 t_3\right)} \right)^2$$

which we can write more concisely as

$$\left(\frac{u^2 - v^2}{8w} - \frac{\left(1 + t_1^2\right)\left(1 - t_2^2 t_3^2\right)}{w}\right)^2 + \left(\frac{2uv}{8w} - \frac{\left(1 + t_1^2\right)\left(t_2 + t_3\right)\left(1 + t_2 t_3\right)}{w}\right)^2 = \left(1 - \frac{u^2 + v^2}{8w} - \frac{\left(t_1 - t_2\right)\left(t_1 - t_3\right)\left(1 + t_2 t_3\right)}{w}\right)^2.$$

Now after establishing common denominators, the numerators of the terms in this equation will be

$$A \equiv u^{2} - v^{2} - 8(1 + t_{1}^{2})(1 - t_{2}^{2}t_{3}^{2})$$

= $-t_{1}^{2}t_{2}^{2}t_{3}^{2} + t_{1}^{2}t_{2}^{2} + t_{1}^{2}t_{3}^{2} + 9t_{2}^{2}t_{3}^{2} - 4t_{1}^{2}t_{2}t_{3} - 4t_{1}t_{2}^{2}t_{3}$
 $-4t_{1}t_{2}t_{3}^{2} + 4t_{1}t_{2} + 4t_{1}t_{3} + 4t_{2}t_{3} - 9t_{1}^{2} - t_{2}^{2} - t_{3}^{2} + 1$

$$B \equiv 2uv - 8 \left(1 + t_1^2\right) \left(t_2 + t_3\right) \left(1 + t_2 t_3\right)$$

= $-2t_1^2 t_2^2 t_3 - 2t_1^2 t_2 t_3^2 + 6t_1 t_2^2 t_3^2 - 6t_1^2 t_3 - 6t_2^2 t_3 - 6t_2 t_3^2$
 $-6t_1^2 t_2 + 2t_1 t_2^2 + 2t_1 t_3^2 + 24t_1 t_2 t_3 + 6t_1 - 2t_2 - 2t_3$

and

$$C \equiv 8w - (u^2 + v^2) - 8(t_1 - t_2)(t_1 - t_3)(1 + t_2t_3)$$

= $-t_1^2 t_2^2 t_3^2 - t_1^2 t_2^2 - t_1^2 t_3^2 - 9t_2^2 t_3^2 - 8t_1^2 t_2 t_3 + 8t_1 t_2^2 t_3$
+ $8t_1 t_2 t_3^2 + 8t_1 t_2 + 8t_1 t_3 - 8t_2 t_3 - 9t_1^2 - t_2^2 - t_3^2 - 1$

and then $A^2 + B^2 = C^2$ is a valid Pythagorean identity in t_1, t_2 and t_3 . This establishes the tangency of C_N and C_1 , and the cases of C_N and C_2 , and C_N and C_3 , follow symmetrically.

Feuerbach's theorem is then a consequence because we know tangency of oriented_circles implies tangency of the associated circles. Note that our proof extends the oriented version, which is more powerful than the original, to general fields.

We may now well ask: what is the geometric meaning of "oriented_circles" over more general fields, for example the complex numbers, or a finite field? We do not have a good answer to this interesting question at this point. It appears that our understanding of universal geometry is still in early stages!

11 Points of tangency

We can now use the Oriented tangency theorem to find the points of tangency of the incircles with the nine-point circle to be:

$$J_{0} \equiv C_{0}C_{N} = \frac{r_{N}[x_{0}, y_{0}] - r_{0}[x_{N}, y_{N}]}{r_{N} - r_{0}}$$

$$= [r_{0} - r_{N} : r_{0}x_{N} - x_{0}r_{N} : r_{0}y_{N} - y_{0}r_{N}]$$

$$J_{1} \equiv C_{1}C_{N} = \frac{r_{N}[x_{1}, y_{1}] - r_{1}[x_{N}, y_{N}]}{r_{N} - r_{1}}$$

$$= [r_{1} - r_{N} : r_{1}x_{N} - x_{1}r_{N} : r_{1}y_{N} - y_{1}r_{N}]$$

$$J_{2} \equiv C_{2}C_{N} = \frac{r_{N}[x_{2}, y_{2}] - r_{2}[x_{N}, y_{N}]}{r_{N} - r_{2}}$$

$$= [r_{2} - r_{N} : r_{2}x_{N} - x_{2}r_{N} : r_{2}y_{N} - y_{2}r_{N}]$$

$$J_{3} \equiv C_{3}C_{N} = \frac{r_{N}[x_{3}, y_{3}] - r_{3}[x_{N}, y_{N}]}{r_{N} - r_{3}}$$

$$= [r_{3} - r_{N} : r_{3}x_{N} - x_{3}r_{N} : r_{3}y_{N} - y_{3}r_{N}].$$

12 Additional features in finite fields

Our geometrical intuition has been directed for thousands of years by physical constructions on flat surfaces. But with the view of Rational trigonometry, we see that metrical theorems can be investigated even over finite fields. The formulas that we have so far derived In our set-up of the Feuerbach theorem are applicable to arbitrary fields, but there are additional aspects that appear which we can illustrate in the simple case of a finite prime field \mathbb{F}_p .

We assumed that the three original t values that determined the original points on the unit circle were distinct, that no t value satisfies $t^2 = -1$, and that the product of any two t values is not equal to -1. In addition we usually separate the case where the oriented_radius of the nine-point circle is equal to the oriented_radius of any of the incircles, for example if we start with an equilateral triangle.

