Original scientific paper
Accepted 4. 3. 2014.

NGUYEN LE
NORMAN JOHN WILDBERGER

Incenter Circles, Chromogeometry, and the Omega Triangle

Incenter Circles, Chromogeometry, and the Omega Triangle

ABSTRACT

Chromogeometry brings together planar Euclidean geometry, here called *blue* geometry, and two relativistic geometries, called *red* and *green*. We show that if a triangle has four blue Incenters and four red Incenters, then these eight points lie on a green circle, whose center is the green Orthocenter of the triangle, and similarly for the other colours. Tangents to the incenter circles yield interesting additional standard quadrangles and concurrencies. The proofs use the framework of rational trigonometry together with standard coordinates for triangle geometry, while a dilation argument allows us to extend the results also to Nagel and Speiker points.

Key words: triangle geometry, incenter circles, rational trigonometry, chromogeometry, four-fold symmetry, Nagel points, Spieker points, Omega triangle

MSC 2000: 51M05, 51M10, 51N10

Upisane kružnice, kromogeometrija i Omega trokut

SAŽETAK

Kromogeometrija povezuje ravninsku euklidsku geometriju, ovdje zvanu *plavom* geometrijom, te dvije relativističke geometrije, nazvane *crvenom* i *zelenom* geometrijom. Pokazuje se da ukoliko trokut ima četiri plava i četiri crvena središta upisanih (odnosno pripisanih) kružnica, tada tih osam točaka leži na zelenoj kružnici čije je središte zeleni ortocentar trokuta. Vrijede i druge dvije analogne tvrdnje. Tangente na upisane kružnice stvaraju nove zanimljive četverokute i konkurentnosti. Dokazi se provode u okviru racionalne trigonometrije sa standardnim koordinatama za geometriju trokuta. Transformacija diletacije dozvoljava proširenje rezultata na Nagelove i Speikerove točke.

Ključne riječi: geometrija trokuta, upisane kružnice, racionalna trigonometrija, kromogeometrija, četverostruka simetrija, Nagelove točke, Spiekerove točke, Omega trokut

1 Introduction

This paper investigates a surprising connection between three closely related Incenter hierarchies of a fixed planar triangle. The framework here is that of Rational Trigonometry ([7], [8]) which allows a consistent *universal triangle geometry* valid for any symmetric bilinear form, as described in [5], together with the three-fold symmetry of *chromogeometry* ([9], [10]), which connects the familiar Euclidean (*blue*) geometry based on the symmetric bilinear form $x_1x_2+y_1y_2$, and two relativistic geometries (*red* and *green*) based respectively on the bilinear forms $x_1x_2-y_1y_2$ and $x_1y_2+y_1x_2$. By working with the rational notions of quadrance and spread instead of the transcendental notions of distance and angle, the main laws of Rational Trigonometry allow metrical geometry, and so triangle geometry, to be developed in each of these three geometries in a parallel fashion, with mostly identical formulas and theorems.

The first results of this paper concern the four Incenters of a planar triangle in one of the three geometries, and were announced in [5]. As in that paper, we here refer to all four meets of the vertex bisectors, or bilines, as Incenters, so do not distinguish between the classical incenter and the three excenters. If a triangle $\overline{A_1A_2A_3}$ has four blue Incenters I_0^b, I_1^b, I_2^b and I_3^b, then all four points lie both on a *red incenter circle* $\mathcal{C}_r^b$ with center the red Orthocenter H_r, and on a *green incenter circle* $\mathcal{C}_g^b$ with center the green Orthocenter H_g; this is illustrated in Figure 1. Similarly, if a triangle has red Incenters, then these lie both on a green incenter circle $\mathcal{C}_g^r$ with center H_g, and a blue incenter circle $\mathcal{C}_b^r$ with center the blue Orthocenter H_b. If a triangle has green Incenters, these lie both on a blue incenter circle $\mathcal{C}_b^g$ with center H_b, and on a red incenter circle $\mathcal{C}_r^g$ with center H_r. Furthermore, if *both* red and green Incenters exist, then they lie on the *same* blue incenter circle, and similarly for the other colours. The proofs are algebraic, and rely on non-obvious simplifications found by the help of a computer. So the Omega triangle formed by the three Orthocenters $\Omega \equiv \overline{H_bH_rH_g}$, introduced in [9], has an intimate connection with the Incenter hierarchies.

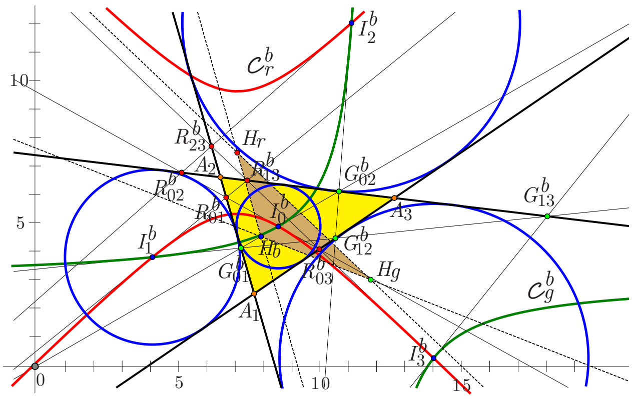

Figure 1: *The four blue Incenters of $\overline{A_1A_2A_3}$ and red and green Incenter Circles*

These facts relate also to elegant classical properties of quadrangles. In [1] Haskell showed that if two quadrangles have the same diagonal triangle, then all eight points of these quadrangles lie on a single conic; and in [11] Woods found a synthetic derivation of the same result. Now it is obvious that the four Incenters of a triangle, with respect to any bilinear form, will form a standard quadrangle in this sense, meaning that the diagonal triangle coincides with the original triangle. As a consequence, if blue and red Incenters exist, then they must lie on a conic. Our assertion is that this conic is actually a green circle $\mathcal{C}_g^b = \mathcal{C}_g^r \equiv \mathcal{C}_g$ with center H_g.

In the case of blue Incenters, the four tangent lines to the red incenter circle $\mathcal{C}_r^b$ at the blue Incenters form a standard quadrilateral, implying that they meet in six points R_{ij}^b, which lie two at a time on the three lines of $\overline{A_1A_2A_3}$, where they are harmonic conjugates with respect to A_1, A_2 and A_3; and similarly the four tangent lines to the green incenter circle $\mathcal{C}_g^b$ at the blue Incenters meet in six points G_{ij}^b on the three lines. This is also seen on the above Figure. Similarly there is a corresponding result when we look at red Incenters, and at green Incenters.

The six lines $A_kR_{ij}^b$, for i, j, k distinct, are the lines of a complete quadrangle, so they meet three at a time at four **quad points** Q_{rj}^b. Similarly, the six lines $A_kG_{ij}^b$ meet three at a time at points Q_{gj}^b. Somewhat remarkably, the four **star lines** $s_j^b \equiv Q_{rj}^b Q_{gj}^b$ form a standard quadrilateral $\overline{s_0^b s_1^b s_2^b s_3^b}$.

This paper also illustrates our novel approach to triangle geometry initiated in [5]; using standard coordinates to establish universal aspects of the subject which are *valid over a general bilinear form*. This employs an affine change of coordinates to place an arbitrary triangle into **standard position**, with vertices at $[0,0]$, $[1,0]$ and $[0,1]$. The various triangle centers and constructions are then expressed in terms of the coefficients a, b and c of the matrix

$$C \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

of the resulting new bilinear form. This allows a systematic augmentation of Kimberling's *Encyclopedia of Triangle Centers* ([2], [3], [4]) to *arbitrary quadratic forms* and *general fields*.

Standard coordinates also have the advantage of yielding surprisingly simple equations for the three coloured Incenter Circles, which turn out to be, after pleasant simplifications,

$$\mathcal{C}_b : Q_b(X) = b_b(2x+2y-1)$$
$$\mathcal{C}_r : Q_r(X) = b_r(2x+2y-1)$$
$$\mathcal{C}_g : Q_g(X) = b_g(2x+2y-1).$$

However the formulas for the star lines s_j^b become rather formidable, but seem to have interesting algebraic aspects. Some intriguing number theoretical questions arise when we inquire into the existence of triangles, over a given field, which have simultaneously blue, red and green Incenters. Studying concrete examples and using empirical computer investigations of Michael Reynolds [6], we make some tentative conjectures on such triangles, both over the rational numbers and over a finite prime field. Finally we extend the results to Spieker and Nagel points by suitable central dilations.

In the rest of this introduction we recall basic facts from [7] and [5] to formulate triangle geometry over a general bilinear form. We then specialize to the blue, red and green geometries, and use standard coordinates to develop formulas for points and lines (always one of our key aims), and to provide explicit computational proofs of the theorems.

1.1 Quadrilaterals and quadrangles

We begin by reminding the reader of some basic facts from the projective geometry of a quadrangle (four points) or quadrilateral (four lines), using a visual presentation to avoid the need to introduce notation.

In Figure 2 we see four blue lines forming a quadrilateral [in this figure colours are not used in a metrical sense, but only as an aide for explanation]. These four blue lines meet in six points, also in blue. These six blue points determine a further three green **diagonal lines**, forming the **diagonal triangle,** in yellow**,** of the original quadrilateral, whose vertices are three green points. Each green point may be joined via a red line to the two blue points not on either of the two green lines it lies on. This produces six red lines, which somewhat remarkably meet three at a time at four red points, giving the **opposite quadrangle** from the original blue quadrilateral. Note that there is a natural correspondence between the four original blue lines and the four red points.

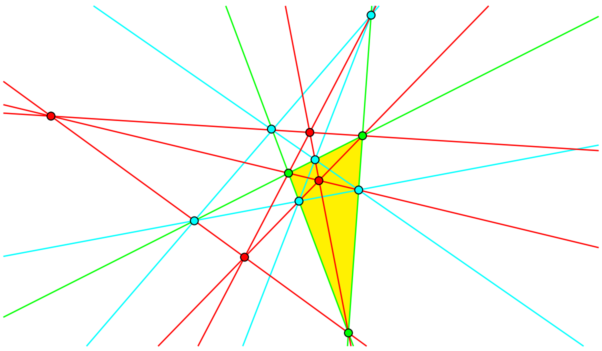

Figure 2: *A quadrilateral and its opposite quadrangle*

The situation is completely symmetric with regard to points and lines. If we had started out with a quadrilateral of four red points, we would join them to form six red lines. These six red lines determine a further three green diagonal points, forming the diagonal triangle of the original quadrilateral, whose sides form three green lines. Each green line meets two of the red lines in two new blue points. These six new blue points lie three at a time on four blue lines, giving the **opposite quadrilateral** from the original red quadrangle.

The diagonal green points on a green line are harmonic conjugates with respect to the two blue points on the same line. The diagonal green lines through a green point are harmonic conjugates with respect to the two red lines through the same point.

There is another more subtle remark to be made here concerning symmetry: each of the three diagonal points is canonically associated to a subdivision of the four original blue lines into two subsets of two: namely those subsets whose joins meet at that diagonal point.

If we start with a triangle, say the yellow triangle in the Figure formed by three green points and three green lines, then any quadrilateral or quadrangle which has that triangle as its diagonal triangle is called **standard**.

