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Visualization of Geodesic Curves, Spheres and Equidistant Surfaces in $\mathbf{S}^2\!\times\!\mathbf{R}$ Space

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ABSTRACT

The $\mathbf{S}^2\!\times\!\mathbf{R}$ geometry is derived by direct product of the spherical plane $\mathbf{S}^2$ and the real line $\mathbf{R}$. In [9] the third author has determined the geodesic curves, geodesic balls of $\mathbf{S}^2\!\times\!\mathbf{R}$ space, computed their volume and defined the notion of the geodesic ball packing and its density. Moreover, he has developed a procedure to determine the density of the geodesic ball packing for generalized Coxeter space groups of $\mathbf{S}^2\!\times\!\mathbf{R}$ and applied this algorithm to them.

E. Molnár showed in [3], that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, V_4, \mathbf{R})$. In our work we shall use this projective model of $\mathbf{S}^2\!\times\!\mathbf{R}$ geometry and in this manner the geodesic lines, geodesic spheres can be visualized on the Euclidean screen of computer.

Furthermore, we shall define the notion of the equidistant surface to two points, determine its equation and visualize it in some cases. We shall also show a possible way of making the computation simpler and obtain the equation of an equidistant surface with more possible geometric meaning. The pictures were made by the Wolfram Mathematica software.

Key words: non-Euclidean geometries, projective geometry, geodesic sphere, equidistant surface

MSC 2010: 53A35, 51M10, 51M20, 52C17, 52C22

Vizualizacija geodetskih krivulja, sfera i ekvidistantnih ploha u prostoru $\mathbf{S}^2\!\times\!\mathbf{R}$

SAŽETAK

$\mathbf{S}^2\!\times\!\mathbf{R}$ geometrija izvodi se kao direktni produkt sferne ravnine $\mathbf{S}^2$ i realnog pravca $\mathbf{R}$. U članku [9], treći je autor odredio geodetske krivulje i geodetske kugle prostora $\mathbf{S}^2\!\times\!\mathbf{R}$, izračunao njihov volumen i definirao pojam popunjavanja geodetskim kuglama i njegovu gustoću. Pored toga, razvio je metodu određivanja gustoće popunjavanja geodetskim kuglama za generalizirane Coxeterove grupe prostora $\mathbf{S}^2\!\times\!\mathbf{R}$ i primijenio taj algoritam na njih.

U [3] je E. Molnar pokazao da homogeni 3-prostori imaju jedinstvenu interpretaciju u projektivnim 3-sferama $\mathcal{PS}^3(\mathbf{V}^4, V_3, \mathbf{R})$. U našem članku koristit ćemo projektivni model $\mathbf{S}^2\!\times\!\mathbf{R}$ geometrije te se na taj način geodetske linije i kugle mogu vizualizirati na euklidskom ekranu računala.

Nadalje, definirat ćemo pojam plohe jednako udaljene od dviju točaka, odrediti njezinu jednadžbu te je vizualizirati u pojedinim slučajevima. Također ćemo pokazati mogući način pojednostavljena računa i dobiti jednadžbe plohe s točnijim geometrijskim značenjem. Slike su napravljene u Wolframovom programu Mathematica.

Ključne riječi: neeuklidske geometrije, projektivna geometrija, geodetske sfere, ekvidistantne plohe

1 On $\mathbf{S}^2\!\times\!\mathbf{R}$ geometry

$\mathbf{S}^2\!\times\!\mathbf{R}$ is derived as the direct product of the spherical plane $\mathbf{S}^2$ and the real line $\mathbf{R}$. The points are described by (P,p) where $P \in \mathbf{S}^2$ and $p \in \mathbf{R}$. The isometry group $Isom(\mathbf{S}^2\!\times\!\mathbf{R})$ of $\mathbf{S}^2\!\times\!\mathbf{R}$ can be derived by the direct product of the isometry group of the spherical plane $Isom(\mathbf{S}^2)$ and the isometry group of the real line $Isom(\mathbf{R})$.

