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# Knotted Tori

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**ABSTRACT**

Paper presents a family of surfaces called knotted tori, which can be regarded as special subset of two-axial surfaces of revolutions of Euler type. Analytic representation of surfaces and some of their specific geometric properties are derived, shaping parameters are discussed and several representatives of interesting shapes are illustrated.

**Key words:** knotted tori, Euler trajectory, two-axial surfaces of revolution

**MSC 2000:** 14J26, 15A04, 53A05

**Torusni čvorovi**

**SAŽETAK**

U članku se prikazuje porodica ploha pod nazivom torusni čvorovi, koja se može promatrati kao podskup dvoosnih rotacijskih ploha Eulerovog tipa. Izvode se analitička prezentacija tih ploha te neka od njihovih geometrijskih svojstava. Raspravlja se o parametrima oblika, a na nekoliko se primjera ilustriraju zanimljivi oblici ploha.

**Ključne riječi:** torusni čvorovi, Eulerova trajektorija, dvoosne rotacijske plohe

## 1 Introduction

Theory of knots is a part of algebraic topology, which studies problem known as "embeddings" of one topological space into another one. Embedding of space $X$ into the space $Y$ is an injective continuous mapping $f : X \to Y$ such that the restriction of $f$ on

$$\underline{f} : X \to f(X) \subseteq Y \tag{1}$$

is a homeomorphism of spaces $X$ and $f\,(X)$.
The easiest form of the above mentioned problem is e.g. inclusion of a unit circle into the three-dimensional Euclidean space $E^3$. In practical terms, this means that: "we take a circle, cut it, knot the created thread and finally glue together again both free ends". Closed space curve created in the described way is a topologically embedded circle into the three-dimensional space.
In the presented paper, the knotted tori will be understood as a regular surface created by a similar knotting of a torus like surface (not solid) in the three-dimensional space $E^3$. Knotted torus is a surface with no self-intersections and singular points, it is closed, and its envelope is again a torus. This means that by knotting a torus no new type of topological structure will be created, or no new type of knot will arise then the original one. Consequently, there exist infinitely many different forms of the knotted tori in the three dimension space.

From the geometric point of view we can assume that a knotted torus is an envelope surface created by the continuous movement of the sphere in the space with the trajectory located on the torus of revolution in such way that no self-intersecting might occur. This curve is in the general case trajectory of the specific composite revolutionary movement about the two skew axes called Euler movement known as the Euler trajectory and described in details in [1].

Simple Euler trajectory in the basic form is a closed space curve without any multiple points illustrated in fig. 1, on the left; its orthographic view in the plane perpendicular to the axis of revolution $^2o$ is a symmetric plane curve, Limaçon of Pascal.

Let $A = (a,0,0,1)$ be a given point on the coordinate axis $x$ that moves in the Euler movement composed from two revolutions, about coordinate axis $z = {}^1o$ and simultaneously about axis $^2o$ parallel to the coordinate axis $x$ in the distance $d \neq 0, {}^2o \parallel x$ determined by equations $y = d, z = 0$. Trajectory of the point movement can be analytically represented by continuous differentiable point function in one real variable $v$, which is on the interval $I = \langle 0,1\rangle \subset R$ in the following form:

$$\mathbf{r}(v) = (a\cos k\pi v,\ a\sin k\pi v\ \cos l\pi v + d(1-\cos l\pi v),$$
$$a\sin k\pi v\ \sin l\pi v - d\ \sin l\pi v,\ 1).$$