1.2 Quadrance, spread and standard coordinates

In this section we briefly summarize the main facts needed from rational trigonometry in the general affine setting (see [7], [8]). We work in the standard two-dimensional vector space V, consisting of row vectors $v = [x,y]$, over a field $\mathbb{F}$.

A **line** l is given by an equation of the form $ax+by+c=0$, or equivalently the proportion $l \equiv \langle a:b:c \rangle$.

We assume a metrical structure determined by a non-degenerate symmetric 2×2 matrix C: this gives a symmetric bilinear form on vectors:

$$v\cdot u \equiv vCu^T.$$

Non-degenerate means $\det C \neq 0$, and implies that if $v\cdot u = 0$ for all vectors u, then $v=0$.

Two vectors v and u are then **perpendicular** precisely when $v\cdot u = 0$. Since the matrix C is non-degenerate, for any vector v there is, up to a scalar, exactly one vector u which is perpendicular to v. Two lines l and m are **perpendicular** precisely when they have perpendicular direction vectors.

The bilinear form determines the main metrical quantity: the **quadrance** of a vector v is the number

$$Q_v \equiv v\cdot v.$$

The **quadrance** between the points A and B is $Q(A,B) \equiv Q_{\overrightarrow{AB}}$. A vector v is **null** precisely when $Q_v = v\cdot v = 0$, in other words precisely when v is perpendicular to itself. A line is **null** precisely when it has a null direction vector. The following basic fact appears in [5].

Theorem 1 (Parallel vectors) *Vectors v and u are parallel precisely when*

$$Q_vQ_u = (v\cdot u)^2.$$

This motivates the following measure of the non-parallelism of two vectors; the **spread** between non-null vectors v and u is the number

$$s(v,u) \equiv 1-\frac{(v\cdot u)^2}{Q_vQ_u} = 1-\frac{(v\cdot u)^2}{(v\cdot v)(u\cdot u)}.$$

The spread $s(v,u)$ is unchanged if either v or u are multiplied by a non-zero number. We define the **spread** between any non-null lines l and m with direction vectors v and u to be $s(l,m) \equiv s(v,u)$. From Theorem 1, the spread between parallel lines is 0. Two non-null lines l and m are perpendicular precisely when the spread between them is 1.

A **circle** is given by an equation of the form $Q(A,X) = K$ for some fixed point A called the **center**, and a number K called the **quadrance**. Note that it is not required that a circle have any points X lying on it: in this case by enlarging the field to a quadratic extension we can guarantee that it does.

The three particular planar geometries we are most interested in come from the **blue**, **red** and **green** bilinear forms given by the respective matrices

$$C_b \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_r \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad C_g \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding formulas for the **blue**, **red** and **green quadrances** between points $A_1 \equiv [x_1,y_1]$ and $A_2 \equiv [x_2,y_2]$

are

$$Q_b(A_1,A_2) = (x_2-x_1)^2 + (y_2-y_1)^2$$
$$Q_r(A_1,A_2) = (x_2-x_1)^2 - (y_2-y_1)^2$$
$$Q_g(A_1,A_2) = 2(x_2-x_1)(y_2-y_1).$$

It will be useful to discuss triangle geometry then in a general setting: suppose $v_1 \circ v_2 \equiv v_1 B v_2^T$ is a symmetric bilinear form, with B a symmetric 2×2 matrix. Suppose $\phi : V \to V$ is a linear transformation given by an invertible 2×2 matrix M, so that $\phi(v) = vM = w$, with inverse matrix N, so that $wN = v$. The new bilinear form $w_1 \cdot w_2 \equiv (w_1 N)\circ(w_2 N)$ then has matrix $D = NBN^T$.

Suppose that $\overline{X_1X_2X_3}$ is a triangle in the vector space V which has a distinguished symmetric bilinear form $\circ$. We may move this triangle by a combination of a translation (which does not effect the bilinear form), and a linear transformation ϕ, so that the triangle is in what we call **standard form**, with points

$$A_1 \equiv [0,0], \quad A_2 \equiv [1,0] \quad \text{and} \quad A_3 \equiv [0,1]$$

and lines

$$l_1 \equiv A_2A_3 = \langle 1:1:-1\rangle$$
$$l_2 \equiv A_1A_3 = \langle 1:0:0\rangle$$
$$l_3 \equiv A_2A_1 = \langle 0:1:0\rangle.$$

Whatever the initial matrix B, the new bilinear form $\cdot$ is given by

$$v\cdot u \equiv vDu^T \quad \text{where} \quad D \equiv NBN^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{1}$$

for some numbers a, b, and c. We may then replace arguments involving the general triangle $\overline{X_1X_2X_3}$ and the bilinear form $\circ$ with ones involving the simpler triangle $\overline{A_1A_2A_3}$. What we prove for the standard triangle $\overline{A_1A_2A_3}$ with bilinear form given by the matrix D will be true for the original triangle with bilinear form given by the original matrix B.

We will assume that D is invertible, so that

$$\Delta \equiv \det D = ac - b^2$$

is non-zero. Another important quantity is the **mixed trace**

$$d \equiv a + c - 2b$$

that appears in many formulas. With these notations, we have the following result from [5].

Theorem 2 (Standard quadrances and spreads) *The quadrances and spreads of $\overline{A_1A_2A_3}$ are*

$$Q_1 \equiv Q(A_2,A_3) = d$$
$$Q_2 \equiv Q(A_1,A_3) = c$$
$$Q_3 \equiv Q(A_1,A_2) = a$$

and

$$s_1 \equiv s(A_1A_2, A_1A_3) = \frac{\Delta}{ac}$$
$$s_2 \equiv s(A_2A_3, A_2A_1) = \frac{\Delta}{ad}$$
$$s_3 \equiv s(A_3A_1, A_3A_2) = \frac{\Delta}{cd}.$$

Furthermore

$$1 - s_1 = \frac{b^2}{ac}, \quad 1 - s_2 = \frac{(a-b)^2}{ad}, \quad 1 - s_3 = \frac{(c-b)^2}{cd}.$$

Note that the centroid of $\overline{A_1A_2A_3}$ is

$$G = \left[\frac{1}{3}, \frac{1}{3}\right].$$

1.3 Bilines, Incenters and some other triangle centers

A **biline** of the non-null vertex $\overline{l_1 l_2}$ is a line b which passes through $l_1 l_2$ and satisfies $s(l_1, b) = s(b, l_2)$. The existence of bilines depends on number theoretical considerations of a particularly simple kind.

Theorem 3 (Existence of Triangle bilines) *The Triangle $\overline{A_1A_2A_3}$ has bilines at each vertex precisely when we can find numbers u, v, w in the field satisfying*

$$ac = u^2, \quad ad = v^2, \quad cd = w^2. \tag{2}$$

In this case we can choose u, v, w so that $acd = uvw$ and

$$du = vw, \quad cv = uw \quad \text{and} \quad aw = uv. \tag{3}$$

We now summarize some basic triangle centers of the standard triangle $\overline{A_1A_2A_3}$, assuming the existence of bilines. These formulas involve the entries a, b, c of D from (1), as well as the secondary quantities u, v and w from (2), satisfying (3). The formulas and proofs are found in [5].

The four Incenters are

$$I_0 = \frac{1}{d+v-w}[-w, v], \qquad I_1 = \frac{1}{d-v+w}[w, -v],$$
$$I_2 = \frac{1}{d+v+w}[w, v], \qquad I_3 = \frac{1}{d-v-w}[-w, -v].$$

Notice that I_1, I_2 and I_3 may be obtained from I_0 by changing signs of: both v and w, just w, and just v respectively. This four-fold symmetry will hold more generally and note that it means that we can generally just record the values of I_0. The Orthocenter H, Circumcenter C and De Longchamps point X_{20} (the orthocenter of the double triangle) are

$$H = \frac{b}{\Delta}[c-b, a-b] \tag{4}$$
$$C = \frac{1}{2\Delta}[c(a-b), a(c-b)]$$
$$X_{20} = \frac{1}{\Delta}\left[b^2 - 2bc + ac, b^2 - 2ab + ac\right].$$

2 The Incenter Circle theorem

Here is the main theorem of the paper, illustrated for green Incenters of the triangle $\overline{A_1A_2A_3}$ in Figure 3. The situation is completely symmetric between the three geometries blue, red and green.

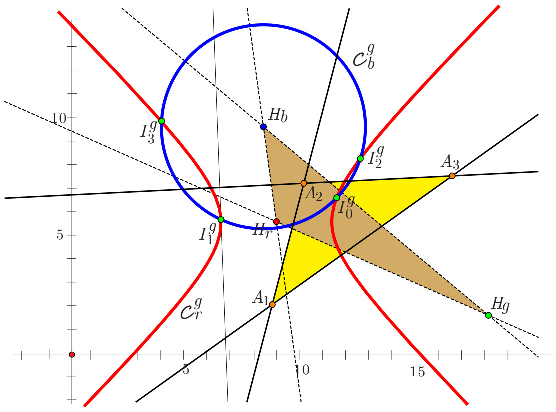

Figure 3: *Green Incenters and the blue and red Incenter Circles*

Theorem 4 (Incenter Circles) *If a triangle $\overline{A_1A_2A_3}$ has four blue Incenters I_0^b, I_1^b, I_2^b and I_3^b, then they all lie both on a red circle C_r^b with center the red Orthocenter H_r, and on a green circle C_g^b with center the green Orthocenter H_g, and similarly for the other colours. Furthermore, if both red and green Incenters exist, then they lie on the same blue circle, so that $C_b^r = C_b^g = C_b$, and similarly for the other colours.*

Proof. To prove that the four blue Incenters I_0^b, I_1^b, I_2^b and I_3^b lie on a red circle C_r^b with center H_r, we need show that

$$Q_r\left(H_r, I_0^b\right) = Q_r\left(H_r, I_1^b\right) = Q_r\left(H_r, I_2^b\right) = Q_r\left(H_r, I_3^b\right).$$

First we find the bilinear forms for the blue, red and green geometries. After translating, and then applying a linear transformation with the matrix M, we send the original triangle to the standard triangle $\overline{A_1A_2A_3}$. If $M^{-1} = N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then the bilinear forms for the blue, red and green geometries become respectively the matrices

$$D_b \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T = \begin{pmatrix} \alpha^2+\beta^2 & \alpha\gamma+\beta\delta \\ \alpha\gamma+\beta\delta & \gamma^2+\delta^2 \end{pmatrix} \equiv \begin{pmatrix} a_b & b_b \\ b_b & c_b \end{pmatrix}$$

$$D_r \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T = \begin{pmatrix} \alpha^2-\beta^2 & \alpha\gamma-\beta\delta \\ \alpha\gamma-\beta\delta & \gamma^2-\delta^2 \end{pmatrix} \equiv \begin{pmatrix} a_r & b_r \\ b_r & c_r \end{pmatrix}$$

$$D_g \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T = \begin{pmatrix} 2\alpha\beta & \alpha\delta+\beta\gamma \\ \alpha\delta+\beta\gamma & 2\gamma\delta \end{pmatrix} \equiv \begin{pmatrix} a_g & b_g \\ b_g & c_g \end{pmatrix}.$$

There are interesting relations between the introduced quantities; for example

$$a_b^2 = a_g^2 + a_r^2, \qquad a_b c_b = b_g^2 + b_r^2,$$
$$a_r c_r = b_b^2 - b_g^2, \qquad a_g c_g = b_b^2 - b_r^2, \qquad c_b^2 = c_g^2 + c_r^2$$

and

$$a_b c_g - 2b_b b_g + c_b a_g = 0, \qquad a_b c_r - 2b_b b_r + c_b a_r = 0,$$

$$a_g c_r - 2b_g b_r + c_g a_r = 0.$$

The determinants of D_b, D_r and D_g are respectively

$$\Delta_b = (\alpha\delta - \beta\gamma)^2, \qquad \Delta_r = \Delta_g = -(\alpha\delta-\beta\gamma)^2 = -\Delta_b$$

and the mixed traces are

$$d_b = (\alpha-\gamma)^2 + (\beta-\delta)^2, \qquad d_r = (\alpha-\gamma)^2 - (\beta-\delta)^2,$$

$$d_g = 2(\alpha-\gamma)(\beta-\delta).$$

Note also the relation $d_b^2 = d_r^2 + d_g^2$.