$$Isom(\mathbf{S}^2\!\times\!\mathbf{R}) := Isom(\mathbf{S}^2) \times Isom(\mathbf{R}); \tag{1.1}$$

The structure of an isometry group $\Gamma \subset Isom(\mathbf{S}^2\!\times\!\mathbf{R})$ is the following: $\Gamma := \{A_i \times \rho_i\}$, where $A_i \times \rho_i := A_i \times (R_i, r_i) := (g_i, r_i)$, $A_i \in Isom(\mathbf{S}^2)$, R_i is either the identity map $\mathbf{1_R}$ of $\mathbf{R}$ or the point reflection $\overline{\mathbf{1}}_\mathbf{R}$. $g_i := A_i \times R_i$ is called the linear part of the transformation $(A_i \times \rho_i)$ and r_i is its translation part. The multiplication formula is the following:

$$(A_1 \times R_1, r_1) \circ (A_2 \times R_2, r_2) = ((A_1 A_2 \times R_1 R_2, r_1 R_2 + r_2). \tag{1.2}$$

E. MOLNÁR has shown in [3], that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, V_4, \mathbb{R})$. In our work we shall use this projective model of $\mathbf{S}^2\times\mathbf{R}$ and the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0), E_1^\infty(\mathbf{e}_1), E_2^\infty(\mathbf{e}_2), E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\}\subset\mathbf{V}^4$ with the unit point $E(\mathbf{e}=\mathbf{e}_0+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3))$ which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Moreover, $\mathbf{y}=c\mathbf{x}$ with $0<c\in\mathbb{R}$ (or $c\in\mathbb{R}\setminus\{0\}$) defines a point $(\mathbf{x})=(\mathbf{y})$ of the projective 3-sphere $\mathcal{PS}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$ are identified). The dual system $\{(e^i)\}\subset V_4$ describes the simplex planes, especially the plane at infinity $(e^0)=E_1^\infty E_2^\infty E_3^\infty$, and generally, $v=u\frac{1}{c}$ defines a plane $(u)=(v)$ of $\mathcal{PS}^3$ (or that of $\mathcal{P}^3$). Thus $0=\mathbf{x}u=\mathbf{y}v$ defines the incidence of point $(\mathbf{x})=(\mathbf{y})$ and plane $(u)=(v)$, as $(\mathbf{x})\mathrm{I}(u)$ also denotes it. Thus $\mathbf{S}^2\times\mathbf{R}$ can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well.

In the later sections we will use some special types of $\mathbf{S}^2\times\mathbf{R}$ isometries, which transforms a fixed $P(1,\alpha,\beta,\gamma)$ point of $\mathbf{S}^2\times\mathbf{R}$ into $(1,1,0,0)$. This will be useful for determining the equidistant surfaces, so now we will compute this transformation.

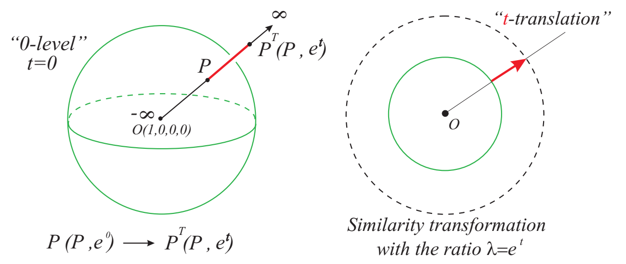

Figure 1: *Translation in* $\mathbf{S}^2\times\mathbf{R}$ *geometry*

$\mathcal{T}=(\mathbf{Id}., T)$ is a fibre translation,

$$P(1,\alpha,\beta,\gamma)\to P^{\mathcal{T}}(1,\alpha',\beta',\gamma')=$$
$$=P^{\mathcal{T}}\Big(1,\frac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}},\frac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}},\frac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\Big). \tag{1.3}$$

($P^{\mathcal{T}}$ has 0 fibre coordinate), (see Fig. 1).