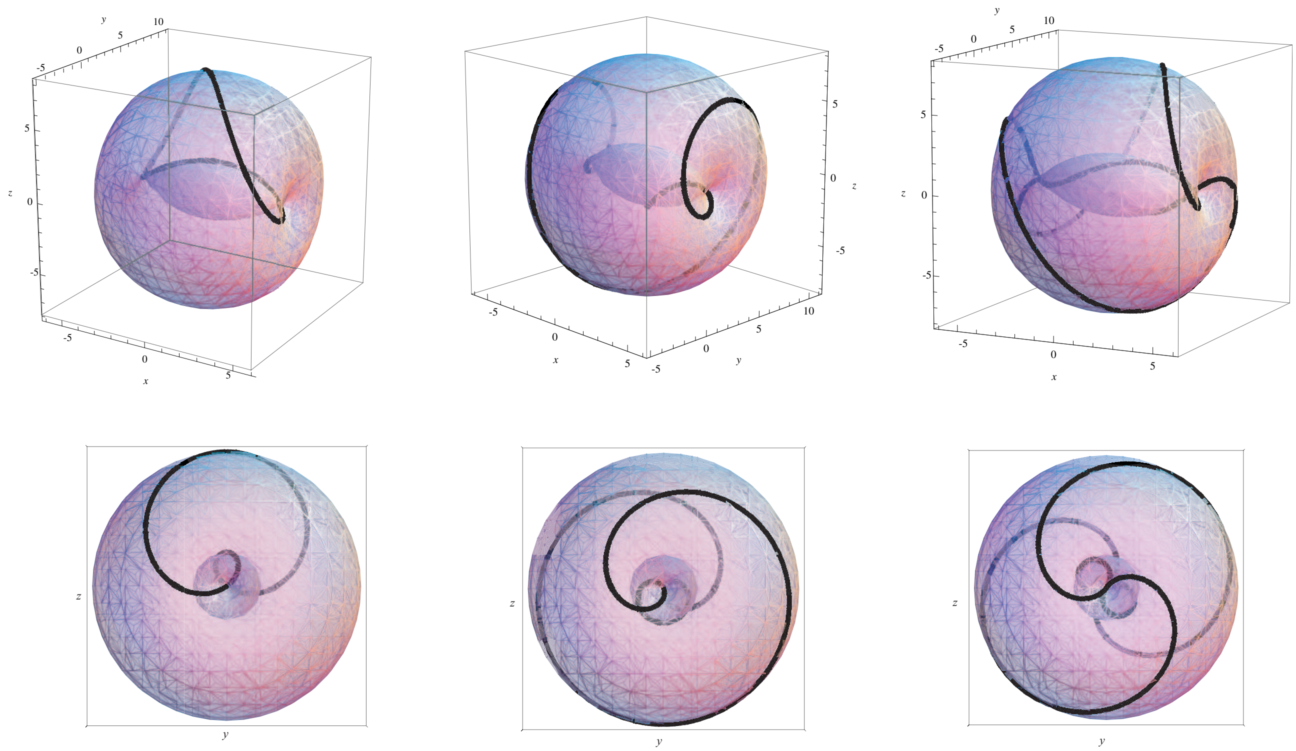

Figure 1: *Forms of Euler trajectories.*

Euler trajectory is located on torus of revolution with the axis in the axis of revolution ${}^2o$, centre in the point $S = (0, d, 0, 1)$ on the coordinate axis $y$ and radius equal to the distance $a$ of the moving point from the axis of revolution ${}^1o$. Coordinates of the Euler trajectory points satisfy the implicit equation of the torus

$$\left(x^2 + (y-d)^2 + z^2 + d^2 - a^2\right)^2 = 4d^2\left((y-d)^2 + z^2\right).$$

Several forms of the Euler trajectory determined by multiples $k$ and $l$ of angular velocities of separate revolutions are illustrated in fig. 1 on the left, in the middle and on the right, with pairs of parameters
$a = 5,\ d = 3,\ (k,l) = (2,2), (2,6), (4,6).$

