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On Central Collineations which Transform a Given Conic to a Circle

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ABSTRACT

In this paper we prove that for a given axis the centers of all central collineations which transform a given proper conic c into a circle, lie on one conic cc confocal to the original one. The conics c and cc intersect into real points and their common diametral chord is conjugate to the direction of the given axis.

Furthermore, for a given center S the axes of all central collineations that transform conic c into a circle form two pencils of parallel lines. The directions of these pencils are conjugate to two common diametral chords of c and the confocal conic through S that cuts c at real points.

Finally, we formulate a theorem about the connection of the pair of confocal conics and the fundamental elements of central collineations that transform these conics into circles.

Key words: central collineation, confocal conics, Apollonian circles

MSC 2010: 51N05, 51A05, 51M15

O perspektivnim kolineacijama koje danu koniku preslikavaju u kružnice

SAŽETAK

U članku je dokazano da središta svih perspektivnih kolineacija koje s obzirom na zadanu os preslikavaju danu koniku c u kružnicu, leže na jednoj konici cc konfokalnoj s početnom konikom. Konike c i cc imaju realna sjecišta, a njihova zajednička dijametralna tetiva konjugirana je smjeru zadane osi kolineacije.

Nadalje je dokazano da osi svih prespektivnih kolineacija koje s obzirom na zadano središte S preslikavaju danu koniku c u kružnicu, čine dva pramena paralelnih pravaca. Smjerovi tih pramenova konjugirani su zajedničkim dijametralnim tetivama konike c i njoj konfokalne konike koja sadrži točku S i realno siječe c.

Na kraju je formuliran teorem koji govori o vezi para konfokalnih konika i temeljnih elemenata perspektivnih kolineacija koje te konike preslikavaju u kružnice.

Ključne riječi: perspektivna kolineacija, konfokalne konike, Apolonijeve kružnice

1 Introduction

Central collineation is a classical and widely studied transformation in projective geometry. Basic properties of this transformation can be found in any textbook of the field, including the well-known fact that it can transform any proper conic into a circle. The standard way of applying this fact in drawing is that a given conic c, the center S and the axis a of the central collineation is chosen in a proper way, using the most suitable, simplest transformation to transfer c to a circle.

In this paper we consider this problem in a more constrained way. Given a conic c and a line a, we try to find all the centers S with which the central collineation with center S and axis a transforms c into a circle. Alternatively, given a conic c and a point S, we try to find all the axes a with which the central collineation with center S and axis a transforms c into a circle. These two problems are solved in Section 3 and 4. Before these results, we briefly present the necessary facts about pencils of circles and confocal conics in Section 2.

The aim of this study is not purely theoretic. Our future purpose is to classify special transformations, called quadratic projections, which are geometric abstractions of omnidirectional vision tools and cameras [2]. This practical equipment is of central importance in robotics, and the geometry of these tools are heavily influenced by the projective geometric issues studied in the present paper.

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2 Pencils of circles and confocal conics

In this section we collect some basic properties of a pencil of circles and a range of confocal conics. Both structures will be used throughout the following sections, so it may be worth to recall the necessary facts about them. These issues can be found in several books of the field, e.g. in [1], [3], [4], [5], [6], [7], [8].

A *pencil of circles* is the set of circles which pass through two given points. It is an elliptic, hyperbolic or parabolic pencil if these intersection points are real and different, imaginary or coinciding, respectively. See fig. 1.

The circles of elliptic, hyperbolic and parabolic pencils cut the central line into the pairs of elliptic, hyperbolic and parabolic involution, respectively. Every point on the radical axis has the same point's circle power with respect to all circles of the pencil. On the radical axis all circles of the pencil induce the same polar involution that is hyperbolic, elliptic or parabolic, if the pencil is elliptic, hyperbolic or parabolic, respectively. Every pencil contains its radical axis and the line at infinity as a splitting circle. Only the hyperbolic pencils contain two imaginary circles that are two pairs of isotropic lines with real intersection points that are called the Poncelet's points.

Appolonian circles are two pencils of circles such that every circle of the first pencil cuts every circle of the second pencil orthogonally, and vice versa. If the first pencil is elliptic, the second is hyperbolic. If the first pencil is parabolic, the second is parabolic, too. See figures 2, 3.