Similarly $\mathcal{R}_z=(\mathbf{R_z},0)$ is a special rotation about z axis with 0 fibre translation, which moves the point $(1,\alpha',\beta',\gamma')$ into the $[x,z]$ plane.

$$P^{\mathcal{T}}(1,\alpha',\beta',\gamma')\to P^{\mathcal{T}\mathcal{R}_z}(1,\alpha'',0,\gamma'')=$$
$$=P^{\mathcal{T}\mathcal{R}_z}(1,\sqrt{\alpha'^2+\beta'^2},0,\gamma'). \tag{1.4}$$

Finally $\mathcal{R}_y=(\mathbf{R_y},0)$ is a special rotation about y axis with 0 fibre translation, which moves the point $(1,\alpha'',0,\gamma'')$ into the $[x,y]$ plane.

$$P^{\mathcal{T}\mathcal{R}_z}(1,\alpha'',0,\gamma'')\to P^{\mathcal{T}\mathcal{R}_z\mathcal{R}_y}(1,1,0,0)=$$
$$=P^{\mathcal{T}\mathcal{R}_z\mathcal{R}_y}(1,\sqrt{\alpha''^2+\gamma''^2},0,0). \tag{1.5}$$

Remark 1.1 *More informations about the isometry group of* $\mathbf{S}^2\times\mathbf{R}$ *and about its discrete subgroups can be found in [1], [2] and [9].*

2 Geodesic curves and spheres of $\mathbf{S}^2\times\mathbf{R}$

E. Molnár [3] has introduced the natural the well-known infinitezimal arc-length square at any point of $\mathbf{S}^2\times\mathbf{R}$ as follows

$$(ds)^2=\frac{(dx)^2+(dy)^2+(dz)^2}{x^2+y^2+z^2}. \tag{2.1}$$

We shall apply the usual geographical coordiantes (ϕ,θ), $(-\pi<\phi\le\pi,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2})$ of the sphere with the fibre coordinate $t\in\mathbf{R}$. We describe points in the above coordinate system in our model by the following equations:

$$x^0=1,\ x^1=e^t\cos\phi\cos\theta,\ x^2=e^t\sin\phi\cos\theta,\ x^3=e^t\sin\theta. \tag{2.2}$$

Then we have $x=\frac{x^1}{x^0}=x^1$, $y=\frac{x^2}{x^0}=x^2$, $z=\frac{x^3}{x^0}=x^3$, i.e. the usual Cartesian coordinates. We obtain by (2.1) and (2.2) that in this parametrization the infinitezimal arc-length square at any point of $\mathbf{S}^2\times\mathbf{R}$ is the following

$$(ds)^2=(dt)^2+(d\phi)^2\cos^2\theta+(d\theta)^2. \tag{2.3}$$

Hence we get the symmetric metric tensor field g_{ij} on $\mathbf{S}^2\times\mathbf{R}$ by components:

$$g_{ij}:=\begin{pmatrix}1&0&0\\0&\cos^2\theta&0\\0&0&1\end{pmatrix}, \tag{2.4}$$

The geodesic curves of $\mathbf{S}^2\times\mathbf{R}$ are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves $\gamma(t(\tau),\phi(\tau),\theta(\tau))$ in our model can be determined by the general theory of Riemann geometry:

By (2.4) the second order differential equation system of the $\mathbf{S}^2\times\mathbf{R}$ geodesic curve is the following [9]:

$$\ddot\phi-2\tan\theta\dot\phi\dot\theta=0,\ \ddot\theta+\sin\theta\cos\theta\dot\phi^2=0,\ \ddot t=0, \tag{2.5}$$

from which we get first an equator circle on $\mathbf{S}^2$ times a line on $\mathbf{R}$ each running with constant velocity c and ω, respectively:

$$t=c\cdot\tau,\ \ \theta=0,\ \ \phi=\omega\cdot\tau,\ \ c^2+\omega^2=1. \tag{2.6}$$

