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ABSTRACT

The moon tilt illusion is the startling discrepancy between the direction of the light beam illuminating the moon and the direction of the sun. The illusion arises because the observer erroneously expects a light ray between sun and moon to appear as a line of constant slope according to the positions of the sun and the moon in the sky. This expectation does not correspond to the reality that observation by direct vision or a camera is according to perspective projection, for which the observed slope of a straight line in three-dimensional space changes according to the direction of observation. Comparing the observed and expected directions of incoming light at the moon, we derive a quantitative expression for the magnitude of the moon tilt illusion that can be applied to all configurations of sun and moon in the sky.

Key words: moon tilt, perspective projection, illusion

MSC 2000: 51N05

Iluzija nagiba mjeseca

SAŽETAK

Iluzija nagiba mjeseca zapanjujući je raskorak između svjetlosne zrake koja osvjetljava mjesec i smjera sunca. Ona se povećava, jer promatrač pogršno očekuje da zraka svjetla između sunca i mjeseca bude pravac konstantnog koeficijenta smjera s obzirom na položaj sunca i mjeseca na nebu. Ovakvo očekivanje ne odgovara stvarnosti kod koje je promatranje s direktnom osi pogleda, ili s kamerom, u skladu s perspektivom (centralnim projiciranjem) za koju se promatrani koeficijent smjera pravca u trodimenzionalnom prostoru mijenja s obzirom na os pogleda. Uspoređujući promatrane i očekivane smjerove zrake usmjerene na mjesec, izvodimo kvantitativan izraz za veličinu iluzije nagiba mjeseca koji se može primijeniti na sve položaje sunca i mjeseca na nebu.

Ključne riječi: nagib mjeseca, perspektiva, iluzija

1 The Nature of the Illusion

The photograph in Figure 1 provides an example of the moon tilt illusion. The moon's illumination is observed to be coming from above, even though the moon is high in the sky and the sun had set in the west one hour before this photo was taken. The moon is 45° above the horizon in the southeast, 80% illuminated by light from the sun striking the moon at an angle of 17° above the horizontal, as shown by the arrow drawn on the photograph. Our intuition (i.e., the incorrect perception that creates the illusion) is that given the relative positions of the sun and the moon, the light from the sun should be striking the moon from below. The moon tilt illusion is thus the perceived discrepancy between the angle of illumination of the moon that we observe (and can capture photographically with a camera pointed at the moon) and the angle that we expect, based on the known locations of the sun and the moon in the sky.

Figure 1: *Photograph of the moon tilt illusion. Picture taken one hour after sunset, with the moon in the southeast and sun already set in the west. Camera pointed upwards 45° from the horizon with bottom of camera parallel to the horizon.*

Rather surprisingly, little mention of the moon tilt illusion (much less a detailed explanation of why it occurs) can be found in astronomy books. Minnaert [1] gives a passing reference: "...the line connecting the horns of the moon, between its first quarter and full moon, for instance, does not appear to be at all perpendicular to the direction from sun to moon; we apparently think of this direction as being a curved line. Fix this direction by stretching a piece of string taut in front of your eye; however unlikely it may have seemed to you at first you will now perceive that the condition of perpendicularity is satisfied". A photograph taken by Lodriguss [2] shows a waxing moon and the setting sun in the same photo. The angle of 23° between the direction of the moon's illumination and the direction of the sun provides a striking illustration of the moon tilt illusion. An article by Schölkopf [3] documents the illusion in an experiment involving 14 subjects by having them indicate their expectation of how the moon's illumination should be oriented with respect to the position of the (visible) sun. He reports that an average discrepancy of 12° is perceived by the subjects between the observable versus expected orientation of the moon's bright limb. Schott's website entitled " 'Falsche' Mondneigung" ('False' Moon Tilt) [4] is devoted to the moon tilt illusion, and features illustrations and useful links. A paper by Glaeser and Schott [5], approaching the phenomenon via the principles of photography, shows that the magnitude of the illusion could in theory be measured through comparison of a close-up shot of the moon with a photograph containing both sun and moon, with the camera directed in a specified direction between them (although no equations are given). However, as they point out, in practice it is not feasible since even a wide-angle lens cannot capture both sun and moon in a photo with azimuth differences for which the illusion can be most clearly observed (between 90° and 180°). Berry [6] proposed a zenith-centered stereoscopic projection of the celestial sphere onto a flat surface with the moon tilt illusion defined as the angle between the projected great circle and a straight moon-sun line drawn on the flat surface "mimicking how we might see the sky when lying on our back looking up". Apparently there still persists a lack of consensus in the literature about the explanation of the moon tilt illusion and disagreement about the best way to measure it.

