Professional paper
Accepted 23.11.2007.

TATIANA OLEJNÍKOVÁ*

Cyclical Surfaces Created by a Conical Helix

Cyclical Surfaces Created by a Conical Helix

ABSTRACT

The paper describes cyclical surfaces created by revolution of a circle about an edge of the trihedron of a conical helix that is moving evenly along the helix. This Euclidean metric transformation is composed from revolution about one of the coordinate axes and transformation of the right-handed coordinate system to the right-handed system of the moving trihedron in every point of the conical helix. This transformation is analytically represented by a functional matrix of 4^{th} order. These surfaces are determined at particular parameter values which have influence on the surface shape. The vector equation of surfaces and some illustrations of this group of surfaces are presented in the paper. The surfaces are illustrated and modelled in the programme Maple.

Key words: cyclical surface, conical helix, trihedron

MSC 2000: 15A04, 53A05, 14J26

Kružne plohe izvedene pomoću zavojnice na stošcu

SAŽETAK

Članak prikazuje kružne plohe koje nastaju rotacijom kružnice oko jednog brida trobrida pratioca na zavojnici stošca, koji se jednoliko giba duž zavojnice. Ova euklidska metrička transformacija sastoji se od rotacije oko jedne od koordinatnih osi i transformacije desno orijentiranog koordinatnog sustava u desno orijentirani sustav gibajućeg trobrida u svakoj točki zavojnice na stošcu. Ta je transformacija analitički prikazana funkcionalnom matricom 4. reda. Plohe su određene posebnim vrijednostima parametra koji utječe na njihov oblik. U članku je predstavljena vektorska jednadžba i nekoliko primjera ove grupe ploha. Primjeri su konstruirani pomoću računalnog programa Maple.

Ključne riječi: kružne plohe, zavojnica na stošcu, trobrid pratilac

1 The vector equation of the cyclical surface created by a conical helix

Let the conical helix s in right-handed coordinate system $(O;x,y,z)$ be located on the circular conical surface with the meridian defined analytically by the vector function $(r_1+u(r_2-r_1),0,uv,1)$, $u\in\mathbf{R}$, $(r_1\neq 0, r_2\neq 0, v\neq 0)$ (Figure 1).

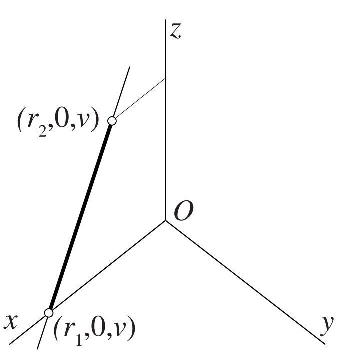

Figure 1: *Meridian of the conical surface*

The helix has the parametrical equations (1), where l is a number of helix screws and the parameter $q=+1$ for right-turned helix and $q=-1$ for left-turned one (Figure 2) [3].

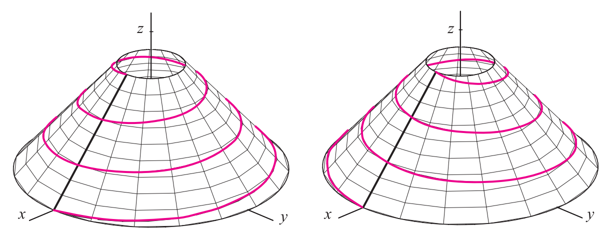

Figure 2: *Conical helix right-turned and left-turned*

$$x=\left(r_1+\frac{\varepsilon(r_2-r_1)}{2l\pi}\right)\cos\varepsilon$$
$$y=q\left(r_1+\frac{\varepsilon(r_2-r_1)}{2l\pi}\right)\sin\varepsilon,\ \varepsilon\in\langle 0,2\pi\rangle. \quad (1)$$
$$z=\frac{v\varepsilon}{21\pi}$$

* The paper was supported by the grant VEGA 1 / 4002/07

The trihedron $(R;t,n,b)$ in the coordinate system $(O;x,y,z)$ is given in every point $R \in s$ with coordinates determined in equation (2), where t is the tangent, b is the binormal and n is the principal normal of the conical helix s (Figure 3). The unit vectors of the trihedron are expressed in equations (3) and (4) [1].