## 2 Knotted tori as cyclical two-axial surfaces of revolution of Euler type, the 1st form

Surfaces in the group of general two-axial surfaces of revolution of the Euler type (complete classification is presented in [2]) can be generated by the movement of the basic circle $g$ located in the plane passing through the first - interior axis of revolution ${}^1o$, and they are represented by vector function

$$\mathbf{r}(u,v) = (a + r\cos 2\pi u,\ 0,\ b + r\sin 2\pi u,\ 1),$$
$$u \in \langle 0,1\rangle,\ a,b,r \in R,\ r \neq 0.$$

Parametric equations of surfaces in this subgroup of surfaces of Euler type are for $(u,v) \in \langle 0,1\rangle^2 \subset R^2$ in the form

$$x(u,v) = (a + r\cos 2\pi u)\cos k\pi v$$
$$y(u,v) = (a + r\cos 2\pi u)\sin k\pi v\ \cos l\pi v -$$
$$-(b + r\sin 2\pi u)\sin l\pi v + d(\cos l\pi v - 1)$$
$$z(u,v) = (a + r\cos 2\pi u)\sin k\pi v\ \sin l\pi v +$$
$$+(b + r\sin 2\pi u)\cos l\pi v + d\sin l\pi v$$

Knotted tori can be created by choosing values of shaping parameters satisfying the following relations

$b = 0,\ \ a < d,\ \ a > r,\ \ k$ and $l$ are even numbers.

Different modifications of knotted tori can be achieved by the choice of other shaping characteristics, multiples of separate angular velocities $k$ and $l$. Number of windings in the direction of the revolution about the second axis is the number of arms equal to $l/2$, number of windings in the direction of the first axis of revolution defines the number of knots, $k/2$.

Simple non-trivial knotted torus (trefoil) is illustrated in fig. 2 on the left, shaping characteristics are $a = 5$, $b = 0$, $d = 10$, $r = 3$, $(k, l) = (6, 4)$. Other illustrated forms are determined by the same shaping parameters $a, b, d, r$, and by values $(k, l) = (10, 4), (14, 4)$, surfaces have two knotted arms; number of knots is 3, 5 and 7.

In fig. 3 knotted tori with values of shaping characteristics $a = 5$, $b = 0$, $d = 12$, $r = 3$, $(k, l) = (8, 6), (10, 6), (14, 6)$ are illustrated, which are created by knotting of three arms 4, 5 and 7 times.

By knotting four arms the next modification of the knotted torus can be created, shaping parameters of the illustrated surfaces are $a = 5$, $b = 0$, d = 17, $r = 2$, $(k, l) = (6, 8), (10, 8), (14, 8)$, so the four arms are knotted 3, 5 and 7 times.