All circles of one pencil induce the same polar involution on its radical axis. This involution is the same as the intersection involution that the corresponding Apollonian pen-

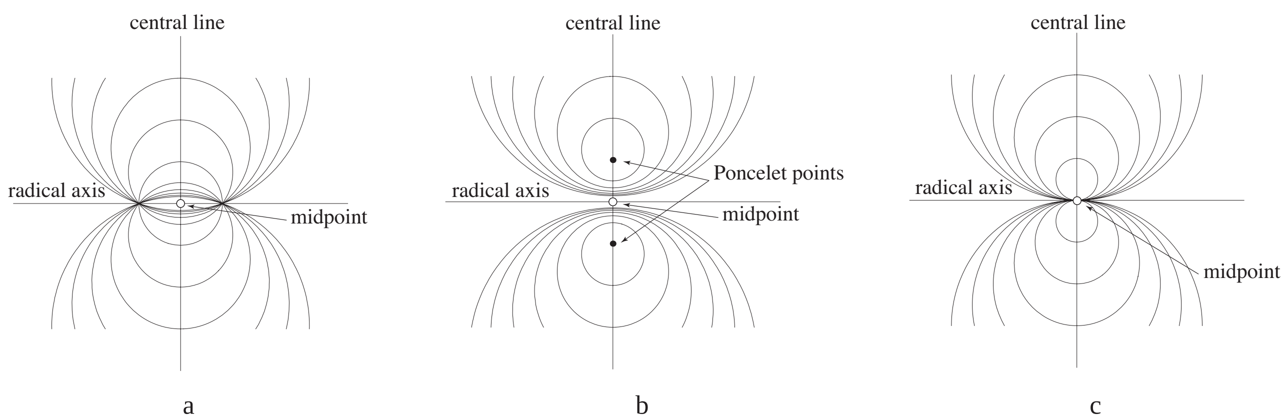

Figure 1: *An elliptic, hyperbolic and parabolic pencil of circles are shown in figures a, b and c, respectively.*

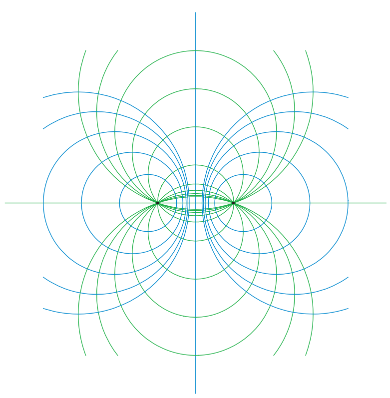

Figure 2: *One family of elliptic and hyperbolic pencils of orthogonal circles*

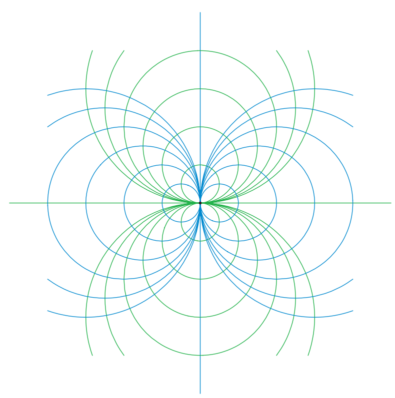

Figure 3: *One family of two orthogonal parabolic pencils*

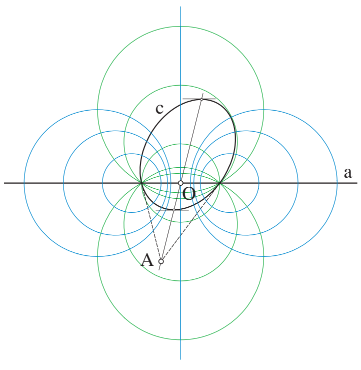

Figure 4: *A is the pole of a with respect to c, and O is the midpoint of the orthogonal pencils of circles. The conic c and the line a with real and different, imaginary or coinciding intersection points, together with the corresponding Apollonian circles, are shown in figures a, b and c, respectively.*

cil induces on its central line. For Apollonian circles the Poncelet's points of the hyperbolic pencil are the Laguerre's points of the elliptic pencil.

For every conic c and a line a such that A^{∞} (the point at infinity of the line a) is an external point of c, exists one family of orthogonal circles such that one pencil cuts the line a at the same points as the conic c, and another cuts the line a into the pairs of polar involution induced by c on a, see fig. 4.

The family of *confocal conics* is a range of conics defined by two pairs of isotropic tangent lines that always have two real intersection points called real foci F_1, F_2. If one focus is the point at infinity, all confocal conics are parabolas. In all other cases a confocal range consists of ellipses and hyperbolas. Through every point P in the plane, two conics of a confocal range pass, and they cut orthogonally. If one of these conics is an ellipse, the other one is a hyperbola and vice versa.

The tangent lines at P to these conics bisect the angles between the tangents from P to any other conic of the confocal system (these tangents can be real and different, imaginary[1] or coinciding). The lines PF_1 and PF_2 are equally inclined to the tangents from P to any one of the conics of the system, see figures 5, 6.