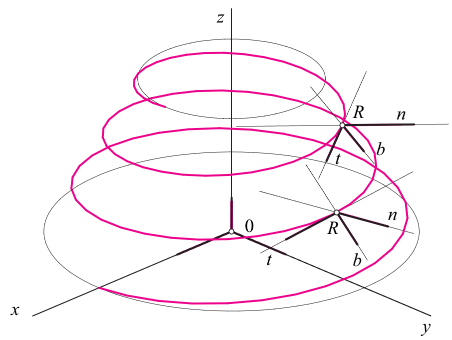

Figure 3: *Thrihedron (R;t,n,b) with unit vectors*

$$R(x_R,y_R,z_R) = \left(\left(r_1+\frac{\varepsilon(r_2-r_1)}{2l\pi}\right)\cos\varepsilon, \left(r_1+\frac{\varepsilon(r_2-r_1)}{2l\pi}\right)\sin\varepsilon, \frac{v\varepsilon}{2l\pi}\right). \tag{2}$$

$$\begin{aligned}
\mathbf{t} &= (a_t,b_t,c_t),\\
a_t &= -\frac{1}{h}\left((r_2-r_1)\cos\varepsilon-(2r_1l\pi+\varepsilon(r_2-r_1))\sin\varepsilon\right),\\
b_t &= -\frac{1}{h}\left(q(r_2-r_1)\sin\varepsilon+(2r_1l\pi+\varepsilon(r_2-r_1))\cos\varepsilon\right),\\
c_t &= -\frac{1}{h}(v), \text{ where}\\
h &= \sqrt{(r_2-r_1)^2+(2r_1l\pi+\varepsilon(r_2-r_1))^2+v^2}.
\end{aligned} \tag{3}$$

$$\begin{aligned}
\mathbf{n} &= (a_n,b_n,c_n) = \frac{\mathbf{tp}\times\mathbf{t}}{|\mathbf{tp}\times\mathbf{t}|},\\
\mathbf{b} &= (a_b,b_b,c_b) = \frac{\mathbf{t}\times\mathbf{n}}{|\mathbf{t}\times\mathbf{n}|}.
\end{aligned} \tag{4}$$

$\mathbf{tp} = ((r_2-r_1)\cos\varepsilon, q(r_2-r_1)\sin\varepsilon, v)$ in equation (4) is the direction vector of the generator of the conical surface passing through the point R.

We will revolve the circle $k_0 = (S,r)$ represented by the vector function $\mathbf{r}(u) = (x(u),y(u),z(u),1)$, $u \in \langle 0,2\pi\rangle$ through the angle $\varepsilon' = m\varepsilon$ about the local coordinate axis x (resp. y, resp. z). The corresponding rotation is represented by the matrix function $T_x(\varepsilon'(\varepsilon))$ (resp. $T_y(\varepsilon'(\varepsilon))$, resp. $T_z(\varepsilon'(\varepsilon))$) in equation (5) [2]. The parameter m is a multiple of the radian velocity of the point R moving on the helix. Right-handed rotation of the circle about the corresponding axis x, y, or z is represented by the parameter $q' = +1$ and left-handed rotation by the parameter $q' = -1$.