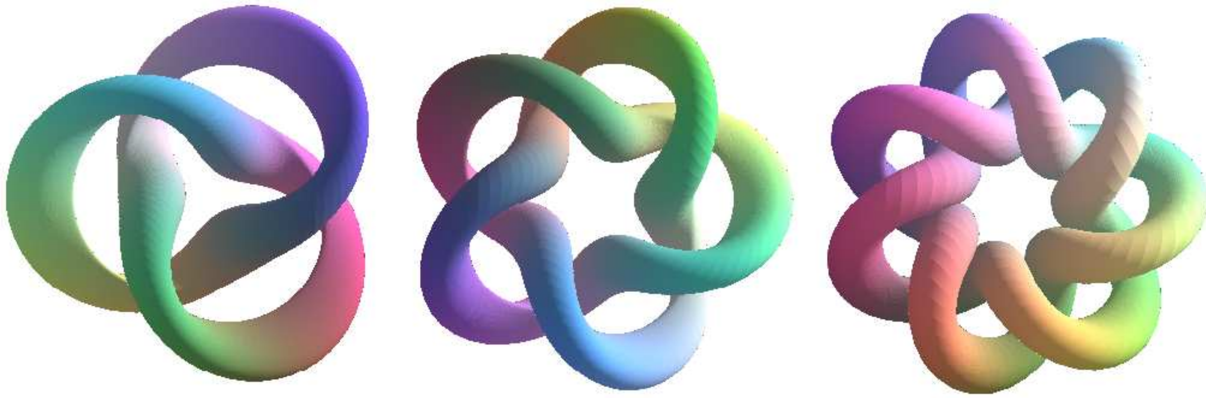

Figure 2: *Knotted tori of the 1st form, with 2 threads.*

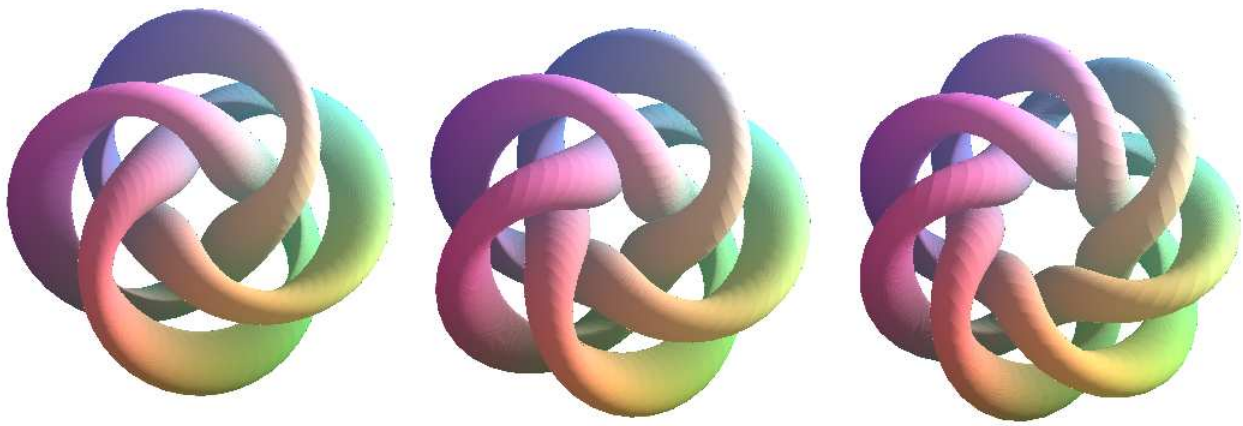

Figure 3: *Knotted tori of the 1st form, with 3 threads.*

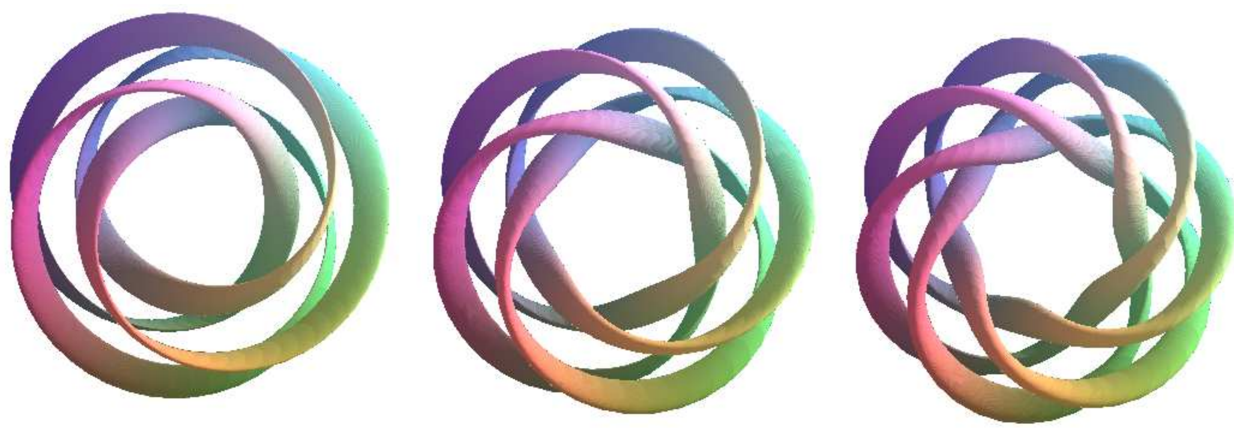

Figure 4: *Knotted tori of the 1st form, with 4 threads.*

## 3 Knotted tori as cyclical two-axial surfaces of revolution of Euler type, the 2nd form

Basic circle of the knotted torus located in the plane of the exterior axis of revolution ${}^2o$ defined by vector function