If the original triangle has four blue Incenters, then the Existence of Triangle bilines theorem shows that we may choose numbers u_b, v_b, w_b satisfying (2) and (3), so that

$$u_b^2 = \left(\alpha^2+\beta^2\right)\left(\gamma^2+\delta^2\right)$$
$$v_b^2 = \left(\alpha^2+\beta^2\right)\left((\alpha-\gamma)^2+(\beta-\delta)^2\right)$$
$$w_b^2 = \left(\gamma^2+\delta^2\right)\left((\alpha-\gamma)^2+(\beta-\delta)^2\right).$$

The blue Incenters are then

$$I_0^b = \frac{1}{d_b+v_b-w_b}[-w_b, v_b], \quad I_1^b = \frac{1}{d_b-v_b+w_b}[w_b, -v_b],$$
$$I_2^b = \frac{1}{d_b+v_b+w_b}[w_b, v_b], \quad I_3^b = \frac{1}{d_b-v_b-w_b}[-w_b, -v_b].$$

In exactly the same fashion

$$I_0^r = \frac{1}{d_r+v_r-w_r}[-w_r, v_r] \quad \text{and} \quad I_0^g = \frac{1}{d_g+v_g-w_g}[-w_g, v_g].$$

According to (4), the respective orthocenters are

$$H_b = \frac{b_b}{\Delta_b}[c_b-b_b, a_b-b_b], \qquad H_r = \frac{b_r}{\Delta_r}[c_r-b_r, a_r-b_r],$$

$$H_g = \frac{b_g}{\Delta_g}[c_g-b_g, a_g-b_g].$$

If we set $e_b \equiv d_b + v_b - w_b$ then

$$\overrightarrow{H_r I_0^b} = -\left(\frac{b_r(c_r - b_r)}{\Delta_r} + \frac{w_b}{e_b}, \frac{b_r(a_r - b_r)}{\Delta_r} - \frac{v_b}{e_b}\right)$$
$$= -\frac{1}{\Delta_r e_b}\left(b_r(c_r - b_r)e_b + \Delta_r w_b, b_r(a_r - b_r)e_b - \Delta_r v_b\right)$$

so that

$$Q_r\left(H_r, I_0^b\right) = \left(\overrightarrow{H_r I_0^b}\right) D_r \left(\overrightarrow{H_r I_0^b}\right)^T$$
$$= \left(\begin{array}{cc}\frac{b_r(c_r-b_r)}{\Delta_r}+\frac{w_b}{e_b} & \frac{b_r(a_r-b_r)}{\Delta_r}-\frac{v_b}{e_b}\end{array}\right)\left(\begin{array}{cc} a_r & b_r \\ b_r & c_r\end{array}\right)\left(\begin{array}{c}\frac{b_r(c_r-b_r)}{\Delta_r}+\frac{w_b}{e_b} \\ \frac{b_r(a_r-b_r)}{\Delta_r}-\frac{v_b}{e_b}\end{array}\right)$$
$$= \frac{1}{\Delta_r^2 e_b^2}\left(\begin{array}{c} a_r\left(b_r(c_r-b_r)e_b+\Delta_r w_b\right)^2 + c_r\left(b_r(a_r-b_r)e_b - \Delta_r v_b\right)^2 \\ +2b_r\left(b_r(c_r-b_r)e_b+\Delta_r w_b\right)\left(b_r(a_r-b_r)e_b-\Delta_r v_b\right)\end{array}\right)$$
$$= \frac{1}{\Delta_r^2 e_b^2}\left(\begin{array}{c} b_r^2(a_r - 2b_r + c_r)\left(a_r c_r - b_r^2\right)e_b^2 \\ -2\Delta_r b_r(v_b - w_b)\left(-b_r^2 + a_r c_r\right)e_b \\ +\Delta_r^2\left(a_r w_b^2 + c_r v_b^2 - 2b_r v_b w_b\right)\end{array}\right).$$

Use the relation $\Delta_r = a_r c_r - b_r^2$ to get

$$Q_r\left(H_r, I_0^b\right) \tag{5}$$
$$= \frac{1}{\Delta_r^2 e_b}\left(\begin{array}{c} b_r^2(a_r - 2b_r + c_r)e_b^2 \\ -2\Delta_r b_r(v_b - w_b)e_b + \Delta_r\left(a_r w_b^2 + c_r v_b^2 - 2b_r v_b w_b\right)\end{array}\right)$$
$$= \frac{\left(\begin{array}{c} 2b_r(b_r - c_r)(a_r - b_r)(v_b d_b - v_b w_b - w_b d_b) \\ + a_r(b_r - c_r)^2 v_b^2 + c_r(a_r - b_r)^2 w_b^2 + b_r^2 d_r d_b^2\end{array}\right)}{\Delta_r^2 e_b}$$

where we have collected v_b^2, w_b^2 and d_b^2 of the numerator of (5), to rewrite it.
Replace $v_b^2 = a_b d_b$, $w_b^2 = c_b d_b$ and $v_b w_b = u_b d_b$ and the values of $a_b, c_b, d_b, a_r, b_r, c_r$ in terms of $\alpha, \beta, \gamma, \delta$ to get the factorization

$$2b_r(b_r - c_r)(a_r - b_r)(v_b - u_b - w_b)d_b + a_r(b_r - c_r)^2 a_b d_b$$
$$+ c_r(a_r - b_r)^2 c_b d_b + b_r^2(a_r + c_r - 2b_r)d_b^2$$
$$= d_b\left(\begin{array}{c} 2b_r(b_r - c_r)(a_r - b_r)(v_b - u_b - w_b) + a_r(b_r - c_r)^2 a_b \\ + c_r(a_r - b_r)^2 c_b + b_r^2(a_r + c_r - 2b_r)d_b\end{array}\right)$$
$$= 2d_b(\alpha\gamma - \beta\delta)\left(\alpha^2 - \alpha\gamma + \gamma^2 + \beta^2 - \beta\delta + \delta^2 - u_b + v_b - w_b\right)$$
$$\times\left(\alpha^2 - \beta^2 - \alpha\gamma + \beta\delta\right)\left(-\gamma^2 + \delta^2 + \alpha\gamma - \beta\delta\right) \tag{6}$$

and also note that

$$(d_b + v_b - w_b)^2 = d_b(a_b + c_b + d_b - 2u_b + 2v_b - 2w_b)$$
$$= 2d_b\left(\alpha^2 - \alpha\gamma + \gamma^2 + \beta^2 - \beta\delta + \delta^2 - u_b + v_b - w_b\right). \tag{7}$$

Combine (6) and (7), to get the surprisingly simple formula

$$Q_r\left(H_r, I_0^b\right)$$
$$= \frac{(\alpha\gamma - \beta\delta)\left(\alpha^2 - \beta^2 - \alpha\gamma + \beta\delta\right)\left(-\gamma^2 + \delta^2 + \alpha\gamma - \beta\delta\right)}{\Delta_r}$$
$$= \frac{b_r(a_r - b_r)(b_r - c_r)}{\Delta_r} \equiv K_r.$$

We may now repeat the calculation to see that $Q_r\left(H_r, I_1^b\right) = Q_r\left(H_r, I_2^b\right) = Q_r\left(H_r, I_3^b\right) = K_r$, showing that indeed the four blue Incenters lie on the red circle $\mathcal{C}_r^b$ with quadrance K_r and center H_r. Note that the expression for K_r depends only on the matrix D_r. Now a similar derivation shows that

$$Q_g\left(H_g, I_i^b\right) = \frac{b_g(a_g - b_g)(b_g - c_g)}{\Delta_g} \equiv K_g, \quad i = 0,1,2,3.$$

Hence the four blue Incenters also lie on a green circle $\mathcal{C}_g^b$ with quadrance K_g and center H_g. Similarly we find that if a triangle has four red Incenters, then they lie on a blue circle $\mathcal{C}_b^r$ with center H_b and quadrance

$$Q_b\left(H_b, I_i^g\right) = Q_b\left(H_b, I_i^r\right) = \frac{b_b(a_b - b_b)(b_b - c_b)}{\Delta_b} \equiv K_b$$

as well as on a green circle $\mathcal{C}_g^r$ with center H_g and quadrance K_g (the same one as above!) Similarly if a triangle has four green Incenters, then they lie on a blue circle $\mathcal{C}_b^g$ with center H_b and quadrance K_b, as well as on a red circle $\mathcal{C}_r^g$ with center H_r and quadrance K_r. The proof is complete. □

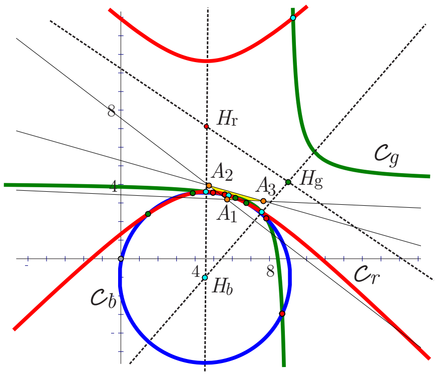

Figure 4: *Three Incenter Circles $\mathcal{C}_b$, $\mathcal{C}_r$ and $\mathcal{C}_g$.*

We now call $\mathcal{C}_b = \mathcal{C}_b^r = \mathcal{C}_b^g$, $\mathcal{C}_r = \mathcal{C}_r^b = \mathcal{C}_r^g$ and $\mathcal{C}_g = \mathcal{C}_g^b = \mathcal{C}_g^r$ the blue, red and green **Incenter Circles** respectively. In Figure 4 we see a (small) triangle $\overline{A_1A_2A_3}$ with its Omega triangle $\overline{H_bH_rH_g}$ and the three Incenter Circles, whose respective meets give the twelve blue, red and green Incenters.