$$\begin{aligned}
T_x(\varepsilon'(\varepsilon)) &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\varepsilon'(\varepsilon) & q'\sin\varepsilon'(\varepsilon) & 0\\ 0 & -q'\sin\varepsilon'(\varepsilon) & \cos\varepsilon'(\varepsilon) & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\\
T_y(\varepsilon'(\varepsilon)) &= \begin{pmatrix} \cos\varepsilon'(\varepsilon) & 0 & q'\sin\varepsilon'(\varepsilon) & 0\\ 0 & 1 & 0 & 0\\ -q'\sin\varepsilon'(\varepsilon) & 0 & \cos\varepsilon'(\varepsilon) & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\\
T_z(\varepsilon'(\varepsilon)) &= \begin{pmatrix} \cos\varepsilon'(\varepsilon) & q'\sin\varepsilon'(\varepsilon) & 0 & 0\\ -q'\sin\varepsilon'(\varepsilon) & \cos\varepsilon'(\varepsilon) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.
\end{aligned} \tag{5}$$

Then we will transform the rotated circle k' given in the local coordinate system $(O;x,y,z)$ into the circle k in the coordinate system $(R;t,n,b)$ where are expressed in (4) (Figure 4) by the transformation given by $\mathbf{M}(\varepsilon)$.

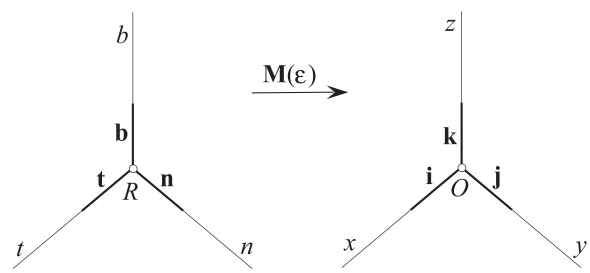

Figure 4: *Transformation (R;t,n,b) into (O;x,y,z)*

The entries in the rows 1, 2 and 3 of $\mathbf{M}(\varepsilon)$ are the coordinates of the unit vectors of the trihedron expressed in equations (3), (4) and the entries in the 4^{th} row are the coordinates of the point R in equation (2)

$$\mathbf{M}(\varepsilon) = \begin{pmatrix} a_t & b_t & c_t & 0\\ a_b & b_b & c_b & 0\\ a_n & b_n & c_n & 0\\ x_R & y_R & z_R & 1\end{pmatrix}. \tag{6}$$

The vector equation of the cyclical surface created by the conical helix is expressed as a product of three matrices for $i = x,y,z$.

$$\mathbf{p}(u,\varepsilon) = \mathbf{r}(u)\cdot\mathbf{T}_i(\varepsilon'(\varepsilon))\cdot\mathbf{M}(\varepsilon). \tag{7}$$

2 Classification of the cyclical surfaces created by a conical helix

The three basic types of cyclical surfaces described above can be characterised with respect to the axis of revolution of the moved circle:

I	the circle rotates about the tangent t,
II	the circle rotates about the binormal b,
III	the circle rotates about the principal normal n

in the point R of the conical helix s.

The circle k_0 in dependence on the axis of revolution $o = x$ (resp. $o = y$, resp. $o = z$) is in:

A	meridian plane,
B	normal plane. (Figure 5)

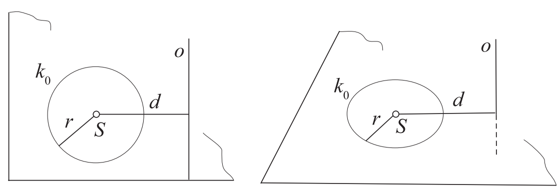

Figure 5: *The circle k_0 in the meridian or normal plane*

The distance $d = |S,o|$ between the centre S of the circle $k_0 = (S,r)$ and the axis of revolution o can be:

a	$d > r$	the circle k_0 does not intersect the axis o,
b	$d = r$	circle k_0 is tangent to the axis o,
c	$d = 0$	the centre of the circle k_0 is on the axis o.

3 Display of the cyclical surfaces created by a conical helix

In this chapter we will display only these cyclical surfaces created by the conical helix, which are applicable in technical practice. Hence we will display cyclical surfaces of types IA, IB, IIA, IIIA. The surfaces of types IIB and IIIB are inapplicable in technical practice in due to their forms.