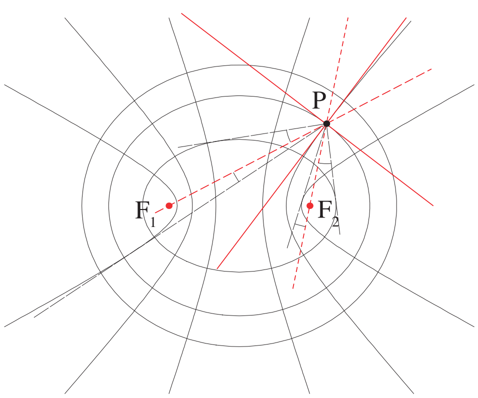

Figure 5: *The range of confocal ellipses and hyperbolas*

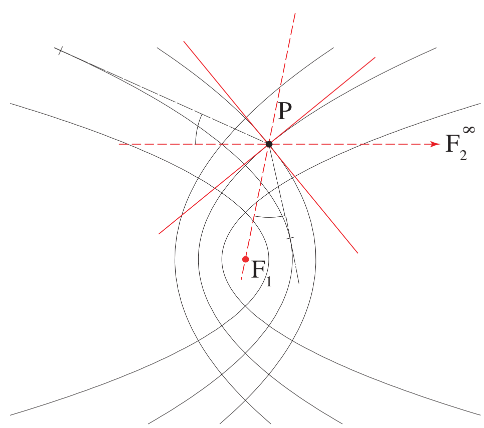

Figure 6: *The range of confocal parabolas*

[1] About the real bisectors of imaginary lines see [6, p. 70].

3 Collineations with given axis

Let c and a be a given conic and a line, and let A^∞ be the point at infinity on the line a. In this section we consider the following problem: where are the centers of all central collineations with a given axis a that transform a conic c into circles?

Lemma 1 *If a point A^∞ lies on the conic c or if it is an internal point of c, there does not exist any central collineation with the axis a that transforms c into a circle.*

PROOF: Since the real or imaginary character of points and lines is invariant under a central collineation, then it transforms interior and exterior of a given conic c into the interior and exterior of the image of c. For every circle, A^∞ is an external point. Therefore, all parabolas and hyperbolas that pass through A^∞ as well as all hyperbolas with A^∞ as an internal point, could not be transformed into a circle by any central collineation with the axis a. $\square$

Theorem 1 *For a given straight line a and a conic c, where $A^\infty \in Ext\, c$, the centers of all central collineations with the axis a that transform c into circles lie on one conic cc that is confocal to c. If a is not parallel to any axis of c, the conic cc is an ellipse, hyperbola or parabola, if c is a hyperbola, ellipse or parabola, respectively.*

PROOF:

• Let a and c ($A^\infty \in Ext\, c$) intersect in two different points D_1, D_2, let $\mathcal{P}(c,a)$ be a pencil of circles that intersect the line a at the same points D_1 and D_2 and let O be the mid-point of $\mathcal{P}(c,a)$. The line a is the radical axis of $\mathcal{P}(c,a)$, the line aa through O orthogonal to a is its central line and the pencil is elliptic or hyperbolic if D_1, D_2 are real or imaginary points, respectively. Let A be the pole of a with respect to c. OA is the diameter of c conjugate to the direction of a. Let T and $\overline{T}$ be the intersection points of the diameter OA and the conic c. If c is a parabola one of these points is the point at infinity. See fig. 7.

For every circle $c' \in \mathcal{P}(c,a)$ there are two central collineations with the axis a that transform c into c'. These collineations transform the diametral chord $T\overline{T}$ of c into the diametral chord T_1T_2 of c' lying on aa. The intersections of corresponding rays TT_1, $\overline{T}T_2$ or TT_2, $\overline{T}T_1$ are the centers S^1 and S^2 of these collineations, respectively. S^1 and S^2 are collinear with A and A', where A' is the pole of c' with respect to a, because polarity is invariant under a central collineation. Since the points T_1 and T_2 correspond involutory (the pair of intersections of $c' \in \mathcal{P}(c,a)$ and aa), then varying c', the mappings $TT_1 \longrightarrow \overline{T}T_2$ and $TT_2 \longrightarrow \overline{T}T_1$ define the same projectivity $(T)\,\overline{\wedge}\,(\overline{T})$. The resulting curve of this projectivity is a conic cc that contains the centers of all central collineations with the axis a which transform c into circles. The points T and $\overline{T}$ that correspond to the splitting circle (c' breaks up into the line a and the line at infinity) are excluding as the centers of requested central collineations.