In this paper, our aim is to derive a quantitative expression for the magnitude of the moon tilt illusion experienced by an upright observer that is straightforward to apply to all configurations of sun and moon in the sky. We model the viewer's expectation of the direction of incoming light using vector geometry, which is appropriate for treating 3D straight lines such as the sun-moon light ray.

2 System of Coordinates and Definitions

Our analysis of the moon tilt illusion is based upon the known locations of the sun and moon in the sky. We adopt topocentric coordinates (instead of right ascension and declination) for the sun and moon, denoted by azimuth (ϕ) [1] and altitude (η). The altitude η is the angle between the sun (or moon) and the observer's local horizon. Recognizing that the altitude angle (η) is the complement of the polar angle (θ), we may rewrite azimuth and altitude (ϕ,η) as spherical coordinates (ϕ,θ). Spherical coordinates for the sun and moon are converted to Cartesian coordinates to allow vector manipulations such as dot and cross products.

2.1 Moon Pointer and Moon Tilt Angle

The moon pointer is defined as the vector *CP* in Figure 2, where *C* is the center of the moon and the vector *CP* has the observed slope of the moon-sun line at point *C*. The demarcation between illuminated and dark portions of the moon is called the terminator. Line *AB* connects the two "horns" of the terminator through the moon's center *C*. The moon pointer *CP* is the perpendicular bisector of line *AB*.

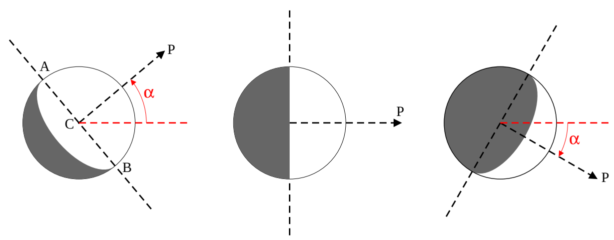

Figure 2: *Definition of moon pointer with* α *angle. From left to right,* $\alpha = 40^\circ$ *(75% illumination),* $\alpha = 0^\circ$ *(50% illumination),* $\alpha = -30^\circ$ *(25% illumination).*

The moon tilt α is the signed angle of the moon's pointer with the horizontal, positive upward and negative downward. An equation for calculating this angle from the locations of the sun and moon is given in Section 4. Using the construction in Figure 2, the angle α may be found experimentally by taking a picture of the moon with the optical axis of the camera pointed at the moon and the bottom of the camera oriented horizontally. For example, for the photo in Figure 1, $\alpha = 17^\circ$.

[1] In physics, the azimuthal angle is defined as positive for counter-clockwise (CCW) rotation from due north (*x*-direction), with the Cartesian coordinates satisfying the right-hand rule. In navigation, azimuth is defined as positive in the clockwise (CW) direction. We will use the CCW notation for calculations but revert to the more familiar navigational CW direction for the presentation of results in Section 5.

3 Cause of Moon Tilt Illusion

When we view the light ray at the moon, which is the only place we can photograph its direction, the slope with the horizontal (α) that we observe is exactly what one would expect from the principles of perspective projection that form the basis of human vision or photography.