In Figure 6 the cyclical surfaces of the named types are shown by rotating the circle k_0 about the corresponding coordinate axis in $(R;t,n,b)$ of the right-turned conical helix with $l = 1$ screw then transformed into the coordinate system $(O;x,y,z)$.

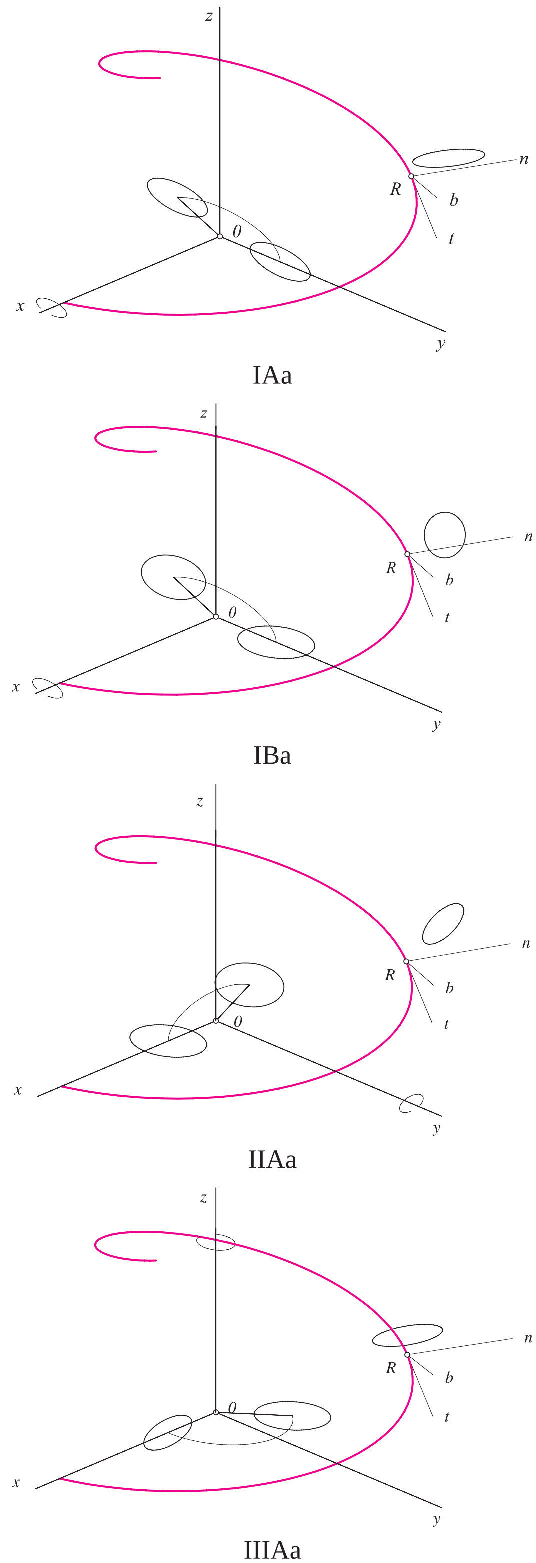

Figure 6: *Types of the cyclical surfaces created by a conical helix*

In Figure 7 a cyclical surfaces of type IA is shown, where the circle k_0 is in the normal plane to the axis of revolution $o = x$. All surfaces shown in Figure 7 have the same parameters $l = 3$, $m = 1$ but different parameter d. If $d > r$ the helix s is not on the surface, if $d = r$ the helix is on the surface and if $d = 0$, then the centre S of the revolving circle is on the helix s. The parameter $q' = +1$ for first three surfaces, $q' = -1$ for forth and fifth surface.