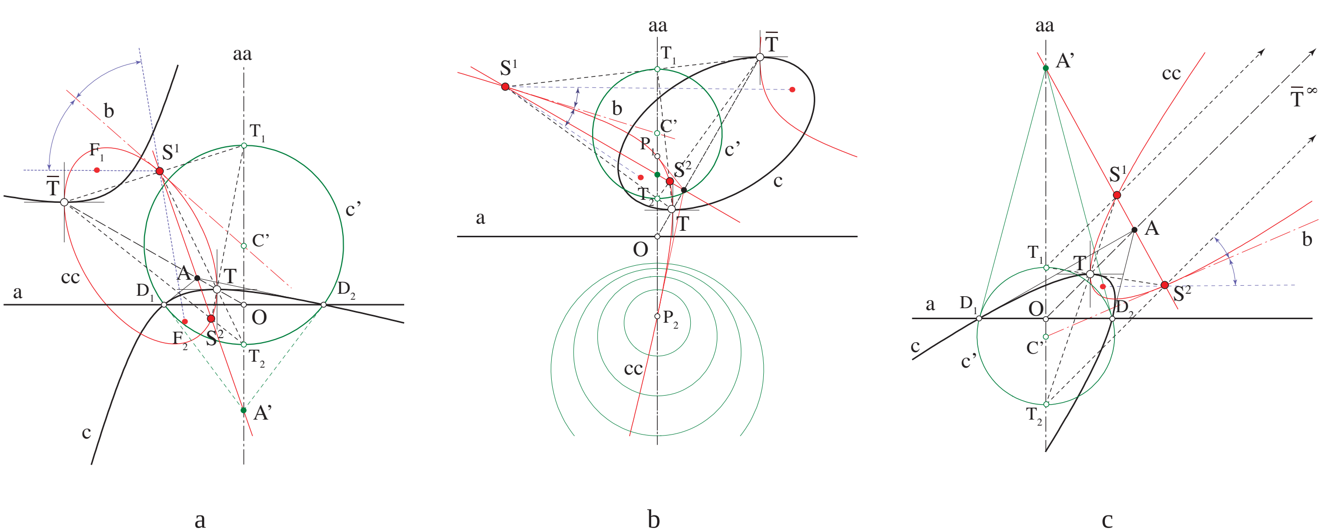

Figure 7: *In figure a, conic c is a hyperbola, D_1, D_2 are real points and cc is an ellipse that cuts aa at imaginary points. In figure b, conic c is an ellipse, D_1, D_2 are imaginary points and cc is a hyperbola that cuts aa at Poncelet's points of $\mathcal{P}(c,a)$. In figure c, conic c is a parabola, D_1, D_2 are real points and cc is a parabola that cuts the line aa at the pair of imaginary points.*

The point S^1 is the intersection of common tangents of c and c' that can be real or imaginary. Therefore, a line b that passes through S^1 and C', where C' is the center of the circle c', bisects the angle $\measuredangle F_1S^1F_2$ (where F_1, F_2 are the foci of c) and it is the tangent line of cc at S^1. Namely, if we suppose that b cuts cc at any other point $S \neq S^2$ (S^1S^2 cuts aa at $A' \neq C'$), then S would be the third solution that transform c into c' which is impossible according to the previous considerations. Thus, the tangent line at every point S^1 of the conic cc bisects the angle $\measuredangle F_1S^1F_2$, i.e. F_1 and F_2 are the foci of cc, see fig. 7. The conic cc cuts the line aa at two double points of the involution $T_1 \longleftrightarrow T_2$, that are imaginary if $\mathcal{P}(c,a)$ is elliptic or real Poncelet's points P_1, P_2 if it is hyperbolic.

Since the conics c and cc intersect each other in the real points T, $\overline{T}$, cc is a hyperbola, ellipse or parabola, if c is an ellipse, hyperbola or parabola, respectively.

• Let a be the tangent of c with a touching point O and let T be another intersection point of c and the diameter of c through O (if c is a parabola, T is the point at infinity). Let c' be a circle with the center C' that touches a at the point O and let T' be another intersection point of the diameter of c' through O and c'. It is clear that there exists a unique central collineation with the center S and the axis a that transform T and T' and the conic c into c', see fig. 8. Since the conics c and c' have common tangent lines t_1, t_2 through S (real and different, imaginary or coinciding) they also have a common bisector b of these tangents through S. This line b also bisects the angle $\measuredangle F_1SF_2$ and is the tangent at S of a conic cc that is confocal to c and passes through the points T and O. Thus, every point on the conic cc is the center of one central collineation that transforms c into a circle. On the other hand, every circle $c' \in \mathcal{P}(c,a)$ defines the unique solution S that is the touching point of the tangent b from C' to the conic cc (another tangent through C' to cc is the line aa with the touching point O), i.e. all solutions lie on the conic cc.

Since the conics c and cc intersect each other in the real points O and T, cc is a hyperbola, ellipse or parabola, if c is an ellipse, hyperbola or parabola, respectively.