$$\mathbf{r}(u) = (a + r\cos 2\pi u,\, b + r\sin 2\pi u,\, 0,\, 1),$$
$$u \in \langle 0,1\rangle,\ a,b,r \in R,\ r \neq 0.$$

determines subgroup of surfaces of Euler type of the second form represented analytically by parametric equations for $(u,v) \in \langle 0,1\rangle^2 \subset R^2$ in the form

$$x(u,v) = (a + r\cos 2\pi u)\cos k\pi v - (b + r\sin 2\pi u)\sin k\pi v$$
$$y(u,v) = (a + r\cos 2\pi u)\sin k\pi v\ \cos l\pi v + $$
$$+ (b + r\sin 2\pi u)\cos k\pi v\ \cos l\pi v + d(\cos l\pi v - 1)$$
$$z(u,v) = (a + r\cos 2\pi u)\sin k\pi v\ \sin l\pi v + $$
$$+ (b + r\sin 2\pi u)\cos k\pi v\ \sin l\pi v + d\sin l\pi v.$$

Different modifications of knotted torus of the 2nd form can be created by shaping characteristics, where $b = 0$, and multiples of angles of revolutions are $k$ and $l$. Number of turns about the second axis of revolution defines the number of arms $l/2$, number of turns about the first axis of revolution gives the number of knots $k/2$.

Knotted tori of the second form are illustrated in fig. 5, values of parameters for corresponding forms illustrated from the left to the right are: $a = 10$, $b = 0$, $d = 17$, $r = 3$, $(k, l) = (4, 6), (4, 10), (4, 14)$.

In fig. 6, samples of modifications determined by parameters $a = 10$, $b = 0$, $d = 17$, $r = 3$, $(k, l) = (6, 8), (6, 10), (6, 14)$ are presented, in fig. 7 illustrations of knotted tori determined by multiples of angular velocities $(k, l) = (8, 6), (8, 10), (8,14)$ are presented.

Some complex forms of knotted tori are extremely interesting in shapes, and they can be utilised as small artefacts, resembling mosaic motifs, laces and embroidered emblems. Samples of these aesthetic objects, which are freely modified forms of knotted tori, are illustrated in fig. 8.