The cause of the moon tilt illusion is simply that the observer is not taking into account the rules of perspective that dictate that the observed slope of the light ray will change when he turns his head to observe the moon and sun. This perceptual disconnect occurs because the observer cannot see the light ray itself, but only its starting position at the sun and the angle at which it strikes the moon. Without any other visual cues to provide more information, he is perceptually unable to envision how the slope of a visible line overhead changes with viewing angle due to perspective projection.

The changing-slope effect due to perspective projection is apparent in a video [7] which scans a long, straight string of lights along the Thames near London's Tower Bridge. All of the lights are at roughly the same distance from the ground. The moving video camera shows the observed slope of the string of lights varying continuously with camera motion: first sloping upwards from the ground on the left, then with zero slope in the middle, and finally sloping downwards to the right. For the moon illusion, the path of the light ray is invisible and we can observe its slope only at one end. If the sun-moon light ray were visible, we would see a straight line of varying slope just like the video and the illusion would vanish.

Knowing that light travels in straight lines in space but 'forgetting' that slope changes as the head turns along a line, the observer expects that when he scans from sun to moon he would see a straight line of constant slope, even though his head has moved. On the basis of this explanation, we calculate the observed angle of the moon tilt (α) and compare it with the expected angle (β) of the moon tilt based upon the known positions of the moon and the sun in the sky. The difference between the observed and expected angles (δ) quantifies the moon tilt illusion.

4 Observed and Expected Slope of Incoming Light

4.1 Observed Moon Tilt (α)

The principles of 2D perspective projection govern the viewing of a 3D line between two objects overhead by the human eye or a camera. The light ray from the sun that illuminates the moon is invisible in the sky, but we can observe its slope with the horizontal where it intersects the moon from the direction of the moon's illumination. The derivation of the tilt angle (α) between the observed incoming direction of light and the horizontal is straightforward but lengthy and is not given here because the equation for the observed tilt from the vertical (χ) is already well known.

Let ϕ_m be the azimuth of the moon and ϕ_s the azimuth of the sun; let $\Delta\phi = |\phi_s - \phi_m|$; let η_m be the altitude of the moon and η_s the altitude of the sun. The angle of the moon's tilt from the horizontal may be derived from an equation for the position angle of the moon's bright limb [8], [9]:

$$\tan\chi = \frac{\cos\eta_s \sin\Delta\phi}{\cos\eta_m \sin\eta_s - \cos\eta_s \sin\eta_m \cos\Delta\phi} \tag{1}$$

χ in this equation is called the position angle of the midpoint of the moon's bright limb measured from the north point of the disk. This may be written:

$$\tan\chi = \frac{\sin\Delta\phi}{\cos\eta_m \tan\eta_s - \sin\eta_m \cos\Delta\phi} \tag{2}$$

The desired angle with the horizontal (α) is the complement of χ so:

$$\boxed{\tan\alpha = \frac{\cos\eta_m \tan\eta_s - \sin\eta_m \cos\Delta\phi}{\sin\Delta\phi}} \tag{3}$$

4.2 Expected Moon Tilt (β)

An observer bases his expectation of the incoming direction of light at the moon on his knowledge of the 3D positions of the sun and moon as they appear to him in the sky, i.e., according to their height difference and horizontal distance apart. For example, in Figure 1, the upright viewer sees the light illuminating the moon from above, but he expects the light to come from below the horizontal, since the moon is higher than the sun. In the sky, there is an absence of visual cues by which the viewer could evaluate the distance of an object; thus the direction of light from sun to moon is assessed from their relative altitudes and azimuths as though sun and moon were equidistant [2] from the viewer. We represent this expected direction of light as a 3D vector **v**, given by the difference of the unit vectors from the observer to the sun ($\hat{\mathbf{s}}$) and the moon ($\hat{\mathbf{m}}$):

$$\mathbf{v} = \hat{\mathbf{s}} - \hat{\mathbf{m}} \tag{4}$$

[2] This assumption is a natural consequence of the 2D perspective-projection basis of human vision. Since objects are projected bigger or smaller when closer or farther away, objects of apparent equal size will be judged as equidistant, in the absence of additional visual cues such as clarity or brightness. We note that even if observers take into account that the sun is much farther away from the earth than the moon, they will still experience an illusion by not considering perspective distortion. For example, for a setting sun they would expect the moon (in any position) to be illuminated from the horizontal, leading to an illusion equal to the observed α tilt.