• If the line a is perpendicular to one axis of the conic c, all centers of central collineations that transform c into circles lie on the central line aa of the pencil $\mathcal{P}(c,a)$. Namely, for every $c' \in \mathcal{P}(c,a)$ there exists $S \in aa$ that is the intersection of common tangent lines of c and c' (these tangent lines can be real or imaginary). The point S is the center of central collineation that transforms c into c'. The line aa is the part of a splitting conic that is confocal to c. In the cases when c is an ellipse or hyperbola, the confocal conic splits into the axes of c, and if c is a parabola it splits into its axis and the line at infinity. □

In the case when c is an ellipse, the conic cc is a hyperbola with two real points at infinity and the pencil of circles $\mathcal{P}(c,a)$ contains two circles into which the given ellipse is transformed by an affinity. The construction of these circles is a solution of a classical task in constructive geometry: The centers of these circles are the intersection points of the line aa and one circle, where the diameter of this circle is formed by the intersection points of the given line a and the axes of the ellipse.

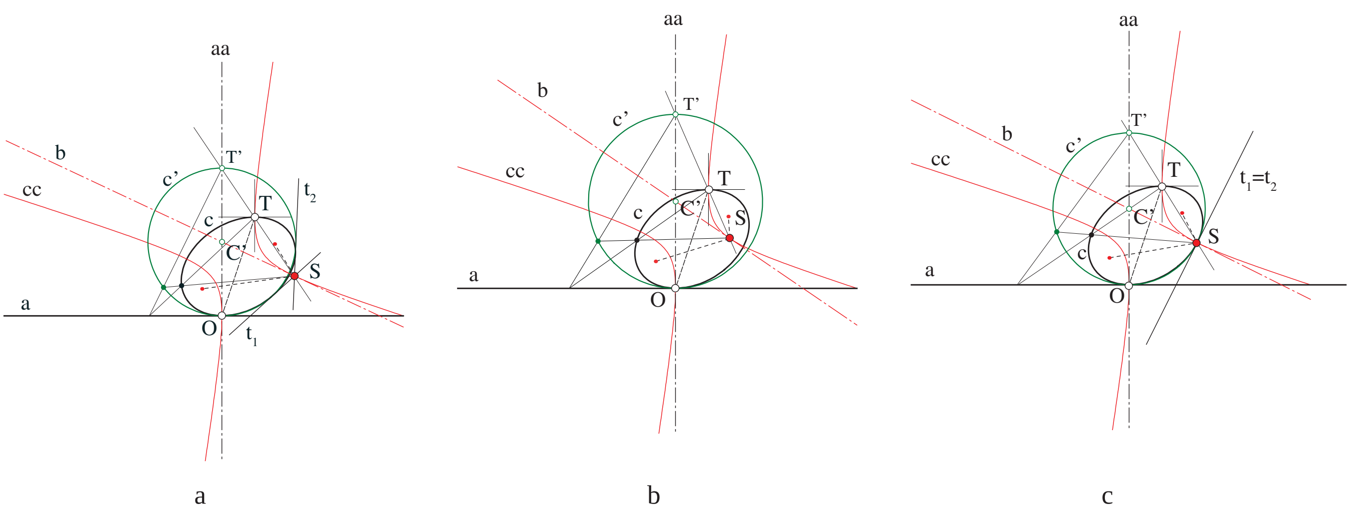

Figure 8: *The illustrations of the above described construction for $S \in \text{Ext}\, c$, $S \in \text{Int}\, c$ and $S \in c$ are shown in figures a, b and c, respectively.*

4 Collineations with given center

For a given point S and a conic c, the following problem is considered in this section: where are the axes of all central collineations with given center S that transform a given proper conic c into circles?

In the following, by cc_S we denote the confocal conic to c that passes through S and cuts c at real points.

Theorem 2 *For a given point S and a conic c, the axes of all central collineations with the center S that transform c into circles form two pencils of parallel lines with directions conjugate to two common diametral chords of c and cc_S.*

PROOF: Let c and S be any conic and a point and let b be the bisector of the tangent lines of c through S. If c is an ellipse or parabola, b bisects the internal angle $\measuredangle F_1SF_2$ and if c is a hyperbola b bisects the external angle $\measuredangle F_1SF_2$. For every point $C' \in b$ there exists one circle c' such that c and c' have common tangent lines t_1, t_2 through S (real and different, imaginary or coinciding). Let the common diametral chords of c and cc_S be $X\overline{X}$ and $Y\overline{Y}$ (if c is a parabola $\overline{X}$ and $\overline{Y}$ coincide with the point at infinity) and let x and y be the tangent lines at X and Y, respectively. The line b is the tangent line of cc_S at S. Let us consider any two lines $a_1 \parallel x$, $S \notin a_1$, and $a_2 \parallel y$, $S \notin a_2$ and let O_1 and O_2 be the midpoints of the polar involutions that conic c induces on a_1 and a_2, respectively. According to Theorem 1, there are two central collineations with the center S and axes a_1 and a_2 that transform c into circles c'_1 and c'_2, respectively. The centers of these circles are the intersection points of b and the lines aa_1 and aa_2, perpendicular to a_1 and a_2 through the points O_1 and O_2, respectively. Thus, for every c and S there are two pencils of parallel lines such that every central collineation with the center S and the axis that belongs to one of the pencils transforms c into a circle. See fig. 9.