We can assume, that the starting point of a geodesic curve is $(1,1,0,0)$, because we can transform a curve into an arbitrary starting point, moreover, unit velocity can be assumed. Then we get the equation systems of a geodesic curve, visualized in Fig. 2 in our Euclidean model:

$$\begin{aligned} \tau &\to (x(\tau),y(\tau),z(\tau)),\\ x(\tau) &= e^{\tau\sin v}\cos(\tau\cos v),\\ y(\tau) &= e^{\tau\sin v}\sin(\tau\cos v)\cos u,\\ z(\tau) &= e^{\tau\sin v}\sin(\tau\cos v)\sin u,\\ &\text{with fixed } -\pi < u \le \pi,\quad -\frac{\pi}{2} \le v \le \frac{\pi}{2}. \end{aligned} \tag{2.7}$$

Remark 2.1 *Thus we have also harmonized the scales along the base sphere and the fibre lines.*

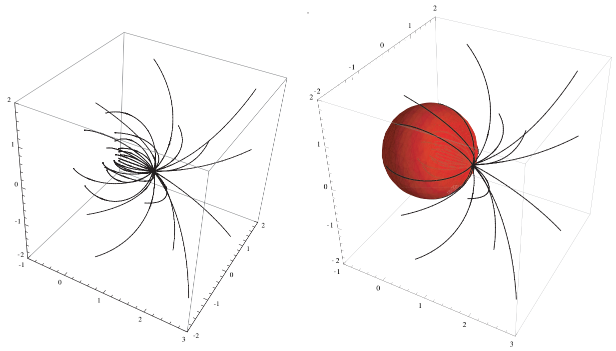

Figure 2: *Geodesic curves with the base sphere, ("the spider")*

Definition 2.2 *The distance $d(P_1,P_2)$ between the points P_1 and P_2 is defined by the arc length of the geodesic curve from P_1 to P_2.*

Definition 2.3 *The geodesic sphere of radius ρ (denoted by $S_{P_1}(\rho)$) with centre at the point P_1 is defined as the set of all points P_2 in the space with the condition $d(P_1,P_2)=\rho$. Moreover, we require that the geodesic sphere is a simply connected surface without selfintersection in $\mathbf{S}^2\times\mathbf{R}$ space.*

Remark 2.4 *We shall see that this last condition depends on radius ρ.*

Definition 2.5 *The body of the geodesic sphere of centre P_1 and of radius ρ in the $\mathbf{S}^2\times\mathbf{R}$ space is called geodesic ball, denoted by $B_{P_1}(\rho)$, i.e. $Q\in B_{P_1}(\rho)$ iff $0\le d(P_1,Q)\le\rho$.*

Remark 2.6 *Henceforth, typically we choose $(1,1,0,0)$ as centre of the sphere and its ball, by the homogeneity of $\mathbf{S}^2\times\mathbf{R}$.*

From (2.7) follows that $S(\rho)$ is a simply connected surface in $\mathbf{E}^3$ if and only if $\rho\in[0,\pi)$, because if $\rho\ge\pi$ then there is at least one $v\in[-\frac{\pi}{2},\frac{\pi}{2}]$ so that $y(\tau,v)=z(\tau,v)=0$, i.e. selfintersection would occur.

Thus we obtain the following

Proposition 2.7 *The geodesic sphere and ball of radius ρ (with the above requirements) exists in the $\mathbf{S}^2\times\mathbf{R}$ space if and only if $\rho\in[0,\pi]$.*