The observer naively expects to view $\mathbf{v}$ without any perspective distortion. If the observer faced the vertical plane containing the sun and moon directly, the slope of $\mathbf{v}$ in this plane is simply the height difference of sun and moon divided by the horizontal distance between them. However, the observer must face the moon for his observation of the illusion. (If our eyes deviate from the azimuth of the moon, the *observed* angle α of the moon tilt would change). With knowledge of the position of the sun and the moon relative to his orientation facing the moon, the observer expects his view of $\mathbf{v}$ as it strikes the moon to be determined by this orientation. This is simply the orthogonal projection of $\mathbf{v}$ on the vertical plane at the moon. The vector $\mathbf{n}$ normal to the vertical projection plane is:

$$\mathbf{n} = m_x \hat{\mathbf{x}} + m_y \hat{\mathbf{y}} \tag{5}$$

where m_x and m_y are the x and y components of the unit $\hat{\mathbf{m}}$ vector. The unit normal vector is:

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{\mathbf{n}}{\sqrt{m_x^2 + m_y^2}} \tag{6}$$

The projection $\mathbf{v_p}$ on the vertical plane is:

$$\mathbf{v_p} = \mathbf{v} - \mathbf{v_n} \tag{7}$$

where $\mathbf{v_n}$ is perpendicular to the vertical plane with:

$$\mathbf{v_n} = (v \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \tag{8}$$

The horizontal unit vector lying in the vertical plane is $\hat{\mathbf{h}} = \hat{\mathbf{n}} \times \hat{\mathbf{z}}$. Since the tangent of an angle between two vectors is equal to the ratio of the cross and dot products, it follows that the desired angle β between $\mathbf{v_p}$ and $\hat{\mathbf{h}}$ is given by:

$$\tan\beta = \frac{|\mathbf{v_p} \times (\hat{\mathbf{n}} \times \hat{\mathbf{z}})|}{\mathbf{v_p} \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{z}})} \tag{9}$$

This formula for β was chosen to avoid having to normalize $\mathbf{v_p}$. As shown in Appendix A, Eq. (9) may be written in terms of the Cartesian components of the unit moon vector $\hat{\mathbf{m}}$ and the unit sun vector $\hat{\mathbf{s}}$:

$$\tan\beta = \frac{|s_z - m_z| \sqrt{m_x^2 + m_y^2}}{s_x m_y - s_y m_x} \tag{10}$$

As shown in Appendix B, conversion of the Cartesian components of the moon and sun vectors to angles yields:

$$\tan\beta = -\frac{|\sin\eta_m - \sin\eta_s|}{\cos\eta_s \sin(\Delta\phi)} \tag{11}$$

This equation for β applies to waxing and waning moons in both hemispheres. The nuisance of insuring that the angle is in the right quadrant can be avoided by writing Eq. (11) in the form:

$$\boxed{\tan\beta = -\frac{(\sin\eta_m - \sin\eta_s)}{\cos\eta_s \sin(\Delta\phi)}} \tag{12}$$

where it is understood that $\Delta\phi = |\phi_s - \phi_m|$ and $|\Delta\phi| \leq 180°$. The sign convention for the β pointer is the same as for the α pointer: a positive value for β corresponds to a direction upward from the horizontal and a negative value corresponds to a direction downward from the horizontal, pointing east or west depending on the location of the sun. Typically the altitude of the moon is higher than that of the sun and β is negative.