On the other hand, if c is transformed to a circle by a central collineation with the center S then the polar involution induced by c on the vanishing line is transformed to the circular involution on the line at infinity. The isotropic lines through S cut c into four imaginary points and among six lines that join them only two are real. These two real sides of the complete quadrangle (determined with the four intersections of c and the isotropic lines through S) are the vanishing lines v_1 and v_2 of the requested central collineations. Thus, there are only two directions for axis of central collineatios with the center S that transform c into circles. The lines v_1, v_2 pass through the pole of b with respect to c and must be excluded from the pencils (a_1) and (a_2) as the axes of central collineations because they correspond with the isotropic lines through S in the correspondences $a_1 \longrightarrow c'_1$ and $a_2 \longrightarrow c'_2$, respectively. $\square$

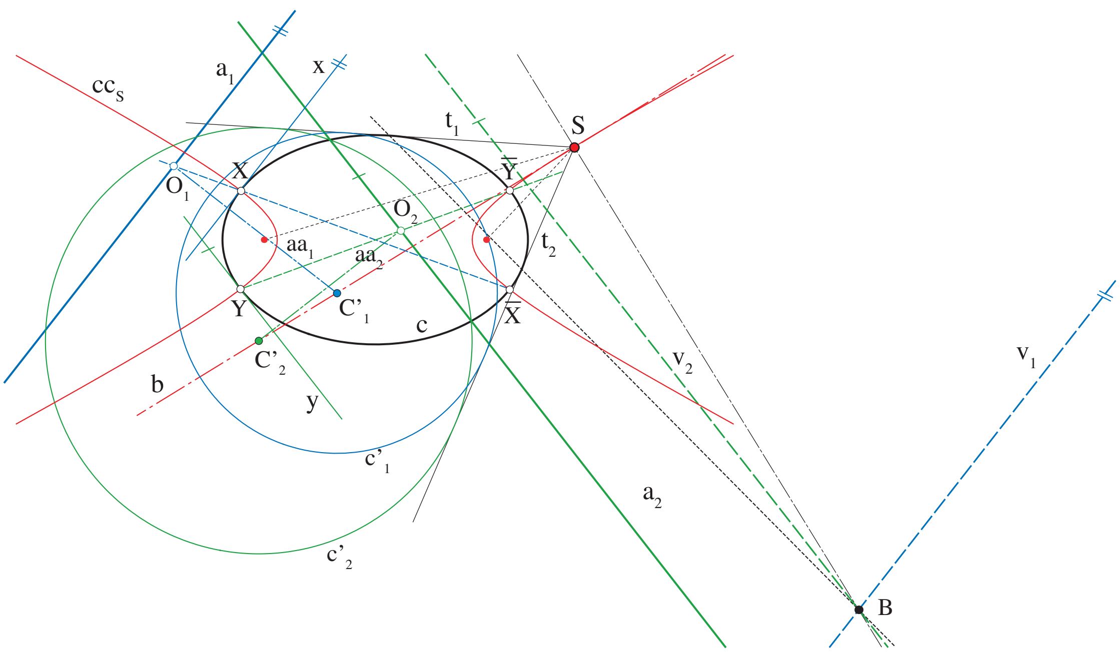

Figure 9: *Circles c'_1 and c'_2 are the images of c under the central collineations with the center S and axes a_1 and a_2 with vanishing lines v_1 and v_2, respectively. For any other central collineation with center S that transforms c into a circle, the vanishing line is either v_1 or v_2.*

5 Joint collineations of confocal conics

Finally we provide a consequence of the above discussed theorems proving a strong relationship among Apollonian circles, confocal conics and the fundamental elements of central collineations which transform them to circles.

If c_1 and c_2 are proper conics with a center C and common diametral chords $X\overline{X}$ and $Y\overline{Y}$, then for every point $A \in X\overline{X}$, $A \neq C$ ($B \in Y\overline{Y}$, $B \neq C$) the pair of polar lines a_1, a_2 (b_1, b_2) of A (B) with respect to c_1 and c_2, respectively, are orthogonal lines intersecting at a point $\overline{A} \in X\overline{X}$ ($\overline{B} \in Y\overline{Y}$) that is conjugate to A (B) with respect to the conics c_1 and c_2. See fig. 10.