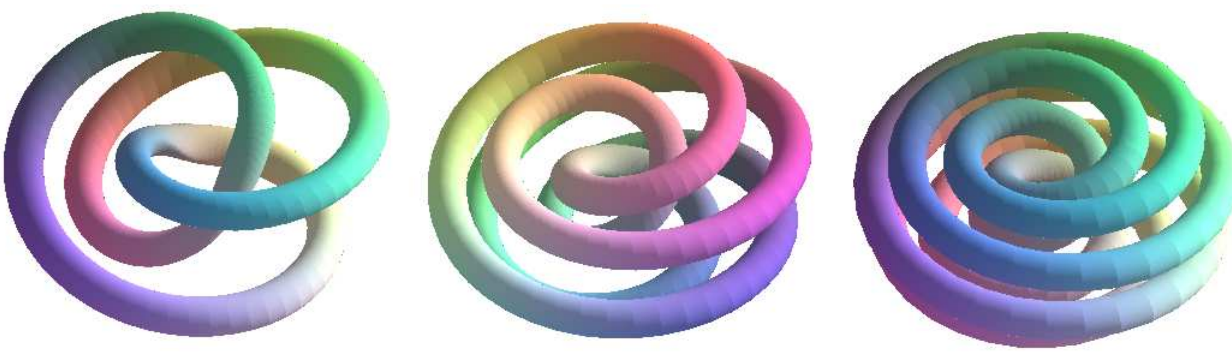

Figure 5: *Knotted tori of the 2nd form, two-folded.*

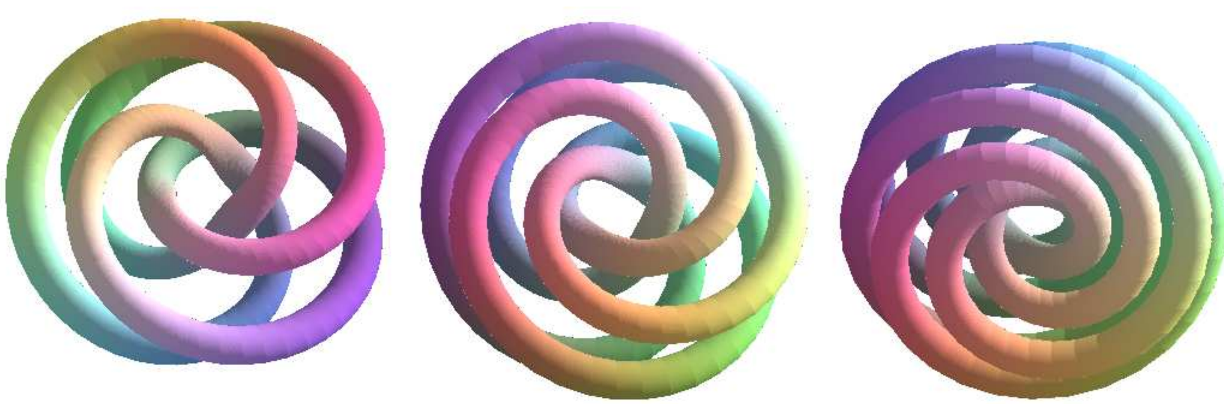

Figure 6: *Knotted tori of the 2nd form, three-folded.*

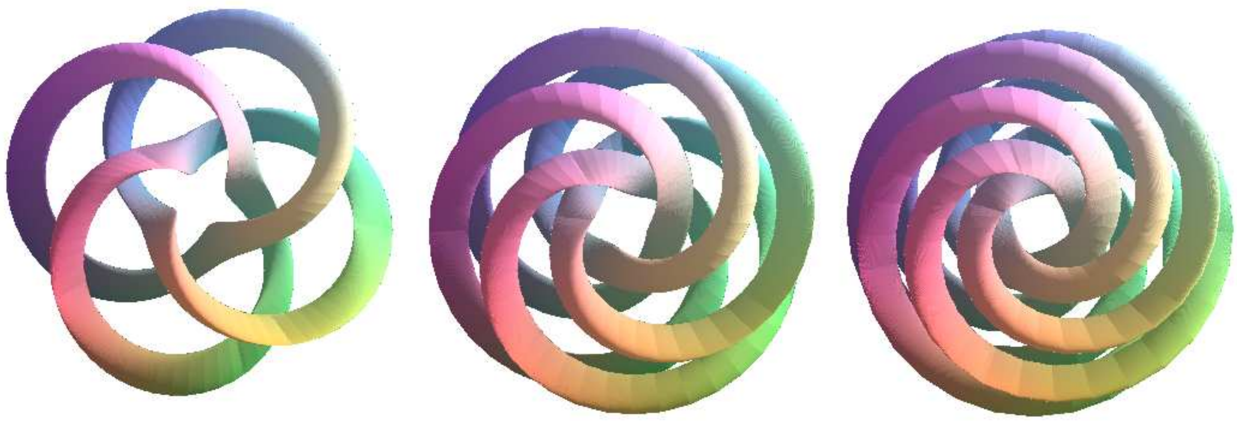

Figure 7: *Knotted tori of the 2nd form, four-folded.*

## 4 Conclusions

There exist infinitely many different forms of knotted tori, and not all of them can be created as special two-axial surfaces of revolution of Euler type, whereas many of them can be created on the base of other principles. There is available a rich source database of materials dealing with knotted tori, where this topic is discussed from different viewpoints and on different theoretical or practical levels. Book [3] is an introduction to the theory of knots and topology of surfaces, where common links between these two areas and their connections to the graph theory and theory of groups are given. Book is written in a very good style, there are a lot of illustrations in there, and just the basic knowledge of secondary school mathematics is expected from readers. Book [4] brings a historical survey of the theory of knots, showing that creation of knots can be regarded as one of the oldest and most widespread techniques in art, weaving, sea-sailing, coding, ciphering, building and architecture, and in many other applications and practical activities influencing the development of human civilisation and culture.