4.3 Magnitude of Moon Tilt Illusion

The moon tilt illusion is defined as the difference (δ) between the slope angle of the observed moon-sun line (α) and slope angle of the expected moon-sun line (β):

$$\boxed{\delta = \alpha - \beta} \tag{13}$$

We may apply this equation to the photograph in Figure 1. The locations of the sun and moon are the altitudes $\eta_m = 45°$, $\eta_s = -15°$, and an azimuth difference $\Delta\phi = 128°$. The illumination of the moon in the photograph is 80%, which agrees with the calculated value [9]. From Eq. (3), $\alpha = 17°$, which is confirmed by the photograph. Eq. (12) gives $\beta = -52°$ and from Eq. (13), $\delta = 17 - (-52) = 69°$, consistent with the viewer's expectation that the incoming light should be strongly angled from below the horizontal.

5 Discussion

We have presented a method for calculating the magnitude of the moon tilt illusion as the degree difference (δ) between the observed direction of the incoming light and the expected direction of incoming light. The model identifies sun/moon configurations ranging from no illusion (when the sun and moon are either close together or both on the horizon) to the strongest illusion (at the equator, when the moon is above the horizon and the azimuthal difference between moon and sun is 180°).

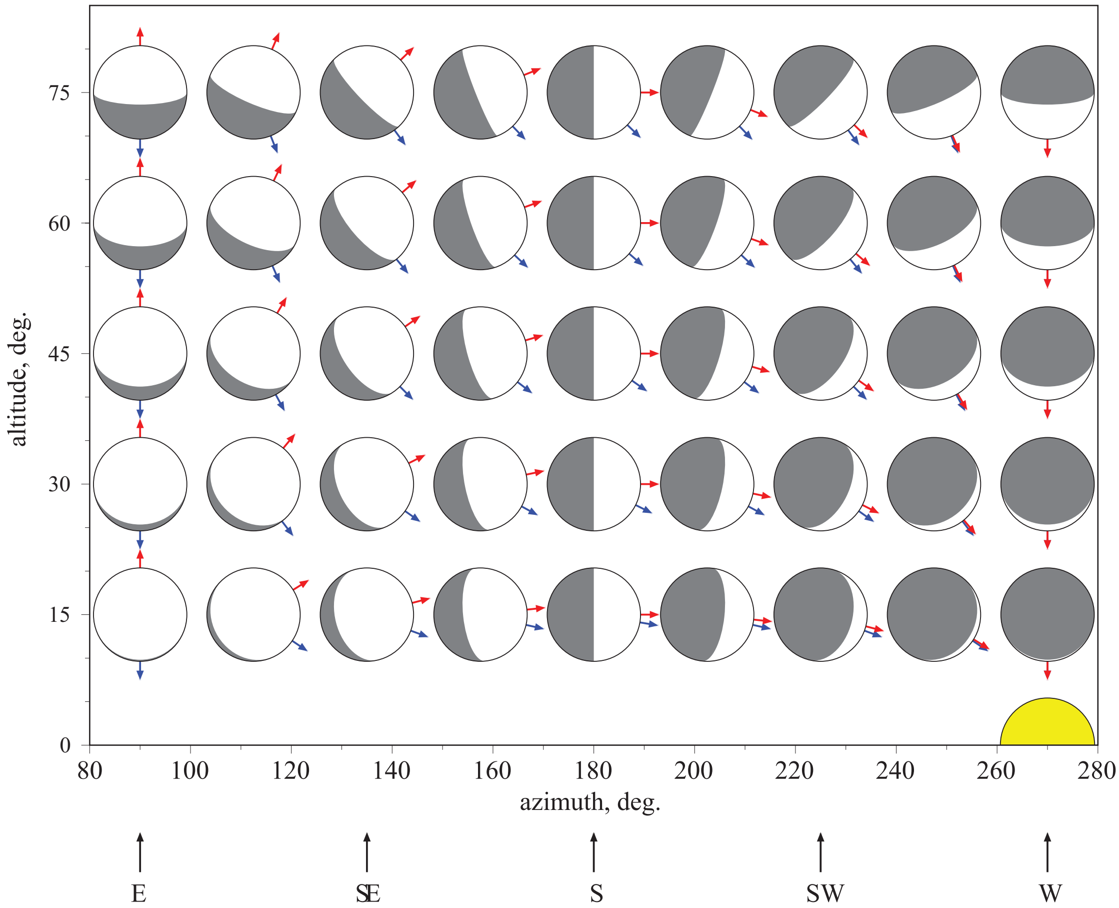

Figure 3: *Moon tilt illusion for waxing phases in northern hemisphere. Sun is setting due west. Red line is observed slope and blue line is expected slope of moon-sun line. Azimuth measured CW from north.*

We focus on cases where the sun and the moon are both visible in the sky, as this allows the observer to evaluate the positions of each. The moon is visible at twilight. Shown in Figure 3 is a chart for a waxing moon with a setting sun in the northern hemisphere. The magnitude of the moon tilt illusion is δ, the degree difference between the observed (red) arrow and the expected (blue) arrow. A set of four charts for waxing or waning moon in the northern and southern hemisphere could be constructed to cover all similar situations. Whether or not a particular configuration is visible depends on the latitude of the observer. For example in Figure 3 for a waxing moon, the horizontal "boat" crescent moon at high altitude in the west is observed near the equator but not in temperate zones. The chart is for the sun setting due west, which occurs at all latitudes during the spring and fall equinoxes. Since the moon tilt (α or β) depends on the *difference* of azimuths ($\Delta\phi$), corrections can be made for the sun setting at azimuths other than 270° by translating the entire set of images horizontally to the right or left.