2.1 Equations of Incenter Circles

Theorem 5 (Incenter Circles equations) *In standard coordinates with $X = [x,y]$, the blue, red and green Incenter circles, when they exist, have respective equations*

$$\mathcal{C}_b : Q_b(X) = b_b(2x + 2y - 1)$$
$$\mathcal{C}_r : Q_r(X) = b_r(2x + 2y - 1)$$
$$\mathcal{C}_g : Q_g(X) = b_g(2x + 2y - 1).$$

Proof. The derivation of these equations, using the formulas established above for the orthocenters H_r and coloured Incenters, is somewhat involved algebraically, although the basic idea is simple. We show how to find the equation of the red Incenter Circle $\mathcal{C}_r$, with center H_r, which four blue Incenters and four green Incenters lie on if they exist. From the definition of a red circle, we get the equation $Q_r(H_r,X)=K_r$, and then substitute the values of H_r and K_r to get

$$\begin{pmatrix} \frac{b_r(c_r-b_r)}{\Delta_r}-x & \frac{b_r(a_r-b_r)}{\Delta_r}-y \end{pmatrix}\begin{pmatrix} a_r & b_r \\ b_r & c_r \end{pmatrix}\begin{pmatrix} \frac{b_r(c_r-b_r)}{\Delta_r}-x \\ \frac{b_r(a_r-b_r)}{\Delta_r}-y \end{pmatrix} = \frac{b_r(a_r-b_r)(b_r-c_r)}{\Delta_r}$$

or after expansion

$$\frac{1}{\Delta_r^2}\begin{pmatrix} a_r(b_r(c_r-b_r)-\Delta_r x)^2+c_r(b_r(a_r-b_r)-\Delta_r y)^2 \\ +2b_r(b_r(c_r-b_r)-\Delta_r x)(b_r(a_r-b_r)-\Delta_r y) \end{pmatrix} = \frac{b_r(a_r-b_r)(b_r-c_r)}{\Delta_r}.$$

This may be rewritten, using $\Delta_r = a_r c_r - b_r^2$, in the form

$$\frac{1}{\Delta_r}(\Delta_r^2 a_r x^2 + 2\Delta_r^2 b_r xy + \Delta_r^2 c_r y^2 + \Delta_r b_r^2(a_r-2b_r+c_r) - 2\Delta_r^2 b_r x - 2\Delta_r^2 b_r y) = b_r(a_r-b_r)(b_r-c_r).$$

Now cancel Δ_r, and rearrange to get

$$\Delta_r a_r x^2 + 2\Delta_r b_r xy + \Delta_r c_r y^2 - 2\Delta_r b_r x - 2\Delta_r b_r y + b_r(a_r c_r - b_r^2) = 0$$

or more simply

$$a_r x^2 + 2b_r xy + c_r y^2 - 2b_r x - 2b_r y + b_r = 0$$

which has the form stated in the theorem. The same kind of calculation establishes the formulas for $\mathcal{C}_b$ and $\mathcal{C}_g$. $\square$

Note that the equations for the Incenter Circles $\mathcal{C}_b, \mathcal{C}_r$ and $\mathcal{C}_g$ allow them to be defined *whether or not* the corresponding Incenters exist! Incenters then exist precisely as meets of these Incenter Circles: for example the blue Incenters $I_0^b, I_1^b, I_2^b, I_3^b$ are just the meets of $\mathcal{C}_r$ and $\mathcal{C}_g$, if these exist in the field in which we work.

2.2 Tangent lines of Incenter Circles

Now we consider tangent lines to Incenter circles. Figure 5 shows the four blue Incenters of $\overline{A_1A_2A_3}$, together with the red and green Incenter Circles passing through them, namely $\mathcal{C}_r$ and $\mathcal{C}_g$. At each of the four Incenters I_i^b, $i=1,2,3,4$ we have the tangent lines t_{ri}^b and t_{gi}^b to the red and green Incenter Circles $\mathcal{C}_r$ and $\mathcal{C}_g$ respectively.

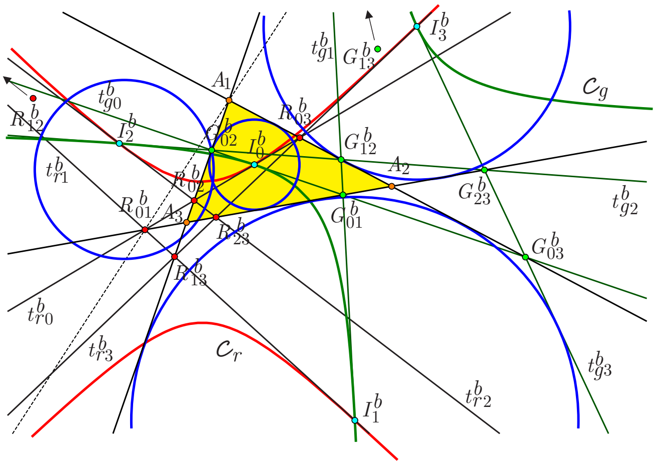

Figure 5: *Incenter tangent meets*

Theorem 6 (Incenter tangent meets) *The tangent lines $t_{r0}^b, t_{r1}^b, t_{r2}^b, t_{r3}^b$ to the red Incenter circle $\mathcal{C}_r$ at the blue Incenters form a standard quadrilateral, as do the tangent lines $t_{g0}^b, t_{g1}^b, t_{g2}^b, t_{g3}^b$ at the green Incenter circle $\mathcal{C}_g$. The same holds for the red and green Incenters, if they exist.*

This implies that the meets $R_{01}^b \equiv t_{r0}^b t_{r1}^b$ and $R_{23}^b \equiv t_{r2}^b t_{r3}^b$ lie on $l_1 = A_2A_3$, and are harmonic conjugates with respect to A_2 and A_3. Similarly $R_{02}^b \equiv t_{r0}^b t_{r2}^b$ and $R_{13}^b \equiv t_{r1}^b t_{r3}^b$ lie on $l_2 = A_1A_3$, and are harmonic conjugates with respect to A_1 and A_3; and $R_{03}^b \equiv t_{r0}^b t_{r3}^b$ and $R_{12}^b \equiv t_{r1}^b t_{r2}^b$ lie on $l_3 = A_1A_2$, and are harmonic conjugates with respect to A_1 and A_2. The points $G_{01}^b \equiv t_{g0}^b t_{g1}^b$ and $G_{23}^b \equiv t_{g2}^b t_{g3}^b$ lie on l_1, and are harmonic conjugates with respect to A_2 and A_3. Similarly $G_{02}^b \equiv t_{g0}^b t_{g2}^b$ and $G_{13}^b \equiv t_{g1}^b t_{g3}^b$ lie on l_2, and are harmonic conjugates with respect to A_1 and A_3, and $G_{03}^b \equiv t_{g0}^b t_{g3}^b$ and $G_{12}^b \equiv t_{g1}^b t_{g2}^b$ lie on l_3, and are harmonic conjugates with respect to A_1 and A_2.

Proof. We prove the result for the meets G_{ij}^b of the green tangent lines t_{gi}^b associated to the blue Incenters; the other cases are similar. We find the joins of a blue Incenter I_i^b and the green Orthocenter H_g to be

$$H_g I_0^b = \left\langle \begin{matrix} (b_g-a_g)\,b_g d_b + (c_g-b_g)\,a_g v_b + (a_g-b_g)\,b_g w_b : \\ (c_g-b_g)\,b_g d_b + (c_g-b_g)\,b_g v_b + (a_g-b_g)\,c_g w_b : \\ (b_g-c_g)\,b_g v_b + (b_g-a_g)\,b_g w_b \end{matrix} \right\rangle$$

$$H_g I_1^b = \left\langle \begin{matrix} (b_g-a_g)\,b_g d_b - (c_g-b_g)\,a_g v_b - (a_g-b_g)\,b_g w_b : \\ (c_g-b_g)\,b_g d_b - (c_g-b_g)\,b_g v_b - (a_g-b_g)\,c_g w_b : \\ -(b_g-c_g)\,b_g v_b - (b_g-a_g)\,b_g w_b \end{matrix} \right\rangle$$

$$H_g I_2^b = \left\langle \begin{matrix} (b_g-a_g)\,b_g d_b + (c_g-b_g)\,a_g v_b - (a_g-b_g)\,b_g w_b : \\ (c_g-b_g)\,b_g d_b + (c_g-b_g)\,b_g v_b - (a_g-b_g)\,c_g w_b : \\ (b_g-c_g)\,b_g v_b - (b_g-a_g)\,b_g w_b \end{matrix} \right\rangle$$

$$H_g I_3^b = \left\langle \begin{matrix} (b_g-a_g)\,b_g d_b - (c_g-b_g)\,a_g v_b + (a_g-b_g)\,b_g w_b : \\ (c_g-b_g)\,b_g d_b - (c_g-b_g)\,b_g v_b + (a_g-b_g)\,c_g w_b : \\ -(b_g-c_g)\,b_g v_b + (b_g-a_g)\,b_g w_b \end{matrix} \right\rangle.$$

The tangent line t_{gi}^b is the line green perpendicular to $H_g I_i^b$ passing through I_i^b. These we may calculate to be

$$t_{g0}^b = \left\langle \begin{matrix} (a_g-b_g)\,u_b + b_g v_b + (a_g-2b_g)\,w_b + d_b b_g - c_b\,(a_g-b_g) : \\ (c_g-b_g)\,u_b + (2b_g-c_g)\,v_b - b_g w_b + d_b b_g + a_b\,(b_g-c_g) : \\ b_g\,(-v_b+w_b-d_b) \end{matrix} \right\rangle$$

$$t_{g1}^b = \left\langle \begin{matrix} (a_g-b_g)\,u_b - b_g v_b - (a_g-2b_g)\,w_b + b_g d_b - c_b\,(a_g-b_g) : \\ (c_g-b_g)\,u_b - (2b_g-c_g)\,v_b + b_g w_b + d_b b_g + a_b\,(b_g-c_g) : \\ b_g\,(v_b-w_b-d_b) \end{matrix} \right\rangle$$

$$t_{g2}^b = \left\langle \begin{matrix} -(a_g-b_g)\,u_b + b_g v_b - (a_g-2b_g)\,w_b + b_g d_b - c_b\,(a_g-b_g) : \\ -(c_g-b_g)\,u_b + (2b_g-c_g)\,v_b + b_g w_b + d_b b_g + a_b\,(b_g-c_g) : \\ b_g\,(-v_b-w_b-d_b) \end{matrix} \right\rangle$$

$$t_{g3}^b = \left\langle \begin{matrix} -(a_g-b_g)\,u_b - b_g v_b + (a_g-2b_g)\,w_b + b_g d_b - c_b\,(a_g-b_g) : \\ -(c_g-b_g)\,u_b - (2b_g-c_g)\,v_b - b_g w_b + d_b b_g + a_b\,(b_g-c_g) : \\ b_g\,(v_b+w_b-d_b) \end{matrix} \right\rangle.$$