We may express the vector equation of this cyclical surface of type IA from equation (7), where the vector function of the circle k_0 in the normal plane to the axis $o = x$ is $\mathbf{r}(u) = (0, r\cos u + d, r\sin u, 1)$, $u \in \langle 0, 2\pi\rangle$ and we get the corresponding matrix for $i = x$ from equations (5), where $a_t, b_t, \cdots, c_n$ are expressed in equations (3), (4), x_R, y_R, z_R are the coordinates of the point R in the equation (2)

$$\mathbf{p}(u,\varepsilon) = (0, r\cos t + d, r\sin t, 0, 1) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varepsilon'(\varepsilon) & q'\sin\varepsilon'(\varepsilon) & 0 \\ 0 & -q'\sin\varepsilon'(\varepsilon) & \cos\varepsilon'(\varepsilon) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_t & b_t & c_t & 0 \\ a_b & b_b & c_b & 0 \\ a_n & b_n & c_n & 0 \\ x_R & y_R & z_R & 1 \end{pmatrix}.$$

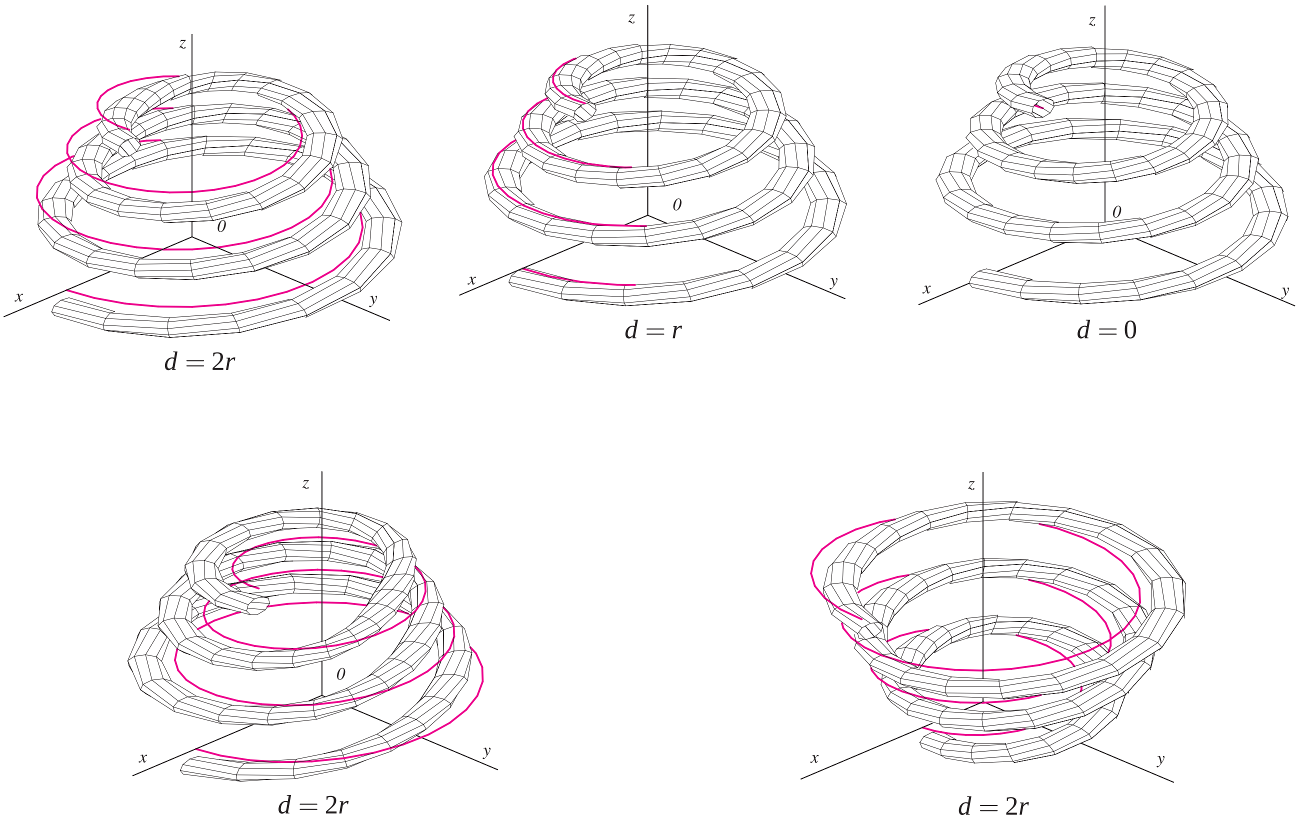

Figure 7: *Cyclical surfaces of type IA*

In Figure 8 cyclical surfaces of type IB are shown, where the circle k_0 is in the meridian plane of the axis of revolution $o = x$. The first two surfaces displayed in Figure 8 have the same parameters $l = 3$, $m = 10, d > r$, and $q = +1$ or $q = -1$. The helix s is not on these surfaces. Third and fourth surfaces have parameters $l = 3$, $m = 15$, $q = +1$ or $q = -1$. These surfaces are applicable only with parameter $m > 1$ in technical practice.

We may express the vector equation of this cyclical surface of type IB similarly as for the surface of type IA, where the vector function of the circle k_0 lying in the meridian plane of the axis $o = x$ is $\mathbf{r}(u) = (r\cos u + d, r\sin u, 0, 1)$, $u \in \langle 0, 2\pi\rangle$.

In Figure 9 cyclical surfaces of types IIB and IIIB are shown, where the circle is in the meridian plane of the axis of revolution $o = y$ or $o = z$. First three surfaces are created by a circle rotating about the binormal and other two about the principal normal of the conical helix s. All surfaces are displayed with the helix s with parameters $l = 3$, $q = +1$.