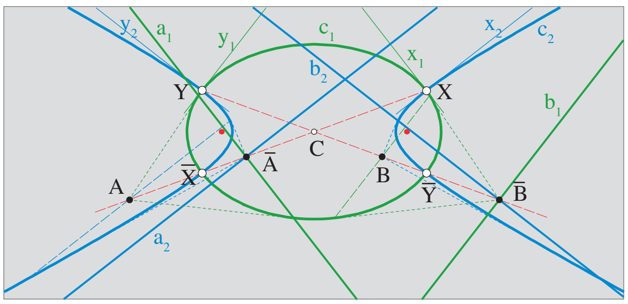

Figure 10: *The polar lines a_1 and a_2 (b_1 and b_2) are parallel to the tangent lines at X (Y) of conics c_1 and c_2, respectively.*

Theorem 3 *Let c_1 and c_2 be proper confocal conics with a center C and common diametral chords $X\overline{X}$ and $Y\overline{Y}$. Let a_1 and a_2 (b_1 and b_2) be any pair of lines conjugate to $X\overline{X}$ ($Y\overline{Y}$) with respect to c_1 and c_2, respectively, intersecting at a point $\overline{A} \in X\overline{X}$, $\overline{A} \neq C$ ($\overline{B} \in Y\overline{Y}$, $\overline{B} \neq C$). Then, the centers S_1, S_2 of all central collineations with axis a_1, a_2 (b_1, b_2) that transform c_1, c_2 into circles lie on the conic c_2, c_1, respectively.*

Varying $S_1 \in c_1$ and $S_2 \in c_2$ the image circles form Apollonian circles with central lines a_2, a_1 (b_2, b_1).

The tangent lines at S_1 and S_2 of conics c_1 and c_2 cut the central lines a_2, a_1 (b_2, b_1) into the centers of circles that correspond to the touching points S_1 and S_2 as the centers of collineations, respectively.

PROOF: The proof follows directly from the proof of theorem 1, properties that are illustrated in fig. 4 and fig. 10 and the fact that confocality of conics and orthogonality of circles are symmetric relations. See figures 11, 12 and 13. □

In the following figures the properties listed in theorem 3 are presented. These figures as well as all previous are produced in the program *Mathematica 7*.