We could verify directly that these four lines form a standard quadrilateral. But we prefer to verify that the meets of these four tangent lines agree with the following meets with the side lines of $\overline{A_1A_2A_3}$:

$$\begin{aligned} G_{01}^b &\equiv t_{g0}^b t_{g1}^b = t_{g0}^b l_1 \\ &= \frac{1}{\lambda_{01}}\left[(b_g-c_g)(-u_b+v_b+a_b),(a_g-b_g)(-u_b-w_b+c_b)\right] \\ G_{23}^b &\equiv t_{g2}^b t_{g3}^b = t_{g2}^b l_1 \\ &= \frac{1}{\lambda_{23}}\left[(b_g-c_g)(u_b+v_b+a_b),(a_g-b_g)(u_b+w_b+c_b)\right] \\ G_{02}^b &\equiv t_{g0}^b t_{g2}^b = t_{g0}^b l_2 = \frac{1}{\lambda_{02}}\left[0,b_g(-v_b+w_b-d_b)\right] \\ G_{13}^b &\equiv t_{g1}^b t_{g3}^b = t_{g1}^b l_2 = \frac{1}{\lambda_{13}}\left[0,b_g(-v_b+w_b+d_b)\right] \\ G_{03}^b &\equiv t_{g0}^b t_{g3}^b = t_{g0}^b l_3 = \frac{1}{\lambda_{03}}\left[b_g(v_b-w_b+d_b),0\right] \\ G_{12}^b &\equiv t_{g1}^b t_{g2}^b = t_{g1}^b l_3 = \frac{1}{\lambda_{12}}\left[b_g(v_b-w_b-d_b),0\right] \end{aligned}$$

where

$$\begin{aligned} \lambda_{01} &= (c_g-a_g)\,u_b + (b_g-c_g)\,v_b + (b_g-a_g)\,w_b \\ &\quad + (a_b b_g - a_b c_g + c_b a_g - c_b b_g) \\ \lambda_{23} &= (a_g-c_g)\,u_b + (b_g-c_g)\,v_b \\ &\quad + (a_g-b_g)\,w_b + (a_b b_g - a_b c_g + c_b a_g - c_b b_g) \\ \lambda_{02} &= (b_g-c_g)\,u_b + (c_g-2b_g)\,v_b + b_g w_b \\ &\quad + (a_b c_g - 2a_b b_g + 2b_b b_g - c_b b_g) \\ \lambda_{13} &= (c_g-b_g)\,u_b + (c_g-2b_g)\,v_b + b_g w_b \\ &\quad + (2a_b b_g - a_b c_g - 2b_b b_g + c_b b_g) \\ \lambda_{03} &= (a_g-b_g)\,u_b + b_g v_b + (a_g-2b_g)\,w_b \\ &\quad + (a_b b_g - 2b_b b_g - c_b a_g + 2c_b b_g) \\ \lambda_{12} &= (b_g-a_g)\,u_b + b_g v_b + (a_g-2b_g)\,w_b \\ &\quad + (2b_b b_g - a_b b_g + c_b a_g - 2c_b b_g)\,. \end{aligned}$$

The fact that $A_2, A_3, G_{01}^b, G_{23}^b$ form a harmonic range etc. is an immediate consequence of a well known fact about standard quadrilaterals in projective geometry, since we have shown that the points A_1, A_2 and A_3 are diagonal points of the quadrilateral formed by the four green tangent lines. $\square$

Following the construction of the red lines in the introductory section on Quadrangles and quadrilaterals, we join a point G_{ij}^b with the triangle point A_k opposite to the triangle line that it lies on; giving six lines $A_k G_{ij}^b$:

$$A_1 G_{01}^b = \left\langle \begin{matrix} (a_g-b_g)(u_b+w_b-c_b) : \\ (b_g-c_g)(-u_b+v_b+a_b) : \\ 0 \end{matrix} \right\rangle$$

$$A_1 G_{23}^b = \left\langle \begin{matrix} (b_g-a_g)(u_b+w_b+c_b) : \\ (b_g-c_g)(u_b+v_b+a_b) : \\ 0 \end{matrix} \right\rangle$$

$$A_2 G_{02}^b = \left\langle \begin{matrix} b_g(-v_b+w_b-d_b) : \\ (b_g-c_g)u_b + (c_g-2b_g)v_b + b_g w_b - b_g d_b + a_b(c_g-b_g) : \\ b_g(v_b-w_b+d_b) \end{matrix} \right\rangle$$

$$A_2 G_{13}^b = \left\langle \begin{matrix} b_g(-v_b+w_b+d_b) : \\ (c_g-b_g)u_b + (c_g-2b_g)v_b + b_g w_b + b_g d_b + a_b(b_g-c_g) : \\ b_g(v_b-w_b-d_b) \end{matrix} \right\rangle$$

$$A_3 G_{03}^b = \left\langle \begin{matrix} (a_g-b_g)u_b + b_g v_b + (a_g-2b_g)w_b + b_g d_b + c_b(b_g-a_g) : \\ b_g(v_b-w_b+d_b) : \\ b_g(-v_b+w_b-d_b) \end{matrix} \right\rangle$$

$$A_3 G_{12}^b = \left\langle \begin{matrix} (b_g-a_g)u_b + b_g v_b + (a_g-2b_g)w_b - b_g d_b + c_b(a_g-b_g) : \\ b_g(v_b-w_b-d_b) : \\ b_g(-v_b+w_b+d_b) \end{matrix} \right\rangle.$$

Theorem 7 (Quad points) *The triples* $\{A_1G_{23}^b, A_2G_{13}^b, A_3G_{12}^b\}$, $\{A_1G_{23}^b, A_2G_{02}^b, A_3G_{03}^b\}$, $\{A_1G_{01}^b, A_2G_{13}^b, A_3G_{03}^b\}$ *and* $\{A_1G_{01}^b, A_2G_{02}^b, A_3G_{12}^b\}$ *of lines are concurrent in the respective points* $Q_{g0}^b, Q_{g1}^b, Q_{g2}^b$ *and* Q_{g3}^b, *called the* ***blue/green quad points.*** *The triples* $\{A_1R_{23}^b, A_2R_{13}^b, A_3R_{12}^b\}$, $\{A_1R_{23}^b, A_2R_{02}^b, A_3R_{03}^b\}$, $\{A_1R_{01}^b, A_2R_{13}^b, A_3R_{03}^b\}$ *and* $\{A_1R_{01}^b, A_2R_{02}^b, A_3R_{12}^b\}$ *are also concurrent in the respective points* $Q_{r0}^b, Q_{r1}^b, Q_{r2}^b$ *and* Q_{r3}^b, *called the* ***blue/red quad points****. Similar results hold for the red and green Incenters, if they exist.*

Proof. We verify this for the blue/green quad points: this is a consequence of the projective geometry of the complete quadrilateral we mentioned in the first section—if the original four tangent lines are regarded as the blue lines in Figure 6, then the quad points Q_{gj}^b correspond to the red points. However we want to find explicit formulas and check things directly. The quad point Q_{gj}^b is naturally associated to the Incenter I_j^b. After some calculation, we find that these are

$$Q^b_{g0} = \frac{b_g}{\lambda_0}[(b_g - c_g)(d_b u_b - (b_b - c_b) v_b),$$
$$(a_g - b_g)((c_b - b_b) w_b + c_b d_b)]$$

$$Q^b_{g1} = \frac{b_g}{\lambda_1}[(b_g - c_g)((a_b - b_b) v_b + a_b d_b),$$
$$(a_g - b_g)(d_b u_b + (a_b - b_b) w_b)]$$

$$Q^b_{g2} = \frac{b_g}{\lambda_2}[(c_g - b_g)(a_b w_b - b_b v_b),$$
$$(b_g - a_g)(b_b w_b - c_b v_b)]$$

$$Q^b_{g3} = \frac{b_g}{\lambda_3}[(b_g - c_g)(-d_b u_b + (d_b + b_b) v_b - a_b w_b + a_b d_b),$$
$$(b_g - a_g)(d_b u_b - c_b v_b + (d_b + b_b) w_b - c_b d_b)]$$

where

$$\begin{aligned}
\lambda_0 &= (b_g - c_g)(b_g d_b + (b_b - c_b)(a_g - b_g)) u_b \\
&\quad - (b_g - c_g)(b_b b_g + c_b a_g - 2c_b b_g) v_b \\
&\quad - b_g (a_g - b_g)(b_b - c_b) w_b + c_b b_g (a_g - b_g) d_b \\
\lambda_1 &= (a_g - b_g)(b_g d_b + (a_b - b_b)(b_g - c_g)) u_b \\
&\quad + b_g (b_g - c_g)(a_b - b_b) v_b \\
&\quad + (a_g - b_g)(2a_b b_g - a_b c_g - b_b b_g) w_b + a_b b_g (b_g - c_g) d_b \\
\lambda_2 &= b_b (b_g - c_g)(a_g - b_g) u_b \\
&\quad + b_g (b_b b_g + c_b a_g - b_b c_g - c_b b_g) v_b \\
&\quad - b_g (a_b b_g + b_b a_g - a_b c_g - b_b b_g) w_b - a_b c_b (b_g - c_g)(a_g - b_g) \\
\lambda_3 &= \left((d_b + b_b)\left(b_g^2 + a_g c_g\right) - a_g b_g (2d_b + b_b) - b_b b_g c_g\right) u_b \\
&\quad + (b_g ((b_g - c_g)(a_b - b_b) + c_b (2a_g - b_g)) - c_b a_g c_g) v_b \\
&\quad + ((d_b + b_b)(b_g - a_g) b_g - a_b a_g (b_g - c_g)) w_b \\
&\quad + b_g^2 (a_b (d_b - c_b) - c_b d_b) + b_g (c_b a_g (d_b + a_b) - a_b c_g (d_b - c_b)) \\
&\quad - a_g c_g a_b c_b.
\end{aligned}$$

We may then check directly that for example Q^b_{g0} is incident with $A_3 G^b_{12}$ by computing

$$\begin{aligned}
&((b_g - a_g) u_b + b_g v_b + (a_g - 2b_g) w_b - b_g d_b + c_b (a_g - b_g)) \cdot \\
&\qquad \cdot \left(\frac{b_g (b_g - c_g)(d_b u_b - (b_b - c_b) v_b)}{\lambda_0}\right) \\
&+ b_g (v_b - w_b - d_b)\left(\frac{b_g (a_g - b_g)((c_b - b_b) w_b + c_b d_b)}{\lambda_0}\right) \\
&+ b_g (-v_b + w_b + d_b) \\
&= \frac{b_g (a_g - b_g)(b_g - c_g)\begin{pmatrix} d_b u_b^2 + (b_b - a_b) u_b w_b - b_b d_b u_b \\ -c_b v_b^2 + b_b v_b w_b + c_b (a_b - b_b) v_b \end{pmatrix}}{\begin{pmatrix} -(b_g - c_g)(b_g d_b + (b_b - c_b)(a_g - b_g)) u_b \\ + (b_g - c_g)(b_b b_g + c_b a_g - 2c_b b_g) v_b \\ + b_g (a_g - b_g)(b_b - c_b) w_b - c_b b_g (a_g - b_g) d_b \end{pmatrix}} \\
&= 0
\end{aligned}$$

since $d_b u_b^2 + (b_b - a_b) u_b w_b - b_b d_b u_b - c_b v_b^2 + b_b v_b w_b + c_b (a_b - b_b) v_b = 0$ by using (2), and similarly for the other indices. In a parallel fashion, we find that the four blue/red quad points Q^b_{rj} have exactly the same formulas as the Q^b_{gj}, except for the replacements $a_g \longrightarrow a_r$, $b_g \longrightarrow b_r$ and $c_g \longrightarrow c_r$, and similarly for the other colours red and green. □

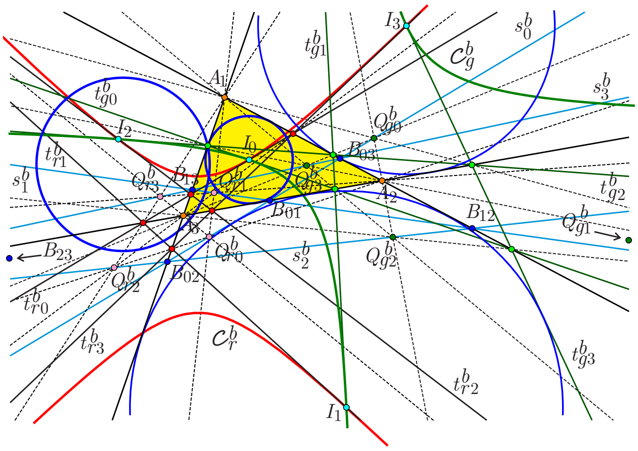

Figure 6: *Quad points and star lines*

Now introduce the **blue star line** s^b_j to be the join of the corresponding blue/red quad point Q^b_{rj} and the blue/green quad point Q^b_{gj}, and similarly for the other colours. There are then four blue star lines s^b_0, s^b_1, s^b_2 and s^b_3.
The **blue star point** B_{ij} is the meet of the two blue star lines s^b_i and s^b_j, that is $B_{ij} \equiv s^b_i s^b_j$, and similarly for the other colours.
Note that following the introductory section on Quadrangles and quadrilaterals, we use the correspondence between the Q^b_{gj}; and the tangent lines t^b_{gj}; and the Incenters I^b_j to match up the indices.