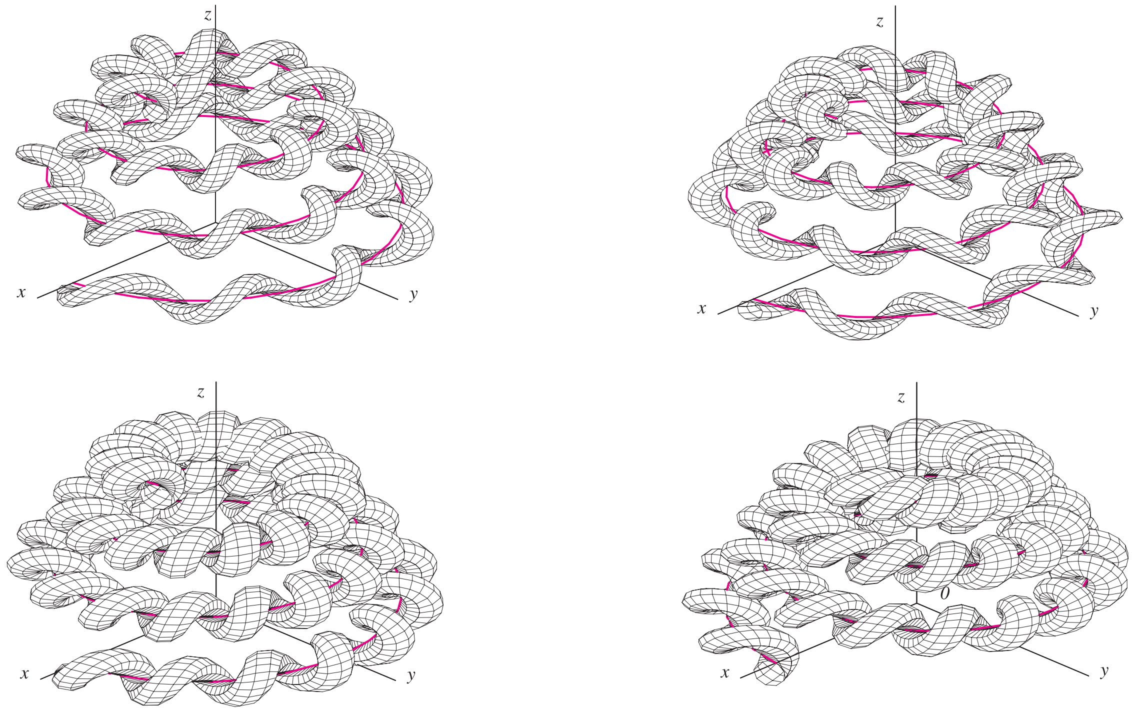

Figure 8: *Cyclical surfaces of type IB*

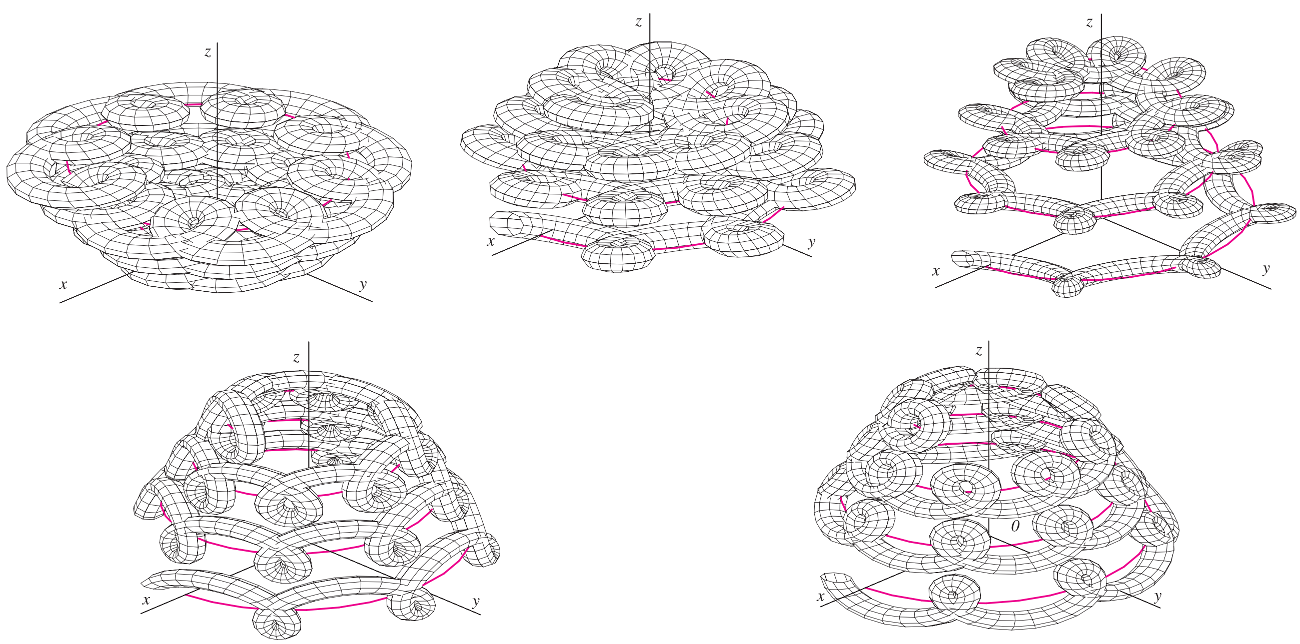

Figure 9: *Cyclical surfaces of type IIB and IIIB*

4 Conclusion

Many of these surfaces may be used in design practice as constructive or ornamental structural components. In Figure 10 there are displayed some illustrations of combinations of these cyclical surfaces with interesting and beautiful aesthetic forms.