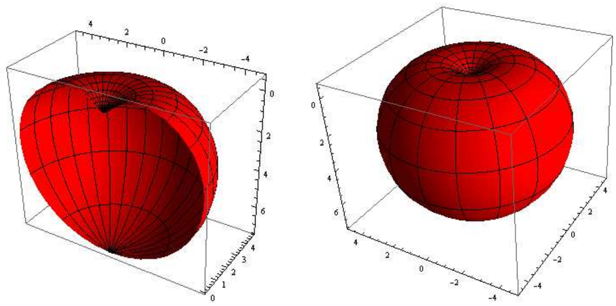

Figure 3: *Geodesic half sphere and sphere of radius $\rho=2$*

3 Equidistant surfaces in $\mathbf{S}^2\times\mathbf{R}$ geometry

3.1 What about parameters of a given geodesic curve?

It can be assumed by the homogeneity of $\mathbf{S}^2\times\mathbf{R}$ that one of endpoints of a given geodesic curve segment is $P_1(1,1,0,0)$ to simplify our calculations. The other endpoint is given by its homogeneous coordinates $P_2(1,a,b,c)$. We consider the geodesic curve segment $\mathcal{G}_{P_1P_2}$. Our first goal is to calculate parameters τ,u,v belonging to $\mathcal{G}_{P_1P_2}$. We get by (2.7) the following important observation:

$$\sqrt{a^2+b^2+c^2}=e^{\tau\sin v}. \tag{3.1}$$

We obtain from (2.7) and (3.1) the following equations and the parameter v :

$$\begin{aligned} &\sqrt{a^2+b^2+c^2}\cos(\tau\cos v)=a \quad\Rightarrow\\ &\Rightarrow \tau\cos v=\arccos\left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right), \end{aligned} \tag{3.2}$$

$$\log\sqrt{a^2+b^2+c^2}=\tau\sin v, \tag{3.3}$$

$$\tan v = \frac{\log\sqrt{a^2+b^2+c^2}}{\arccos\left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right)} \Rightarrow$$
$$\Rightarrow v = \arctan\left(\frac{\log\sqrt{a^2+b^2+c^2}}{\arccos\left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right)}\right), \tag{3.4}$$

if $P_2 \notin x \Leftrightarrow a \neq 0,\ b = c = 0$.

Remark 3.1 *If $P_2 \in x$, then $v = \frac{\pi}{2}$, $u = 0$ moreover the geodesic curve is the Euclidean segment P_1P_2, and its equidistant surface is a Euclidean sphere in our model.*

Thus we get the parameter τ from (3.1) and (3.3):

$$\tau = \frac{\log\sqrt{a^2+b^2+c^2}}{\sin v} \quad \text{if } v \neq 0. \tag{3.5}$$

The parameter u of the given geodesic curve $\mathcal{G}_{P_1P_2}$ can be expressed by (2.7):

$$\frac{z(\tau)}{y(\tau)} = \frac{c}{b} = \tan u \rightarrow u = \arctan\left(\frac{c}{b}\right). \tag{3.6}$$

Remark 3.2 *If $v = 0$ then*
$\tau = \arccos\left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right) = \arccos(a)$ *and* $u = \arctan\left(\frac{c}{b}\right)$.

3.2 The equation of the equidistant surface

We want to find an equation for the surface $\mathcal{S}_{P_1P_2}$ consisting of all points that are equidistant from the point P_1 and P_2.

Definition 3.3 *An equidistant surface $\mathcal{S}_{P_1P_2}$ of two arbitrary given points $P_1, P_2 \in \mathbf{S}^2\times\mathbf{R}$ consists of all $X \in \mathbf{S}^2\times\mathbf{R}$ points, for which $d(P_1,X) = d(X,P_2)$ holds. Moreover, we require that this surface is a simply connected surface without selfintersection in $\mathbf{S}^2\times\mathbf{R}$ space.*

The varying point $X(1,x,y,z) \in \mathcal{S}_{P_1P_2}$ satisfies the following equation (see formulas (1.1), (1.2), (1.3), Fig. 4):

$$d(P_1,X) = d(X,P_2) = d(X^{\mathcal{T}\mathcal{R}_z\mathcal{R}_y}, P_2^{\mathcal{T}\mathcal{R}_z\mathcal{R}_y}) = \\ = d(X^{\mathcal{T}\mathcal{R}_z\mathcal{R}_y}, P_1), \quad \forall X \in \mathcal{S}_{P_1P_2} \tag{3.7}$$