Theorem 8 (Star quadrilateral) *The four blue star lines form a standard quadrilateral* $\overline{s^b_0 s^b_1 s^b_2 s^b_3}$. *This holds also for the other colours.*

Proof. The proof we have is surprisingly complicated. The star lines s^b_j have quite involved formulas; for example we find that

$$s^b_0 = Q^{b0}_g Q^{b0}_r =$$

$$\left\langle \begin{pmatrix} E_0 d_b u_b + F_0 c_b v_b : \\ b_g b_r \left((b_b - c_b)^2 + c_b d_b\right) \cdot \\ \cdot (a_g b_r - b_g a_r - a_g c_r + c_g a_r + b_g c_r - c_g b_r) d_b u_b \\ -2c_b b_g b_r (a_g b_r - b_g a_r - a_g c_r + c_g a_r + b_g c_r - c_g b_r)(b_b - c_b) d_b v_b \\ -a_b (b_r - c_r)(b_g - c_g)(a_g b_r - b_g a_r)\left((b_b - c_b)^2 + c_b d_b\right) w_b \\ +2a_b c_b (b_r - c_r)(b_g - c_g)(a_g b_r - b_g a_r)(b_b - c_b) d_b \end{pmatrix} : \begin{pmatrix} -b_g b_r d_b (a_g b_r - b_g a_r - a_g c_r + c_g a_r + b_g c_r - c_g b_r) \cdot \\ \cdot \left(\left((b_b - c_b)^2 + c_b d_b\right) u_b - 2c_b (b_b - c_b) v_b\right) \end{pmatrix} \right\rangle$$

where E_0 and F_0 are both homogeneous polynomials of degree 6 in the variables a_i, b_i and c_i, with the former having 32 terms and the latter 46 terms. After some trial and error we can present these in the somewhat pleasant, but still mysterious, forms:

$$\begin{aligned}E_0 = & -b_g b_r\left(a_g c_r - c_g a_r - b_g c_r + c_g b_r\right)\cdot \\ & \cdot\left(b_b^2 + 4c_b^2 + a_b c_b - 6 b_b c_b\right) \\ & + b_g b_r\left(a_g b_r - b_g a_r\right)\left(b_b^2 + 2c_b^2 + a_b c_b - 4 b_b c_b\right) \\ & - 2c_b\left(a_g c_g b_r^2 - a_r c_r b_g^2 + a_g a_r c_r b_g - a_g c_g a_r b_r\right)\left(b_b - c_b\right)\end{aligned}$$

and

$$\begin{aligned}F_0 = & \left(a_g c_g b_r^2 - a_r c_r b_g^2 + a_g a_r c_r b_g - a_g c_g a_r b_r\right)\cdot \\ & \cdot\left(b_b^2 - 4 b_b c_b + 2c_b^2 + a_b c_b\right) \\ & + b_g b_r\left(a_g c_r - c_g a_r - b_g c_r + c_g b_r\right)\cdot \\ & \cdot\left(-5b_b^2 - 4c_b^2 + 2a_b b_b - 3a_b c_b + 10 b_b c_b\right) \\ & - 2b_g b_r\left(a_g b_r - b_g a_r\right)\left(b_b - c_b\right)\left(a_b - 2b_b + c_b\right).\end{aligned}$$

We can then calculate the blue star points, for example

$$B_{03} = \left[\frac{\begin{pmatrix} b_g b_r d_b\left(a_g b_r - b_g a_r - a_g c_r + c_g a_r + b_g c_r - c_g b_r\right)\cdot \\ \cdot\left(\left(\left(b_b - c_b\right)^2 + c_b d_b\right) u_b - 2c_b\left(b_b - c_b\right) v_b\right)\end{pmatrix}}{E_0 d_b u_b + F_0 c_b v_b}, 0\right]$$

from which clearly B_{03} lies on l_3. The computations are similar for the other indices, and the other colours. $\square$

3 Explicit examples and some conjectures

3.1 An example over $\mathbb{Q}\left(\sqrt{30}, \sqrt{217}, \sqrt{741}, \sqrt{2470}, \sqrt{82\,297}\right)$

We will now explore in detail a particular triangle which has both blue, red and green Incenters; for us this is not only an important tool for checking the consistency of our formulas, but also a way to get a sense of the level of complexity of various constructions. In fact this kind of explicit calculation of examples is much to be encouraged in this subject: especially as working over concrete fields, including finite fields and explicit extension fields of the rationals, allows us to appreciate the number theoretic aspects of our geometrical investigations. For example, finding a triangle with blue, red and green Incenters approximately is easy with a geometry package: finding a concrete example and working out all the points precisely is more challenging.

In particular we were unable, despite a reasonable computer search, to find *any* triangles with purely rational points that have blue, red and green Incenters! We would like to thank Michael Reynolds for his contributions to this search. So we tentatively conjecture that *there are no such triangles*.

In any case, to get an explicit example we use an algebraic extension field of the rationals; so by $\sqrt{30}$ we mean an appropriate symbol in the extension field $\mathbb{Q}\left(\sqrt{30}\right)$ etc.. Note that although our use of square roots is entirely algebraic, our representation of these square roots as approximate rational numbers (we prefer to avoid discussion of "real numbers"), necessarily brings an *approximate aspect into our diagrams*.

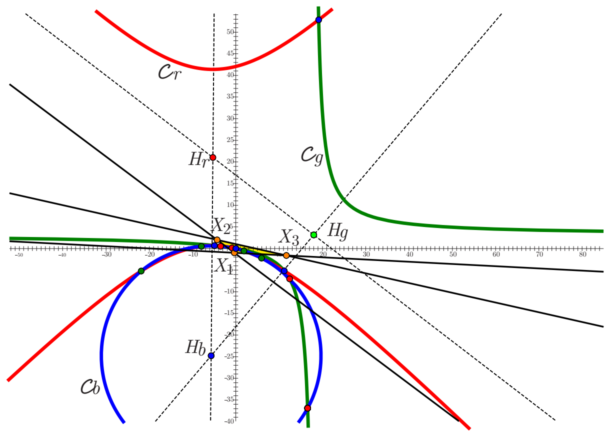

Figure 7: *An example triangle* $\overline{X_1X_2X_3}$

Example 1 *One may check that the basic Triangle with points*

$$X_1 \equiv [-21/59, -58/59], \quad X_2 \equiv [-13/3, 2] \quad \textit{and}$$

$$X_3 \equiv [35/3, -8/5]$$

in $\mathbb{Q}\left(\sqrt{30}, \sqrt{217}, \sqrt{741}, \sqrt{2470}, \sqrt{82\,297}\right)$ *has both blue, red and green Incenters. After translation by* $(21/59, 58/59)$ *we obtain* $\widetilde{X}_1 = [0,0]$, $\widetilde{X}_2 = [-704/177, 176/59]$ *and* $\widetilde{X}_3 = [2128/177, -182/295]$. *The matrix* N *and its inverse* M, *where*

$$N = \begin{pmatrix} -\frac{704}{177} & \frac{176}{59} \\ \frac{2128}{177} & -\frac{182}{295}\end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix} \qquad \textit{and}$$

$$M = N^{-1} = \begin{pmatrix} \frac{13}{704} & \frac{5}{56} \\ \frac{95}{264} & \frac{5}{42}\end{pmatrix}$$

respectively send $[1,0]$ *and* $[0,1]$ *to* $\widetilde{X}_2$ *and* $\widetilde{X}_3$, *and* $\widetilde{X}_2$ *and* $\widetilde{X}_3$ *to* $[1,0]$ *and* $[0,1]$. *From now on we discuss only the standard triangle* $\overline{A_1A_2A_3}$ *associated to* $\overline{X_1X_2X_3}$; *to convert back into the original coordinates, we would multiply by* N *and translate by* $(-21/59, -58/59)$. *The bilinear forms in these new standard coordinates, for the blue, red and green* *geometries respectively, are given by matrices*

$$D_b = N\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}N^T = \begin{pmatrix}\frac{774\,400}{31\,329} & -\frac{7778\,848}{156\,645}\\ -\frac{7778\,848}{156\,645} & \frac{113\,507\,716}{783\,225}\end{pmatrix} = \begin{pmatrix}a_b & b_b\\ b_b & c_b\end{pmatrix}$$
$$D_r = N\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}N^T = \begin{pmatrix}\frac{216\,832}{31\,329} & -\frac{7202\,272}{156\,645}\\ -\frac{7202\,272}{156\,645} & \frac{112\,911\,484}{783\,225}\end{pmatrix} = \begin{pmatrix}a_r & b_r\\ b_r & c_r\end{pmatrix}$$
$$D_g = N\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}N^T = \begin{pmatrix}-\frac{247\,808}{10\,443} & \frac{2000\,768}{52\,215}\\ \frac{2000\,768}{52\,215} & -\frac{774\,592}{52\,215}\end{pmatrix} = \begin{pmatrix}a_g & b_g\\ b_g & c_g\end{pmatrix}.$$

The determinants of D_b, D_r and D_g are $\Delta_b = \frac{97\,140\,736}{87\,025}$ and $\Delta_r = \Delta_g = -\frac{97\,140\,736}{87\,025}$, while the mixed traces are $d_b = \frac{6724}{25}$, $d_r = \frac{6076}{25}$ and $d_g = -\frac{576}{5}$. The orthocenters of $\overline{A_1A_2A_3}$ are

$$H_b = \left[-\frac{8825\,537}{1019\,520}, -\frac{84\,337}{25\,488}\right], H_r = \left[\frac{87\,833\,227}{11\,214\,720}, \frac{55\,537}{25\,488}\right],$$
$$H_g = \left[\frac{7105}{3894}, \frac{377}{177}\right].$$