The limits of δ, the magnitude of the illusion, are 0° for a new moon and 180° for a full moon. Near new moon, the δ angle is too small to be visible with the naked eye. For the crescent moon with under 90° azimuth difference between sun and moon, the magnitude of the illusion (δ) is small and the illusion is unimpressive, since the observed (red) and expected (blue) light directions are both below the horizontal. At half moon (sun-moon azimuth difference of 90°, moon at 180° on the chart), the discrepancy between the observed and expected directions becomes very noticeable since the observed light direction (red) is horizontal but the expected light direction(blue) is from below. For the gibbous moon at sunset or sunrise with azimuth difference greater than 90°, the illusion becomes striking since the moon is unambiguously lit from above the horizontal and the position of the sun is below the horizontal. The illusion is particularly impressive at sunset when the gibbous moon is at high altitude in the southwest or at sunrise when the gibbous moon is at high altitude in the southeast (both cases for the northern hemisphere). If illumination exceeds about 90 percent, the direction of the red moon pointer may become difficult for the observer to discern.

In addition to the setting sun configuration in Figure 3, another interesting case occurs when the sun and moon are at the same (non-zero) altitude. Although the moon is lit from above the horizontal, the observer would expect to see the light travel horizontally from the sun to the moon. Our model gives $\beta = 0$ and $\delta = \alpha$.

In Figure 3, we note that for a particular elevation and setting sun at 270°, the expected beta is the same at moon azimuths of (180 + x) and (180 - x) degrees. For example, at 60° elevation, the angle of the blue sun-moon arrow is $-50.8°$ at moon azimuths of 135° and 225°. Moving from right to left at fixed altitude on Figure 3, the blue arrow moves CCW at first but switches to a CW movement after passing the 180° azimuth. Looking at Figure 3, instead of symmetry about 180°, one might expect the blue arrow indicating the direction of the sun to continue turning CCW when moving right to left at constant altitude. However, Figure 3 is a 2D representation of the 3D position of the sun relative to the moon. As the moon-sun azimuth difference increases beyond 90° from right to left at constant altitude, the sun moves *behind* the observer, causing the projected slope of the moon-sun vector to move in a CW direction. Facing the elevated moon and with the setting sun directly behind him, the observer would expect the light to illuminate the moon from below. The actual illumination is directly from above. Thus on the equator at sunset and particularly at high moon altitudes for which the moon is lit from above, observers experience a spectacular moon tilt illusion of magnitude 180°.