It is clear by the above equation (3.7), that the length of the geodesic curve $\mathcal{G}_{P_1,X}$ is equal to the length of the geodesic line $\mathcal{G}_{X^{\mathcal{T}\mathcal{R}_z\mathcal{R}_y}_{\mathcal{S}_{P_1P_2}},P_1}$, thus the τ parameters (can be determined by (3.5) and Remark 3.1) of the above geodesic curves are equal. Finally we have got the equation of the equaidistant surface $\mathcal{S}_{P_1P_2}$ after major simplification in model of $\mathbf{S}^2\times\mathbf{R}$ geometry:

$$4\arccos^2\left(\frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}\sqrt{x^2+y^2+z^2}}\right)+ \\ +\log^2\left(\frac{a^2+b^2+c^2}{x^2+y^2+z^2}\right) = \\ = 4\arccos^2\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)+\log^2(x^2+y^2+z^2). \tag{3.8}$$

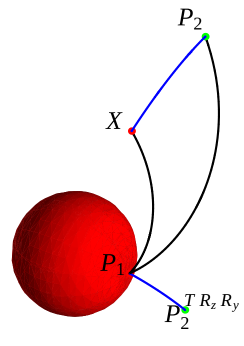

Figure 4: *It is clear, that $X \in \mathcal{S}_{P_1P_2}$ if and only if $d(P_1,X) = d(P_1,P_2^{\mathcal{T}\mathcal{R}_z\mathcal{R}_y})$ holds.*

Using formula (3.8) we can visualize the equidistant surface by computer (Fig. 5).

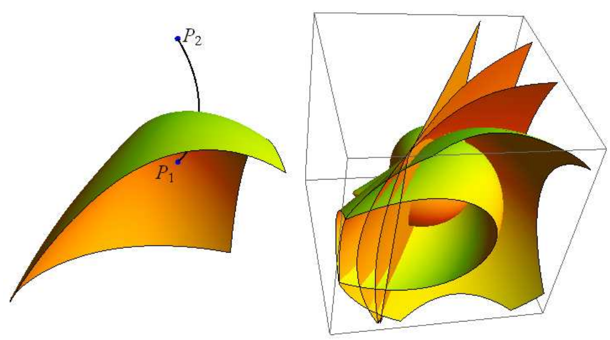

Figure 5: *An equidistant surface, and how to change the equidistant surfaces $\mathcal{G}_{P_1,P_2}$ with fixed $P_1(1,1,0,0)$ and varying $P_2 : (1,4,0,0) \rightarrow (1,-\frac{1}{2},\frac{\sqrt{3}}{2},0)$ along their geodesic curve.*

Remark 3.4 *The behavior of the equidistant surfaces at (and near) the origin requires more discussion.*

3.3 Further examination in Euclidean sense

Equation (3.8) can be simplified by using some Euclidean concepts and notations. In this subsection we will use these simplifications to find some Euclidean geometric meaning behind the equation.

Let us note the Euclidean location vectors of the points $P_1(1,1,0,0)$, $P_2(1,a,\ b,\ c)$ and $X(1,x,y,z)$ by $\mathbf{e}(1,0,0)$, $\mathbf{a}(a,b,c)$ and $\mathbf{x}(x,y,z)$ respectively. Substituting the usual denotation of Euclidean scalar product $\langle\ ,\ \rangle$ into equation (3.8) we obtain its following form:

$$4\arccos^2\left(\frac{\langle\mathbf{a},\mathbf{x}\rangle}{\sqrt{\langle\mathbf{a},\mathbf{a}\rangle}\sqrt{\langle\mathbf{x},\mathbf{x}\rangle}}\right)+\log^2\left(\frac{\langle\mathbf{a},\mathbf{a}\rangle}{\langle\mathbf{x},\mathbf{x}\rangle}\right)= \\ =4\arccos^2\left(\frac{\langle\mathbf{x},\mathbf{e}\rangle}{\sqrt{\langle\mathbf{x},\mathbf{x}\rangle}}\right)+\log^2\left(\langle\mathbf{x},\mathbf{x}\rangle\right). \tag{3.9}$$