It is of some interest to investigate cases over a finite field when such assumptions are not necessarily holding. This leads to situations where aspects of the Feuerbach framework hold, but others do not which will be unfamiliar to the student of Euclidean geometry over the real or rational numbers. Finite field geometry is a rich ground which holds the promise to enrich the subject and strengthen ties to number theory and combinatorics.

To focus the discussion, we consider the case of working in the field \mathbb{F}_{17} . When one of our formulas does not apply, because a denominator in the affine expression is 0, we substitute a blank -.

13 Examples in \mathbb{F}_{17}



Figure 9: *Example* 1 with t values of 3, 7, 15 in \mathbb{F}_{17}

With these *t*-values, letting $t_1 = 3$, $t_2 = 7$ and $t_3 = 15$, we obtain the following:

$$\begin{array}{ll} I_0 = [0,0] & r_0 = 1 \\ A_1 = [8,14] & I_1 = [11,15] & r_1 = 11 \\ A_2 = [2,10] & I_2 = [9,11] & r_2 = 12 \\ A_3 = [13,2] & I_3 = [5,6] & r_3 = 2 \\ & N = [2,9] & r_N = 1 \end{array} \begin{array}{ll} J_0 = - \\ J_1 = [13,5] \\ J_2 = [6,15] \\ J_3 = [16,12] \end{array}$$

Since $r_0 = r_N$, the expressions for the coordinates of the corresponding point of tangency J_0 will be undefined.

Remark 5 Such a case does not arise when working in the rational field. This is because only one of the incircles has a radii with the same sign as that of the nine-point circle, and that incircle can only have the same radius as the nine-point circle if their centres coincide (which only occurs when the starting triangle is equilateral).



13.2 Example 2: *t***-values of** {2,7,11}

Figure 10: *Example 2 with t values of* 2, 7, 11 *in* \mathbb{F}_{17}

With these *t*-values, letting $t_1 = 2$, $t_2 = 7$ and $t_3 = 11$, we obtain the following:

Since the radius of the nine-point circle is not the same as the radius of any of the incircles this time, none of the *t*-values are a square root of -1 and no product of two *t*values is -1, we can show that the nine-point circle is tangent to the four incircles of the triangle $\overline{A_1A_2A_3}$. However, we do run into the issue of J_1 and J_3 coinciding.

Remark 6 While a proof is not attempted here, it is suspected that there does not exist a case where all the tangent points exist and are distinct in \mathbb{F}_{17} , based on extensive (but not exhaustive) testing of different combinations of t-values in \mathbb{F}_{17} . This is most likely a result of \mathbb{F}_{17} being "too small" for all J points to be distinct.



13.3 Example 3: *t***-values of** {4,7,15}

Figure 11: *Example* 3 with t values of 4,7,15 in \mathbb{F}_{17}

With these *t*-values, we have exactly one square root of -1 and no products for -1. Letting $t_1 = 4$, $t_2 = 7$ and $t_3 = 15$, we obtain the following:

	$I_0 = [0, 0]$	$r_0 = 1$	$I_{2} = [2, 2]$
$A_1 = [8, 14]$	$I_1 = [0, 0]$	$r_1 = 16$	$J_0 = [3, 3]$
$A_2 = [6, 7]$	$I_2 = [3, 12]$	$r_2 = 8$	$J_1 = [5, 5]$
$A_3 = [2, 8]$	$I_3 = [3, 12]$	$r_{3} = 9$	$J_2 = [3, 3]$
5 []]	N = [3, 3]	$r_N = 0$	$J_3 = [3,3]$

As expected, since we have a square root of -1 in our *t*-values, we get overlapping incentres and a nine-point radius of zero, resulting in all the points of tangency concentrating at the nine-point centre. These overlapping coordinates are a result of the factor $(1 + t_1^2)$ going to zero since t_1 is a square root of -1 in our field. This means that I_1 has both coordinates go to zero while I_2 and I_3 simplify to get the same coordinates. This case results in the nine-point circle becoming a null_circle. Notably, since we do not have any products of *t*-values giving -1, we are still able to find coordinates for all the significant points.

14 A complete finite field example in \mathbb{F}_{23}

If we choose our *t*-values in \mathbb{F}_{23} to be $t_1 = 2$, $t_2 = 5$ and $t_3 = 13$, we obtain the following:

	$I_0 = [0, 0]$	$r_0 = 1$	L = [10, 4]
$A_1 = [6, 17]$	$I_1 = [22, 1]$	$r_1 = 18$	$J_0 = [10, 4]$ L = [3, 4]
$A_2 = [11, 21]$	$I_2 = [19, 7]$	$r_2 = 7$	$J_1 = [3, 4]$ $L_1 = [1, 6]$
$A_3 = [18, 9]$	$I_3 = [1, 12]$	$r_3 = 8$	$J_2 = [1, 0]$
	N = [18, 21]	$r_N = 13$	$J_3 = [0, 10]$

Since we can find a nice example in \mathbb{F}_{23} without any significant points overlapping, we know that the "smallest" prime finite field for the Feuerbach theorem (which we can think of as the finite field of least order such that all significant points can be distinct with an appropriate choice of *t*-values) will be less that or equal to \mathbb{F}_{23} . It is perhaps interesting to consider this kind of question, which has both a geometrical, a combinatorial, and a number theoretic aspect simultaneously.



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