Eq. (12) for the calculation of the expected angle β depends upon the locations (azimuth and altitude) of the moon and sun in the sky. β is the angle of the sun-moon vector with the horizontal as projected upon a vertical plane perpendicular to the azimuth of the moon. Consider some of the limits for β which are independent of the projection plane used for its calculation and depend only on the geometry of the configuration. When the moon and sun have the same altitude, $\beta = 0$. When the moon and sun have the same azimuth with the moon above the sun, $\beta = -90°$. In the limit as the moon approaches the sun (new moon), $\beta = \alpha$ for any angle of approach. When the moon has a non-zero altitude and the moon-sun azimuth difference is 180°, the moon-sun vector strikes the moon from below so that $\beta = -90°$. Values of β from Eq. (12) conform to these limits.

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Notation

$\mathbf{m}$	moon vector OM
$\mathbf{n}$	normal vector to vertical plane
$\mathbf{s}$	sun vector OS
$\mathbf{x,y,z}$	Cartesian coordinate vectors
$\mathbf{v}$	moon-sun direction, Eq. (4)
$\mathbf{v_n}$	projection of $\mathbf{v}$ on $\mathbf{n}$
$\mathbf{v_p}$	projection of $\mathbf{v}$ on vertical plane
α	observed angle of moon pointer with horizontal
β	expected angle of moon pointer with horizontal
δ	difference of observed and expected angles of moon pointer with horizontal
η	altitude of moon or sun
θ	polar angle of moon or sun in spherical coordinates
ϕ	azimuth of moon or sun
χ	position angle of moon's bright limb
$\hat{}$	"hat" symbol for unit vector

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Appendix A. β from vector components

Reduce the vector equation:

$$\tan\beta = \frac{|\mathbf{v_p} \times (\hat{\mathbf{n}} \times \hat{\mathbf{z}})|}{\mathbf{v_p} \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{z}})}$$

to component form.

$$\mathbf{v_p} = \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$

$$\mathbf{v_p} \times (\hat{\mathbf{n}} \times \hat{\mathbf{z}}) = (\mathbf{v_p} \cdot \hat{\mathbf{z}})\hat{\mathbf{n}} - (\mathbf{v_p} \cdot \hat{\mathbf{n}})\hat{\mathbf{z}}$$

But $\mathbf{v_p}$ is perpendicular to $\hat{\mathbf{n}}$ so:

$$\begin{aligned}\mathbf{v_p} \times (\hat{\mathbf{n}} \times \hat{\mathbf{z}}) &= (\mathbf{v_p} \cdot \hat{\mathbf{z}})\hat{\mathbf{n}} \\ &= (\mathbf{v} \cdot \hat{\mathbf{z}})\hat{\mathbf{n}} - (\mathbf{v} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \hat{\mathbf{z}})\hat{\mathbf{n}} \\ &= (\mathbf{v} \cdot \hat{\mathbf{z}})\hat{\mathbf{n}}\end{aligned}$$

because $\hat{\mathbf{n}}$ is perpendicular to $\hat{\mathbf{z}}$.

$$\mathbf{v} = \hat{\mathbf{s}} - \hat{\mathbf{m}} = (s_x - m_x)\hat{\mathbf{x}} + (s_y - m_y)\hat{\mathbf{y}} + (s_z - m_z)\hat{\mathbf{z}}$$

$$|\mathbf{v_p} \times (\hat{\mathbf{n}} \times \hat{\mathbf{z}})| = |\mathbf{v} \cdot \hat{\mathbf{z}}| = |v_z| = |s_z - m_z|$$