If we apply the well known Euclidean angle formula then $\cos(\varepsilon)=\frac{\langle\mathbf{a},\mathbf{x}\rangle}{\sqrt{\langle\mathbf{a},\mathbf{a}\rangle\langle\mathbf{x},\mathbf{x}\rangle}}$ and $\cos(\delta)=\frac{x}{\sqrt{\langle\mathbf{x},\mathbf{x}\rangle}}$ where the angles ε and δ are the angles $\measuredangle(\mathbf{a},\mathbf{x})$ and $\measuredangle(\mathbf{x},\mathbf{e})$ in Euclidean sense $(0\le\varepsilon,\delta\le\pi)$.

$$4\varepsilon^2+\log^2(\langle\mathbf{a},\mathbf{a}\rangle\langle\mathbf{x},\mathbf{x}\rangle^{-1})=4\delta^2+\log^2(\langle\mathbf{x},\mathbf{x}\rangle) \tag{3.10}$$

It is easy to examine some special cases using equation (3.10) of the equidistance surface.

Let us consider that case, where $P_2=(1,a,0,0)$ and $(a\neq 1)$, i.e. P_2 lies on the x-axis and $P_1\neq P_2$.

It is clear that in this case $\varepsilon=\delta$ holds and by (3.10) we obtain:

$$\log^2(\langle\mathbf{a},\mathbf{a}\rangle\langle\mathbf{x},\mathbf{x}\rangle^{-1})=\log^2(\langle\mathbf{x},\mathbf{x}\rangle)\Rightarrow \\ \Rightarrow\sqrt{\langle\mathbf{x},\mathbf{x}\rangle}=\langle\mathbf{a},\mathbf{a}\rangle^{1/4}. \tag{3.11}$$

Thus $\mathcal{S}_{P_1P_2}$ is an Euclidean sphere with centre in the origin of radius $\rho=\langle\mathbf{a},\mathbf{a}\rangle^{1/4}$.

Let us consider that case, where P_2 lies on the base sphere, e.g. $\langle\mathbf{a},\mathbf{a}\rangle=1$.

In this case we get by (3.10) the following equation:

$$4\varepsilon^2+\log^2(\langle\mathbf{x},\mathbf{x}\rangle^{-1})=4\delta^2+\log^2(\langle\mathbf{x},\mathbf{x}\rangle)\Leftrightarrow\varepsilon=\delta. \tag{3.12}$$

This means, if P_2 is on the base sphere, then $\mathcal{S}_{P_1P_2}$ is an Euclidean plane without the origin.

Our projective method gives us a way of investigating homogeneous spaces, which suits to study and solve similar problems (see [7], [8]).

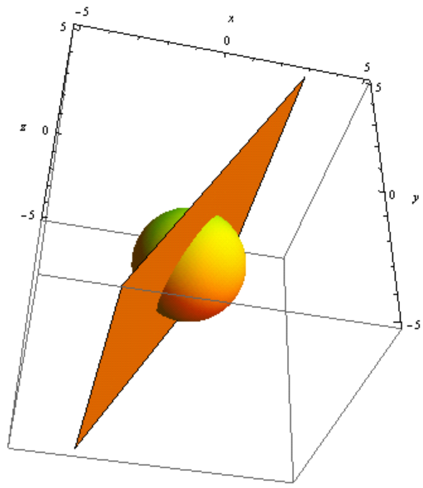

Figure 6: *The equidistant surface* $\mathcal{G}_{P_1,P_2}$ *with fixed* $P_1(1,1,0,0)$ *and* $P_2:(1,4,0,0)$*, in this case the equidistance surface is a spere in Euclidean sense and if* $P_2(1,-\frac{1}{2},\frac{\sqrt{3}}{2},0)$*, in this case the equidistant surface is a plane without origin in Euclidean sense. These are the special cases discussed in Subsection 3.3.*

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