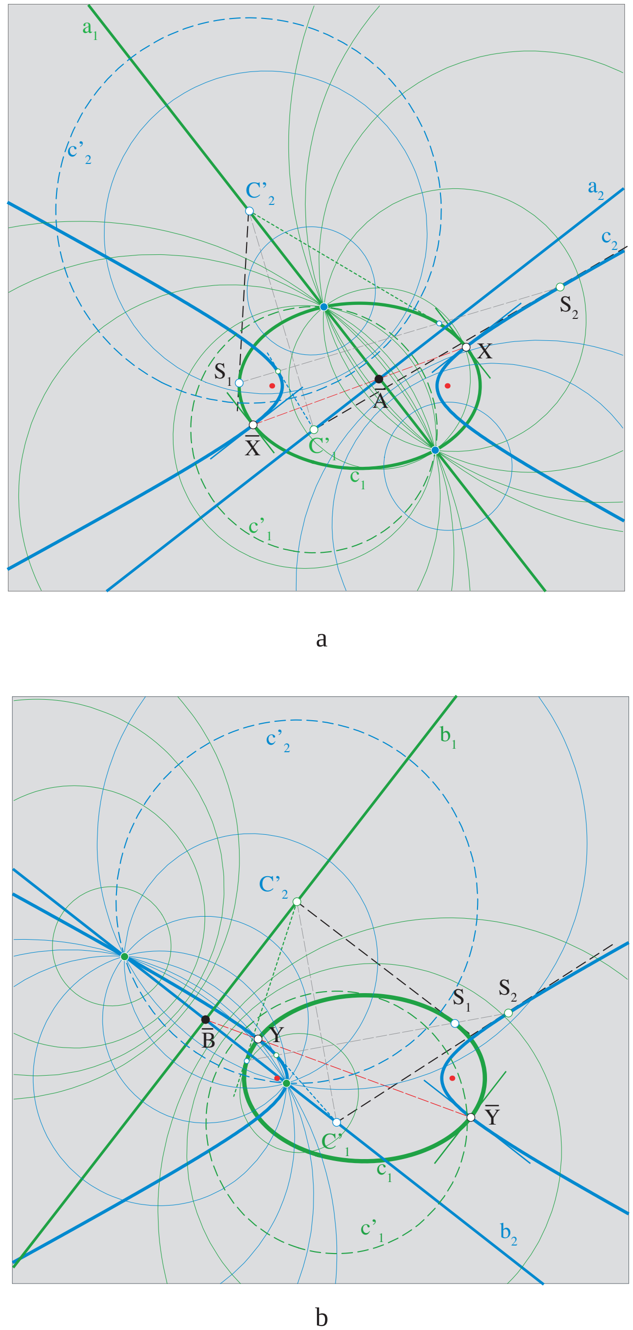

Figure 11: *All central collineations with the green (blue) axis and a center on the blue (green) conic transform the green (blue) conic into green (blue) circles. The directions of the axes in figures a and b are conjugate to the common diametral chords of confocal conics. In both figures, the property that the tangent line at a center of collineation passes through the center of image circle is pointed out for the points S_1 and S_2.*

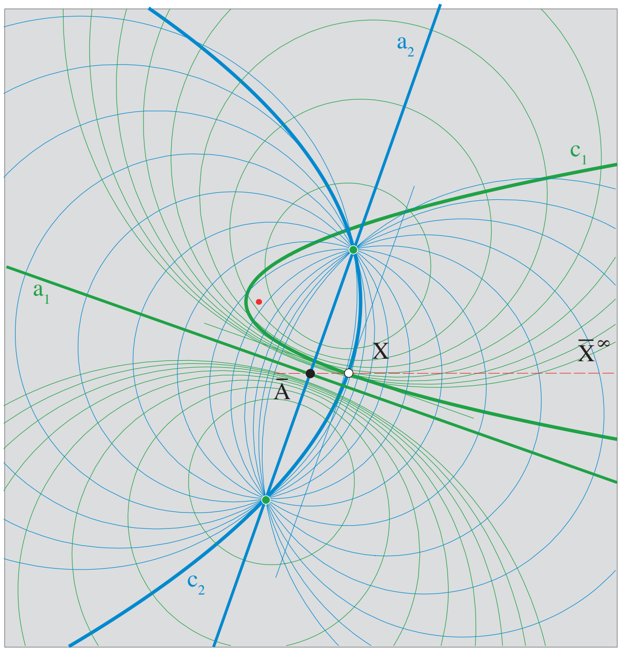

Figure 12: *The illustration of theorem 3 for confocal parabolas c_1 and c_2.*

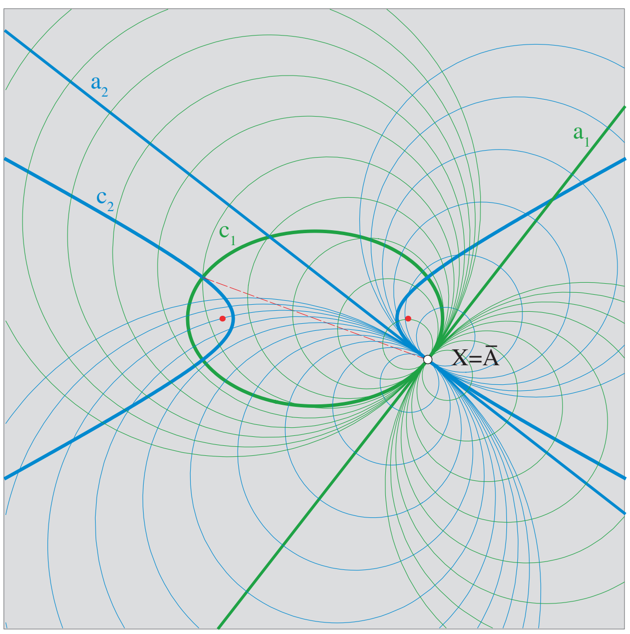

Figure 13: *If the axes a_1 and a_2 coincide with the tangent lines at the intersection point of c_1 and c_2, the image circles form the family of parabolic pencils.*

References

[1] R. Cesarec, Analitička geometrija, Školska knjiga, Zagreb, (1957) (in Croatian).

[2] C. Geyer, K. Daniilidis, Catadioptric projective geometry, Intl. Journal of Computer Vision, 45 (2001), 223—243.

[3] H. Hilton, Plane Algebraic Curves, Oxford Universtity Press, London, (1920)

[4] Encyclopaedia of Mathematics - SpringerLink. Edited by M. Hazewinkel, available on-line: `http://eom.springer.de/s/s091970.htm`

[5] V. Niče, Sintetička geometrija, Školska knjiga, Zagreb, (1956) (in Croatian).

[6] G. Salmon, A Treatise on Conic Sections, Longmans, Green, Reader and Dyer, London, (1869), available on-line: `http://books.google.com`

[7] J. G. Semple, G. T. Kneebone, Algebraic Projective geometry, Oxford University Press, Oxford, (1952)

[8] M. Berger, Geometry I-II, Springer-Verlag, Berlin, (1977)

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