Some interesting and inspiring sources can be found on Internet, where one can see also many interactive applications enabling modelling of knotted tori on-line on web. Web-page at address [5] is entirely designed for bringing information about available source materials with references to all aspects of knotted tori modelling - theory, art, computer modelling, video presentations, on-line galleries.

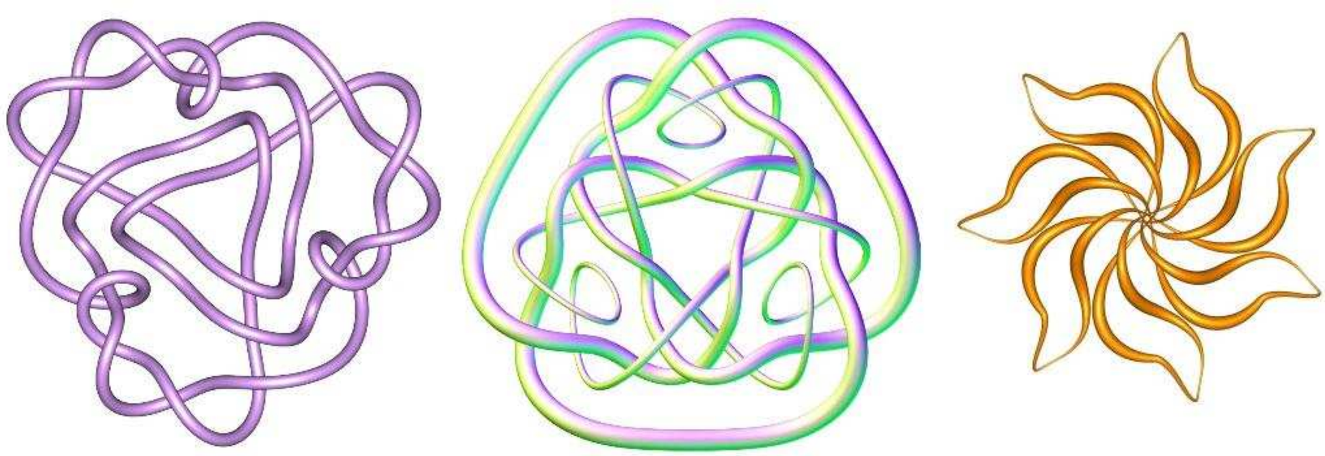

Figure 8 *Knotted tori, free forms.*

## References

[1] D. VELICHOVÁ, Trajectories of composite rotational movements, G−Slovak Journal for Geometry and Graphics, ISSN 1335- X, Vol. 3, No. 5, 2006, pp. 47-64.

[2] D. VELICHOVÁ, Classification of two-axial surfaces, G - Slovak Journal of Geometry and Graphics, ISSN 1335- X, Vol. 4, No. 7, 2007, pp. 63-82.

[3] N. D. GILBERT, T. PORTER, *Knots and Surfaces*, Oxford University Press, 1997 (original 1994).

[4] J. C. TURNER, P. VAN DE GRIEND, *History and Science of Knots*, World Scientific Publishing Co., 1996.

[5] `http://www.earlham.edu/~peters/knotlink.htm`

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