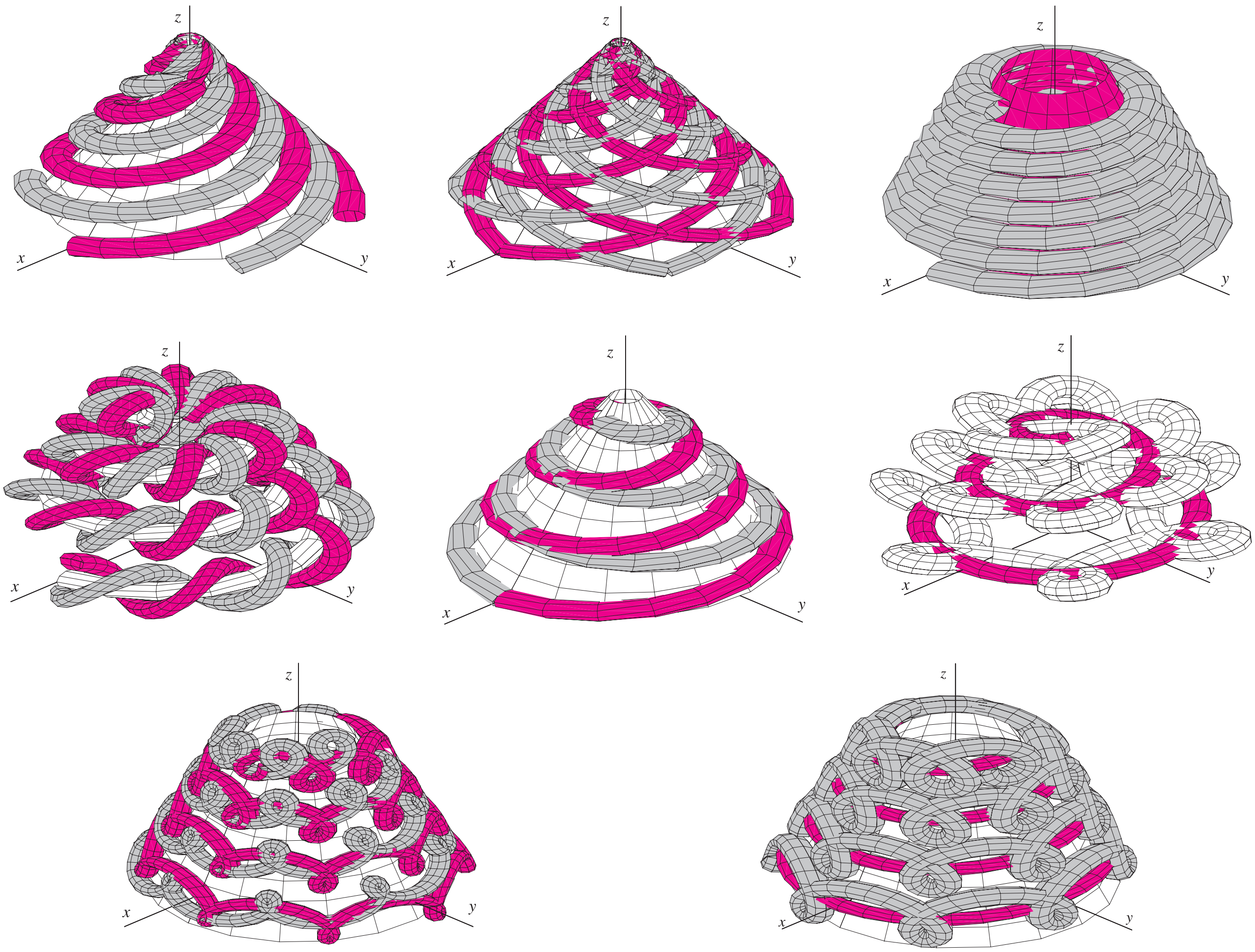

Figure 10: *Combinations of cyclical surfaces of different types*

References

[1] BUDINSKÝ, B., KEPR, B., *Basic of Differential Geometry with Technical Applications*, SNTL - Publishers of Technical Literature, Praha, 1970

[2] GRANAT, L., SECHOVSKY, H., *Computer Graphics*, SNTL - Publishers of Technical Literature, Praha, 1980

[3] MEDEK, V., ZAMOŽIK, J., *Knots and Surfaces*, *Constructive geometry for technics*, ALFA - Publishers of Technical and Economical Literature, Bratislava, 1974

Tatiana Olejníková
Department of Applied Mathematics
Faculty of Civil Engineering
Technical University in Kosice, Slovakia
e-mail: tatiana.olejnikova@tuke.sk