The denominator is the scalar triple product:

$$\begin{aligned}\mathbf{v_p} \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{z}}) &= \mathbf{v} \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{z}}) - (\hat{\mathbf{v}} \cdot \hat{\mathbf{n}})[\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{z}})] \\ &= \mathbf{v} \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{z}})\end{aligned}$$

because $\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{z}}) = (\hat{\mathbf{n}} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{z}} = 0$.

$$\mathbf{n} \times \hat{\mathbf{z}} = (m_x, m_y, 0) \times (0,0,1) = (m_y, -m_x, 0)$$

$$\begin{aligned}\mathbf{v} \cdot (\mathbf{n} \times \hat{\mathbf{z}}) &= [(s_x - m_x), (s_y - m_y), (s_z - m_z)] \cdot (m_y, -m_x, 0) \\ &= s_x m_y - s_y m_x\end{aligned}$$

In terms of the normalized vector $\hat{\mathbf{n}}$:

$$\mathbf{v} \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{z}}) = \frac{s_x m_y - s_y m_x}{\sqrt{m_x^2 + m_y^2}}$$

Substituting results for the numerator and denominator of $\tan\beta$:

$$\tan\beta = \frac{|s_z - m_z| \sqrt{m_x^2 + m_y^2}}{s_x m_y - s_y m_x} \qquad \square$$

Appendix B. β from altitude and azimuth angles

Convert the component formulation for β to altitude and azimuth angles of the sun and the moon:

$$\tan\beta = \frac{|s_z - m_z| \sqrt{m_x^2 + m_y^2}}{s_x m_y - s_y m_x}$$

Cartesian coordinates of the observer-moon and observer-sun unit vectors are:

$$\begin{aligned}\hat{\mathbf{m}} &= m_x\hat{\mathbf{x}} + m_y\hat{\mathbf{y}} + m_z\hat{\mathbf{z}} \\ \hat{\mathbf{s}} &= s_x\hat{\mathbf{x}} + s_y\hat{\mathbf{y}} + s_z\hat{\mathbf{z}}\end{aligned}$$

In terms of altitudes and azimuths:

$$\begin{aligned}&m_x = \cos\eta_m \cos\phi_m; \;\; m_y = \cos\eta_m \sin\phi_m; \;\; m_z = \sin\eta_m \\ &s_x = \cos\eta_s \cos\phi_s; \;\;\; s_y = \cos\eta_s \sin\phi_s; \;\;\; s_z = \sin\eta_s\end{aligned}$$

so:

$$m_x^2 + m_y^2 = \cos^2\eta_m \cos^2\phi_m + \cos^2\eta_m \sin^2\phi_m = \cos^2\eta_m$$

$$\sqrt{m_x^2 + m_y^2} = \cos\eta_m$$

$$(s_z - m_z) = \sin\eta_s - \sin\eta_m$$

$$\begin{aligned}s_x m_y - s_y m_x &= \cos\eta_s \cos\phi_s \cos\eta_m \sin\phi_m \\ &\quad - \cos\eta_s \sin\phi_s \cos\eta_m \cos\phi_m \\ &= \cos\eta_s \cos\eta_m (\sin\phi_m \cos\phi_s - \cos\phi_m \sin\phi_s) \\ &= \cos\eta_s \cos\eta_m \sin(\Delta\phi)\end{aligned}$$

where $\Delta\phi = (\phi_m - \phi_s)$.

$$\begin{aligned}\tan\beta = -\frac{|s_z - m_z| \sqrt{m_x^2 + m_y^2}}{s_x m_y - s_y m_x} &= -\frac{\cos\eta_m |\sin\eta_s - \sin\eta_m|}{\cos\eta_s \cos\eta_m \sin(\Delta\phi)} \\ &= -\frac{|\sin\eta_m - \sin\eta_s|}{\cos\eta_s \sin(\Delta\phi)} \qquad \square\end{aligned}$$