Blue, red and green Incenters exist over $\mathbb{F} = \mathbb{Q}\left(\sqrt{30}, \sqrt{217}, \sqrt{741}, \sqrt{2470}, \sqrt{82\,297}\right)$ and we may choose

$$u_b = \frac{1875\,104}{31\,329}, \quad v_b = \frac{14\,432}{177}, \quad w_b = \frac{873\,628}{4425}$$
$$u_r = \frac{17\,248}{156\,645}\sqrt{82\,297}, \quad v_r = \frac{2464}{885}\sqrt{217}$$
$$w_r = \frac{196}{4425}\sqrt{217}\sqrt{82297}$$
$$u_g = \frac{19\,712}{52\,215}\sqrt{2470}, \quad v_g = \frac{2816}{295}\sqrt{30}, \quad w_g = \frac{448}{295}\sqrt{741}.$$

Then the four blue Incenters, the four red Incenters and the four green Incenters of $\overline{A_1A_2A_3}$ respectively are

$$I_0^b = \left[-\frac{761}{590}, \frac{220}{413}\right] \qquad I_1^b = \left[\frac{5327}{10\,384}, -\frac{25}{118}\right]$$
$$I_2^b = \left[\frac{761}{2112}, \frac{25}{168}\right] \qquad I_3^b = \left[\frac{5327}{270}, \frac{220}{27}\right]$$

$$I_0^r = \begin{bmatrix} \frac{1}{22\,429\,440}(4032\,553\sqrt{217} - 20\,461\sqrt{217}\sqrt{82297} \\ +210\,343\sqrt{82\,297} - 76\,618\,507), \\ \frac{1}{50\,976}(2923\sqrt{217} - 7\sqrt{217}\sqrt{82297} \\ +133\sqrt{82\,297} - 30\,049) \end{bmatrix}$$
$$I_1^r = \begin{bmatrix} \frac{1}{22\,429\,440}(20\,461\sqrt{217}\sqrt{82297} - 4032\,553\sqrt{217} \\ +210\,343\sqrt{82\,297} - 76\,618\,507), \\ \frac{1}{50\,976}(7\sqrt{217}\sqrt{82297} - 2923\sqrt{217} \\ +133\sqrt{82\,297} - 30\,049) \end{bmatrix}$$
$$I_2^r = \begin{bmatrix} \frac{1}{22\,429\,440}(4032\,553\sqrt{217} + 20\,461\sqrt{217}\sqrt{82297} \\ -210\,343\sqrt{82\,297} - 76\,618\,507), \\ \frac{1}{50\,976}(7\sqrt{217}\sqrt{82297} + 2923\sqrt{217} \\ -133\sqrt{82\,297} - 30\,049) \end{bmatrix}$$
$$I_3^r = \begin{bmatrix} \frac{-1}{22\,429\,440}(4032\,553\sqrt{217} + 20\,461\sqrt{217}\sqrt{82297} \\ +210\,343\sqrt{82\,297} + 76\,618\,507), \\ \frac{-1}{50\,976}(7\sqrt{217}\sqrt{82297} - 2923\sqrt{217} \\ -133\sqrt{82\,297} - 30\,049) \end{bmatrix}$$

$$I_0^g = \begin{bmatrix} \frac{203}{7788}\sqrt{741} - \frac{247}{3894}\sqrt{30} + \frac{35}{3894}\sqrt{2470} - \frac{3211}{7788}, \\ \frac{13}{1239}\sqrt{2470} - \frac{29}{177}\sqrt{30} + \frac{20}{1239}\sqrt{741} - \frac{100}{177} \end{bmatrix}$$
$$I_1^g = \begin{bmatrix} \frac{247}{3894}\sqrt{30} - \frac{203}{7788}\sqrt{741} + \frac{35}{3894}\sqrt{2470} - \frac{3211}{7788}, \\ \frac{29}{177}\sqrt{30} + \frac{13}{1239}\sqrt{2470} - \frac{20}{1239}\sqrt{741} - \frac{100}{177} \end{bmatrix}$$
$$I_2^g = \begin{bmatrix} -\frac{247}{3894}\sqrt{30} - \frac{203}{7788}\sqrt{741} - \frac{35}{3894}\sqrt{2470} - \frac{3211}{7788}, \\ -\frac{29}{177}\sqrt{30} - \frac{13}{1239}\sqrt{2470} - \frac{20}{1239}\sqrt{741} - \frac{100}{177} \end{bmatrix}$$
$$I_3^g = \begin{bmatrix} \frac{247}{3894}\sqrt{30} + \frac{203}{7788}\sqrt{741} - \frac{35}{3894}\sqrt{2470} - \frac{3211}{7788}, \\ \frac{29}{177}\sqrt{30} - \frac{13}{1239}\sqrt{2470} + \frac{20}{1239}\sqrt{741} - \frac{100}{177} \end{bmatrix}.$$

The Incenter circle quadrances are

$$K_b = \frac{18\,154\,129\,609}{28\,196\,100}, \qquad K_r = -\frac{11\,681\,819\,191}{28\,196\,100}$$
$$K_g = \frac{1182\,272}{10\,443}.$$

The blue, red and green Incenter Circles themselves have respective equations

$$4840\,000x^2 - 19\,447\,120xy + 28\,376\,929y^2$$
$$+19\,447\,120x + 19\,447\,120y - 9723\,560 = 0$$
$$19\,360x^2 - 62\,524xy + 12\,103y^2$$
$$+62\,524x + 62\,524y - 31\,262 = 0$$
$$193\,600x^2 - 2572\,240xy + 4032\,553y^2$$
$$+2572\,240x + 2572\,240y - 1286\,120 = 0.$$

The four tangent lines t_{gj}^b are

$$t_{g0}^b = \langle 1570 : -11823 : 8323 \rangle$$
$$t_{g1}^b = \langle -127512 : -33761 : 58261 \rangle$$
$$t_{g2}^b = \langle -18216 : -11823 : 8323 \rangle$$
$$t_{g3}^b = \langle -1570 : 4823 : 8323 \rangle.$$

The meets of these four tangent lines agree with the following meets with the side lines of $\overline{A_1A_2A_3}$:

$$G^b_{01} \equiv t^b_{g0}t^b_{g1} = t^b_{g0}l_1 = \left[\frac{3500}{13\,393}, \frac{9893}{13\,393}\right]$$
$$G^b_{23} \equiv t^b_{g2}t^b_{g3} = t^b_{g2}l_1 = \left[-\frac{3500}{6393}, \frac{9893}{6393}\right]$$
$$G^b_{02} \equiv t^b_{g0}t^b_{g2} = t^b_{g0}l_2 = \left[0, \frac{1189}{1689}\right]$$
$$G^b_{13} \equiv t^b_{g1}t^b_{g3} = t^b_{g1}l_2 = \left[0, \frac{1189}{689}\right]$$
$$G^b_{03} \equiv t^b_{g0}t^b_{g3} = t^b_{g0}l_3 = \left[-\frac{8323}{1570}, 0\right]$$
$$G^b_{12} \equiv t^b_{g1}t^b_{g2} = t^b_{g1}l_3 = \left[\frac{8323}{18\,216}, 0\right].$$

The blue/red quad points Q^b_{rj} *associated to* $I^b_0, I^b_1, I^b_2, I^b_3$ *respectively are*

$$Q^b_{r0} = \left[\frac{18\,005\,811\,535}{21\,082\,889\,161}, -\frac{12\,129\,669\,559}{21\,082\,889\,161}\right]$$
$$Q^b_{r1} = \left[-\frac{18\,005\,811\,535}{9330\,605\,209}, \frac{12\,129\,669\,559}{9330\,605\,209}\right]$$
$$Q^b_{r2} = \left[\frac{18\,005\,811\,535}{14\,928\,733\,909}, \frac{12\,129\,669\,559}{14\,928\,733\,909}\right]$$
$$Q^b_{r3} = \left[\frac{18\,005\,811\,535}{45\,342\,228\,279}, \frac{12\,129\,669\,559}{45\,342\,228\,279}\right].$$

The respective blue/green quad points Q^b_{gj} *are*

$$Q^b_{g0} = \left[-\frac{4161\,500}{2654\,777}, \frac{11\,762\,777}{2654\,777}\right]$$
$$Q^b_{g1} = \left[-\frac{4161\,500}{12\,547\,777}, \frac{11\,762\,777}{12\,547\,777}\right]$$
$$Q^b_{g2} = \left[\frac{4161\,500}{10\,977\,777}, \frac{11\,762\,777}{10\,977\,777}\right]$$
$$Q^b_{g3} = \left[\frac{4161\,500}{20\,870\,777}, \frac{11\,762\,777}{20\,870\,777}\right].$$

The blue star lines are then

$$s^b_0 = Q^b_{r0}Q^b_{g0} = \langle 1796\,063\,533\,088 : 868\,804\,574\,039 : -1034\,074\,074\,039 \rangle$$
$$s^b_1 = \langle 272\,084\,614\,990 : 1199\,343\,574\,039 : -1034\,074\,074\,039 \rangle$$
$$s^b_2 = \langle 272\,084\,614\,990 : 868\,804\,574\,039 : -1034\,074\,074\,039 \rangle$$
$$s^b_3 = \langle 1796\,063\,533\,088 : 1199\,343\,574\,039 : -1034\,074\,074\,039 \rangle$$

and they meet at the blue star points

$$B_{01} = \left[\frac{165\,269\,500\,000}{927\,258\,959\,049}, \frac{761\,989\,459\,049}{927\,258\,959\,049}\right]$$
$$B_{23} = \left[-\frac{165\,269\,500\,000}{596\,719\,959\,049}, \frac{761\,989\,459\,049}{596\,719\,959\,049}\right]$$
$$B_{02} = \left[0, \frac{1034\,074\,074\,039}{868\,804\,574\,039}\right], B_{13} = \left[0, \frac{1034\,074\,074\,039}{1199\,343\,574\,039}\right]$$
$$B_{03} = \left[\frac{1034\,074\,074\,039}{1796\,063\,533\,088}, 0\right], B_{12} = \left[\frac{1034\,074\,074\,039}{272\,084\,614\,990}, 0\right].$$

Note the pleasant rationality of the previous objects.

3.2 An example over $\mathbb{F}_{13}$

Now we look at an example over a finite field.

Theorem 9 (Null quadrances incenters) *Suppose that the field* $\mathbb{F}$ *contains an element* i, *where* $i^2 = -1$, *and the characteristic of* $\mathbb{F}$ *is not* 2. *If*

$$K_b \equiv \frac{b_b(a_b - b_b)(b_b - c_b)}{\Delta_b} = K_r \equiv \frac{b_r(a_r - b_r)(b_r - c_r)}{\Delta_r} = K_g \equiv \frac{b_g(a_g - b_g)(b_g - c_g)}{\Delta_g} = 0$$

then the standard Triangle $\overline{A_1A_2A_3}$ *has four distinct blue, red and green Incenters.*

Proof. If $K_b = 0$ then from the definition of the blue incenter circle $\mathcal{C}_b$, which is $Q_b(H_b, X) = K_b$, $\mathcal{C}_b$ is a null circle, so it is a product of lines. Similarly, if $K_r = 0$ then $\mathcal{C}_r$ is a null circle, and if $K_g = 0$ then $\mathcal{C}_g$ is a null circle. These null lines have distinct direction vectors $(1, \pm i), (1, \pm 1)$ and $(1,0), (0,1)$ respectively, and they are never parallel since char$(\mathbb{F}) \neq 2$, so $i \neq \pm 1$. Therefore, any two null circles meet in exactly four points. □

Here is an example found by Michael Reynolds [6] which illustrates explicitly the above theorem.

Example 2 *The triangle* $\overline{X_1X_2X_3}$ *with points* $X_1 \equiv [3,4], X_2 \equiv [1,9]$ *and* $X_3 \equiv [12,3]$ *in* $\mathbb{F}_{13}$ *has four blue, red and green Incenters. In* $\mathbb{F}_{13}$ *the squares are* $0,1,3,4,9,10$ *and* 12, *and in particular* $-1 = 12 = 5^2$ *is a square. After translation by* $(3,4)$ *we obtain* $\widetilde{X}_1 = [0,0], \widetilde{X}_2 = [11,5]$ *and* $\widetilde{X}_3 = [9,12]$. *The matrix N and its inverse M*

$$N = \begin{pmatrix} 11 & 5 \\ 9 & 12 \end{pmatrix}, \qquad M = N^{-1} = \begin{pmatrix} 10 & 11 \\ 12 & 7 \end{pmatrix}$$

send $[1,0]$ *and* $[0,1]$ *to* $\widetilde{X}_2$ *and* $\widetilde{X}_3$, *and* $\widetilde{X}_2$ *and* $\widetilde{X}_3$ *to* $[1,0]$ *and* $[0,1]$ *respectively. The bilinear form in these new standard coordinates for the blue, red and green geometries* *respectively are*

$$D_b = N\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}N^T = \begin{pmatrix}3 & 3\\ 3 & 4\end{pmatrix}$$
$$D_r = N\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}N^T = \begin{pmatrix}5 & 0\\ 0 & 2\end{pmatrix}$$
$$D_g = N\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}N^T = \begin{pmatrix}6 & 8\\ 8 & 8\end{pmatrix}.$$

We can see immediately that $K_b = K_r = K_g = 0$ from the definitions

$$Q_b\left(H_b, I_{ir}\right) = \frac{b_b\left(a_b - b_b\right)\left(b_b - c_b\right)}{\Delta_b} \equiv K_b$$
$$Q_r\left(H_r, I_{ib}\right) = \frac{b_r\left(a_r - b_r\right)\left(b_r - c_r\right)}{\Delta_r} \equiv K_r$$
$$Q_g\left(H_g, I_{ir}\right) = \frac{b_g\left(a_g - b_g\right)\left(b_g - c_g\right)}{\Delta_g} \equiv K_g, \quad i = 0,1,2,3.$$

The four blue, red and green Incenters respectively are

$$I_0^b = [4,8], \quad I_1^b = [3,6], \quad I_2^b = [8,10], \quad I_3^b = [11,4]$$
$$I_0^r = [10,9], \quad I_1^r = [8,2], \quad I_2^r = [6,5], \quad I_3^r = [4,12]$$
$$I_0^g = [9,8], \quad I_1^g = [5,3], \quad I_2^g = [12,11], \quad I_3^g = [2,4]$$

and the blue, red and green Incenter Circles respectively have equations

$$\mathcal{C}_b : (y - x + 1)(x + 3y - 1) = 0$$
$$\mathcal{C}_r : (x - 6y)(x + 6y) = 0$$
$$\mathcal{C}_g : (x + 2y - 2)(x + 5y - 5) = 0.$$

From Michael Reynolds' computer investigations, we tentatively conjecture that for finite fields $\mathbb{F}_p$ where $p \equiv 3 \bmod 4$, there are *no* triangles which have both blue, red and green Incenters, and for finite fields $\mathbb{F}_p$ where $p \equiv 1 \bmod 4$, blue, red and green Incenters exists precisely when $K_b = K_r = K_g = 0$, as in the above example.

4 Spieker circles and Nagel circles

Now we recall from [5] that the central dilation $\delta_{-1/2}$ about the centroid takes the Orthocenter to the Circumcenter, and the Incenters to the *Spieker centers*. In standard coordinates

$$\delta_{-1/2}([x,y]) = (1/2)[1 - x, 1 - y].$$

The inverse central dilation δ_{-2} takes the Orthocenter to the *De Longchamps point* X_{20}, and takes the Incenters to the *Nagel points*. In standard coordinates

$$\delta_{-2}([x,y]) = [1 - 2x, 1 - 2y].$$

Theorem 10 (Spieker circles) *If a triangle has four blue Incenters I_0^b, I_1^b, I_2^b and I_3^b, then the four blue Spieker centers all lie both on a red Spieker circle with center the red Circumcenter C_r, and on a green Spieker circle with center the green Circumcenter C_g. If both say blue and red Incenters exist, then all 8 blue and red Spieker points lie on the same green circle. The same holds for the other colours.*

Proof. We see that if we use the central dilation formula to transform Incenter circles centred at the Orthocenters, we get the Spieker circles centred at Circumcenters, so this theorem is a direct consequence of the Incenter circles theorem and the fact that a dilation preserves circles of any colour. □

Here are the formulas for the coloured Circumcenters in standard coordinates:

$$C_b = \frac{1}{2\Delta_b}\left[c_b\left(a_b - b_b\right), a_b\left(c_b - b_b\right)\right]$$
$$C_r = \frac{1}{2\Delta_r}\left[c_r\left(a_r - b_r\right), a_r\left(c_r - b_r\right)\right]$$
$$C_g = \frac{1}{2\Delta_g}\left[c_g\left(a_g - b_g\right), a_g\left(c_g - b_g\right)\right].$$

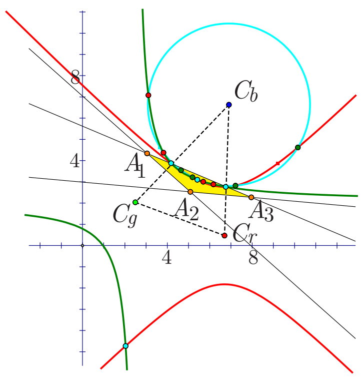

Figure 8: *Blue, red and green Speiker circles*

Theorem 11 (Nagel circles) *If a triangle has four blue Incenters I_0^b, I_1^b, I_2^b and I_3^b, then the four blue Nagel centers all lie both on a red Nagel circle with center the red De Longchamps point X_{20r}, and on a green Nagel circle with center the green De Longchamps point X_{20g}. If both say blue and red Incenters exist, then all 8 blue and red Nagel points lie on the same green circle. The same holds for the other colours.*

Proof. In the same fashion as in the previous theorem, if we use the inverse central dilation δ_{-2} to transform Incenter circles centred at the Orthocenters, we get the Nagel circles centred at De Longchamps points. □

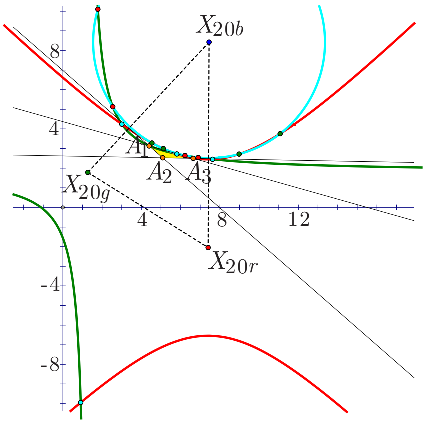

Figure 9: *Blue, red and green Nagel circles*

Here are the formulas for the blue, red and green De Longchamps points:

$$X_{20b} = \frac{1}{\Delta_b}\left[b_b^2 - 2b_b c_b + a_b c_b, b_b^2 - 2a_b b_b + a_b c_b\right]$$

$$X_{20r} = \frac{1}{\Delta_r}\left[b_r^2 - 2b_r c_r + a_r c_r, b_r^2 - 2a_r b_r + a_r c_r\right]$$

$$X_{20g} = \frac{1}{\Delta_g}\left[b_g^2 - 2b_g c_g + a_g c_g, b_g^2 - 2a_g b_g + a_g c_g\right].$$

In Figure 10 we see the relations between the three coloured Orthocenters, Circumcenters and De Longchamps points. The lines joining these are the three coloured Euler lines. Note that the centroids of the triangles of Orthocenters, Circumcenters and De Longchamps points all agree with the centroid G of the original triangle $\overline{A_1A_2A_3}$. We conclude with a simple observation about De Longchamps points.

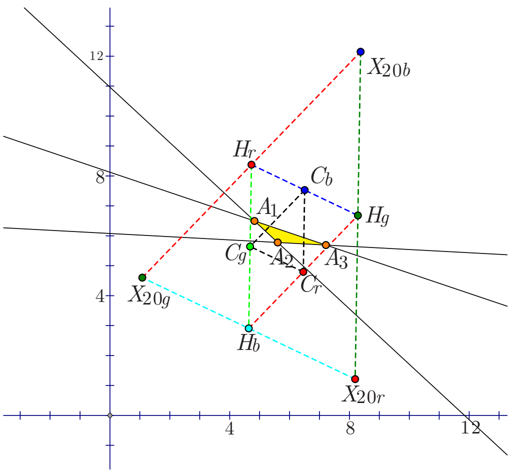

Figure 10: *Blue, red, green Orthocenters, Circumcenters and De Longchamps points*

Theorem 12 (Orthocenters as midpoints) *For any triangle, a coloured orthocenter H is the midpoint of the two De Longchamps points X_{20} of the other two colours.*

Proof. This follows by considering the action of the central dilation δ_{-2} which takes the circumcenter C_i to the orthocenter H_i, and the orthocenter H_i to the De Longchamps point X_{20i}. $\square$

References

[1] M. W. HASKELL, The Construction of Conics under given conditions, *Bulletin of the Amer. Math. Soc.* **11 (5)** (1905), 268–273.

[2] C. KIMBERLING, *Triangle Centers and Central Triangles*, vol. 129 Congressus Numerantium, Utilitas Mathematica Publishing, Winnepeg, MA, 1998.

[3] C. KIMBERLING, *Encyclopedia of Triangle Centers*, `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[4] C. KIMBERLING, Major Centers of Triangles, *Amer. Math. Monthly* **104** (1997), 431–438.

[5] N. LE, N. J. WILDBERGER, Universal Affine Triangle Geometry and Four-fold Incenter Symmetry, *KoG* **16** (2012), 63–80.

[6] M. REYNOLDS, *Personal Communication*, 2013.

[7] N. J. WILDBERGER, *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg Books, Sydney, 2005.

[8] N. J. WILDBERGER, Affine and projective metrical geometry, *arXiv: math/0612499v1*.

[9] N. J. WILDBERGER, Chromogeometry, *Mathematical Intelligencer* **32 (1)** (2010), 26–32.

[10] N. J. WILDBERGER, Chromogeometry and Relativistic Conics, *KoG* **13** (2009), 43–50.

[11] B. M. WOODS, The Construction of Conics under given conditions, *Amer. Math. Monthly* **21 (6)** (1914), 173–180.

Nguyen Le
e-mail: n.h.le@unsw.edu.au

Norman John Wildberger
e-mail: n.wildberger@unsw.edu.au

School of Mathematics and Statistics UNSW
Sydney 2052 Australia